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Research Article

Existence and Iteration of Positive Solutions for One-Dimensional p-Laplacian Boundary Value Problems with Dependence on the First-Order Derivative

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This paper deals with the existence and iteration of positive solutions for the following one-dimensional p-Laplacian boundary value problems: $(\phi_p(u'(t)))' + a(t)f(t,u(t),u'(t)) = 0$, $t \in (0,1)$, subject to some boundary conditions. By making use of monotone iterative technique, not only we obtain the existence of positive solutions for the problems, but also we establish iterative schemes for approximating the solutions.

1. Introduction

In this paper, we are concerned with the existence and iteration of positive solutions for the following one-dimensional p-Laplacian boundary value problems:

$$\big(\phi_p\big(u'(t)\big)\big)' + a(t)f\big(t,u(t),u'(t)\big) = 0, \quad t \in (0,1), \tag{1.1}$$

subject to one of the following boundary conditions:

$$u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 0, \tag{1.2}$$

or

$$\gamma u(0) - \delta u'(0) = 0, \qquad u'(1) = 0, \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$ with $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, $\alpha,\beta,\gamma,\delta > 0$ and f, a satisfy the following:

(H_1) $f : [0,1] \times [0,+\infty) \times (-\infty,0] \to [0,+\infty)$ is continuous;

(H_1^*) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous;

(H_2) $f(t,u,v)$ is nondecreasing in $u - v$ for all $t \in [0,1]$, that is, $f(t,u,v) \geq f(t,x,y)$ for all $u - v \geq x - y$, $t \in [0,1]$;

(H_2^*) $f(t,u,v)$ is nondecreasing in $u + v$ for all $t \in [0,1]$, that is, $f(t,u,v) \geq f(t,x,y)$ for all $u + v \geq x + y$, $t \in [0,1]$;

(H_3) $a : (0,1) \to [0,+\infty)$ is measurable, and $a(t)$ is not identically zero on any compact subinterval of $(0,1)$. Furthermore, $a(t)$ satisfies $0 < \int_0^1 a(t)dt < +\infty$.

Here, a positive solution of (1.1), (1.2) or (1.1), (1.3) means a solution $u^*(t)$ of (1.1), (1.2) or (1.1), (1.3) satisfying $u^*(t) > 0$, $0 < t < 1$.

The boundary value problems (1.1), (1.2) and (1.1), (1.3) deserve a special mention because these forms occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory and turbulent flow of a gas in a porous medium [1].

A consistent account on the existing literature on equation

$$\left(\phi_p(u'(t))\right)' + a(t)f\left(t,u(t)\right) = 0, \quad t \in (0,1), \tag{1.4}$$

is provided and it emphasizes the use of upper and lower solution technique and the fixed point theory, for instance, Krasnoselskii fixed point theorem, the fixed point index of a completely continuous operator with respect to a cone in a Banach space, one may see [2–5] and the references therein. In [6], by using the monotone iterative technique, Ma et al. obtained the existence of monotone positive solution and established the corresponding iterative schemes of (1.4) under the multipoint boundary value condition. However, in their discussion, the nonlinear term f is not involved with the first-order derivative $u'(t)$.

Recently, there is much attention focused on the study of the boundary value problems like (1.1) which the nonlinear term f is involved with the first-order derivative explicitly. In [7], Bai et al. considered the boundary value problems (1.1), (1.2) and (1.1), (1.3) and they proved that problems (1.1), (1.2) and (1.1), (1.3) possessed at least three positive solutions by applying a fixed point theorem due to Avery and Peterson [8]. In [9], the authors also deal with the boundary value problem (1.1), (1.2) via Krasnoselskii fixed point theorem. Here, we should mention that Sun and Ge [10] have got the positive solution of the boundary value problem (1.1), (1.2) by making use of monotone iterative technique.

On the other hand, when f is involved with the first-order derivative explicitly, we can see easily that the results obtained in [1, 7, 9] are only the existence of positive solutions under some suitable conditions. Seeing such a fact, it is an interesting problem which shows how to find these solutions since they exist definitely. Motivated by the above-mentioned results, by making use of the classical monotone iterative technique, we will investigate not only the existence of positive solutions for the boundary value problems (1.1), (1.2) and (1.1), (1.3), but also give iterative schemes for approximating the solutions. Unlike the idea of [10], we will construct a special subset K (see Section 3) and look at $u(t) - u'(t)$ or $u(t) + u'(t)$ as a unit to overcome difficulties when f depends on both $u(t)$ and $u'(t)$. It is worth starting that the first term of our iterative schemes are simple functions which are determined with some linear ordinary equations and cone P (see Remark 3.2). Therefore, the iterative schemes are significant and feasible. At the same time, we will correct some mistakes in [11–13] (see Remark 3.5).

This paper is organized as follows. After this section, some definitions and lemmas will be established in Section 2. In Section 3, we will give our main results Theorems 3.1 and 3.4. Finally, an example is also presented to illustrate our results in Section 4.

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces. We also state some lemmas which are important to proof our main results.

Definition 2.1. Let X be a real Banach space. A nonempty closed set $P \subset X$ is called a cone, if it satisfies the following two conditions:

(i) $a_1 u + a_2 v \in P$, for all $u, v \in P$ and all $a_1 \geq 0$, $a_2 \geq 0$;

(ii) $u \in P$, $-u \in P$ implies $u = 0$.

Definition 2.2. A map τ is said to be concave on $[0,1]$, if

$$\tau(tu + (1-t)v) \geq t\tau(u) + (1-t)\tau(v), \tag{2.1}$$

for all $u, v \in [0,1]$ and $t \in [0,1]$.

Consider the Banach space $X := C^1[0,1]$ equipped with the norm

$$\|u\| := \max_{0\leq t\leq 1}\left[|u(t)| + |u'(t)|\right], \tag{2.2}$$

and define the cone $P \subset X$ by

$$P := \{u(t) \in X : u(t) \geq 0,\ u(t) \text{ is concave and nonincreasing on } [0,1]\}. \tag{2.3}$$

Lemma 2.3 (see [3]). *If $u(t) \in P$, then $u(t) \geq q(t)\max_{0\leq t\leq 1}|u(t)|$, where $q(t) := 1-t$, $t \in [0,1]$.*

Define the operator $A : X \to X$ as follows:

$$(Au)(t) := \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) + \int_t^1 \phi_q\left(\int_0^s a(r)f(r,u(r),u'(r))dr\right)ds, \quad t \in [0,1]. \tag{2.4}$$

Lemma 2.4. *Assume (H_1), (H_3) hold, then $A : P \to P$ is completely continuous.*

Proof. From (H_1), (H_3), it is obviously that $(Au)(t) \geq 0$. Since $(Au)'(t) \leq 0$, we can see that $(Au)'(t)$ is continuous and nonincreasing on $[0,1]$, that is, $(Au)(t)$ is concave on $[0,1]$, so $A : P \to P$ is well defined. The continuity of A is clear because of the continuity of f and a. Now, we will prove that A is compact. Let $\Omega \subset P$ be a bounded set, then there exists D, such that $\Omega \subset \{u(t) \in P : \|u\| \leq D\}$. For any $u(t) \in \Omega$, we have $0 \leq \int_0^1 a(r)f(r,u(r),u'(r))dr \leq \max_{t\in[0,1],u\in[0,D],v\in[-D,0]} f(t,u(t),v(t)) \int_0^1 a(t)dt := E$, then we have

$$|(Au)(t)| \leq \left(\frac{\beta}{\alpha}+1\right)\phi_q(E), \qquad |(Au)'(t)| \leq \phi_q(E), \qquad |(\phi_p((Au)'(t)))'| \leq E. \tag{2.5}$$

The Arzela-Ascoli theorem guarantees that $A\Omega$ is relatively compact, which means that A is compact. Then $A : P \to P$ is completely continuous. □

Lemma 2.5. *Assume (H_1), (H_2) and (H_3) hold. If $\omega_1(t), \omega_2(t) \in P$ such that $\omega_1(t) - \omega_1'(t) \geq \omega_2(t) - \omega_2'(t)$, $t \in [0,1]$, then $(A\omega_1)(t) \geq (A\omega_2)(t)$, $(A\omega_1)'(t) \leq (A\omega_2)'(t)$, $t \in [0,1]$.*

Proof. Noticing that $f(t,u,v)$ is nondecreasing in $u-v$, $t\in[0,1]$, the proof is simple, here we omit it. □

3. Main results

For convenience, we denote

$$C_1 := \min\left\{1,\frac{\beta}{\alpha}\right\},\qquad C_2 := 1+\frac{\beta}{\alpha},\qquad S_1 := \left(1+C_1^{-1}\right)^{-1},\qquad S_2 := 8C_1^{-1}C_2,$$
$$C_3 := \min\left\{1,\frac{\delta}{\gamma}\right\},\qquad C_4 := 1+\frac{\delta}{\gamma},\qquad S_3 := \left(1+C_3^{-1}\right)^{-1},\qquad S_4 := 8C_3^{-1}C_4,$$
$$M_1 := \left[\left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r)dr\right)\right]^{-1},\qquad N_1 := \left[\frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r)dr\right)\right]^{-1},$$
$$M_2 := \left[\left(\frac{\delta}{\gamma}+2\right)\phi_q\left(\int_0^1 a(r)dr\right)\right]^{-1},\qquad N_2 := \left[\frac{\delta}{\gamma}\phi_q\left(\int_{\theta_2}^{1-\theta_2} a(r)dr\right)\right]^{-1}, \tag{3.1}$$

where $\theta_1\in(0,(2C_2)^{-1})\subset(0,1/2)$, $\theta_2\in(0,(2C_4)^{-1})\subset(0,1/2)$. It is easy to see that $0<S_1$, $S_3<1$, $S_2,S_4>8$.

Theorem 3.1. *Assume* (H_1), (H_2), *and* (H_3) *hold. Moreover, suppose that there exist six constants* R_i, r_i, L_i, $i=1,2$, *with* $R_1,r_1,L_1>0$, $R_2,r_2,L_2<0$, $S_1(L_1-L_2)>S_2(r_1-r_2)$ *and* $R_1-R_2:=4\theta_1C_2(r_1-r_2)$, *such that*

(H_4) $\max_{0\le t\le 1}f(t,L_1,L_2)\le\phi_p[(L_1-L_2)S_1M_1]$;

(H_5) $\min_{\theta_1\le t\le 1-\theta_1}f(t,R_1,R_2)\ge\phi_p[(r_1-r_2)S_2N_1]$.

Then the boundary value problem (1.1), (1.2) *has at least two nonincreasing positive solutions* $u^*(t)$ *and* $v^*(t)\in P$ *with*

$$S_2(r_1-r_2)\le\|u^*\|\le S_1(L_1-L_2),\qquad S_2(r_1-r_2)\le\|v^*\|\le S_1(L_1-L_2), \tag{3.2}$$

and $\lim_{n\to\infty}(A^nu_0)(t)=u^*(t)$, $\lim_{n\to\infty}(A^nv_0)(t)=v^*(t)$, *where*

$$u_0(t)=L_1-L_2+C_5e^t,\quad v_0(t)=\left(q(t)+1\right)\left(r_1-r_2\right)+C_6e^t,\quad t\in[0,1], \tag{3.3}$$

C_5 *and* C_6 *are arbitrary constants which satisfy* $-e^{-1}(L_1-L_2)\le C_5\le 0$, $-e^{-1}(r_1-r_2)\le C_6\le 0$.

Proof. We denote a set $K\subset P$ by

$$K := \Big\{u(t)\in P : C_1\max_{0\le t\le 1}\left|u'(t)\right|\le\left|u(t)\right|,\ \max_{0\le t\le 1}\left|u(t)\right|\le C_2\max_{0\le t\le 1}\left|u'(t)\right|,$$
$$S_2\left(r_1+\left|r_2\right|\right)\le\|u\|\le S_1\left(L_1+\left|L_2\right|\right)\Big\}. \tag{3.4}$$

Based on the preceding preliminaries, we can divide our proof into three steps.

Step 1. We first prove $A(K) \subset K$. Let $u(t) \in K$, note that $u(t)$, $u'(t)$ is nonincreasing on $[0,1]$, then

$$0 \le u(t) - u'(t) \le \max_{0\le t\le 1}\left[\left|u(t)\right| + \left|u'(t)\right|\right] = \|u\| \le S_1\left(L_1 - L_2\right) \le L_1 - L_2, \quad t \in [0,1]. \tag{3.5}$$

By Lemma 2.3, we have

$$\begin{aligned}
u(t) - u'(t) &\ge \min_{\theta_1\le t\le 1-\theta_1}\left|u(t)\right| \\
&\ge \min_{\theta_1\le t\le 1-\theta_1} q(t)\max_{0\le t\le 1}\left|u(t)\right| \\
&= \frac{1}{2}\theta_1\left[\max_{0\le t\le 1}\left|u(t)\right| + \max_{0\le t\le 1}\left|u(t)\right|\right] \\
&\ge \frac{1}{2}\theta_1\left[\max_{0\le t\le 1}\left|u(t)\right| + C_1\max_{0\le t\le 1}\left|u'(t)\right|\right] \\
&\ge \frac{1}{2}\theta_1 \min\left\{1, C_1\right\}\|u\| \\
&\ge \frac{1}{2}\theta_1 C_1 S_2\left(r_1 - r_2\right) \\
&= 4\theta_1 C_2\left(r_1 - r_2\right) \\
&= R_1 - R_2, \quad t \in \left[\theta_1, 1-\theta_1\right].
\end{aligned} \tag{3.6}$$

Since $f(t,u,v)$ is nondecreasing in $u - v$, $t \in [0,1]$, and by assumptions (H_4) and (H_5), we obtain

$$\begin{gathered}
0 \le f\left(t, u(t), u'(t)\right) \le f\left(t, L_1, L_2\right) \le \phi_p\left[\left(L_1 - L_2\right)S_1 M_1\right], \quad t \in [0,1], \\
f\left(t, u(t), u'(t)\right) \ge f\left(t, R_1, R_2\right) \ge \phi_p\left[\left(r_1 - r_2\right)S_2 N_1\right], \quad t \in \left[\theta_1, 1-\theta_1\right],
\end{gathered} \tag{3.7}$$

which imply that

$$\begin{aligned}
\|Au\| &= \max_{0\le t\le 1}\left[\left|(Au)(t)\right| + \left|(Au)'(t)\right|\right] \\
&= \max_{0\le t\le 1}\left\{\phi_q\left(\int_0^t a(r)f(r,u(r),u'(r))dr\right) + \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right)\right. \\
&\qquad\left. + \int_t^1 \phi_q\left(\int_0^s a(r)f(r,u(r),u'(r))dr\right)ds\right\} \\
&\le \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&\le \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r)f(r,L_1,L_2)dr\right) \\
&\le \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r)dr\right)S_1(L_1-L_2)M_1 \\
&= S_1(L_1-L_2),
\end{aligned}$$

$$\begin{aligned}
\|Au\| &= \max_{0\le t\le 1}\left[\left|(Au)(t)\right| + \left|(Au)'(t)\right|\right] \\
&\ge \max_{0\le t\le 1}\left|(Au)(t)\right| \\
&\ge \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&\ge \frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r)f(r,R_1,R_2)dr\right) \\
&\ge \frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r)dr\right)S_2(r_1-r_2)N_1 \\
&= S_2(r_1-r_2).
\end{aligned} \tag{3.8}$$

On the other hand, for $u(t) \in K$, we have

$$\begin{aligned}
\left|(Au)(t)\right| &= \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&\quad + \int_t^1 \phi_q\left(\int_0^s a(r)f(r,u(r),u'(r))dr\right)ds \\
&\geq \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&\geq C_1\max_{0\leq t\leq 1}\left|(Au)'(t)\right|, \\
\max_{0\leq t\leq 1}\left|(Au)(t)\right| &= \frac{\beta}{\alpha}\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&\quad + \int_0^1 \phi_q\left(\int_0^s a(r)f(r,u(r),u'(r))dr\right)ds \\
&\leq \left(\frac{\beta}{\alpha}+1\right)\phi_q\left(\int_0^1 a(r)f(r,u(r),u'(r))dr\right) \\
&= C_2\max_{0\leq t\leq 1}\left|(Au)'(t)\right|.
\end{aligned} \tag{3.9}$$

In virtue of (3.8)-(3.9), $A(K) \subset K$.

Step 2. Let $u_0(t) = L_1 - L_2 + C_5e^t$, C_5 is an arbitrary constant which satisfies $-e^{-1}(L_1 - L_2) \leq C_5 \leq 0$, $t \in [0,1]$, then $u_0(t) \geq 0$, $u_0'(t) = C_5e^t \leq 0$, $u_0''(t) = C_5e^t \leq 0$, $t \in [0,1]$. Hence, $u_0(t) \in P$, $u_0(t) - u_0'(t) = L_1 - L_2$, $t \in [0,1]$. Let $u_1(t) = (Au_0)(t)$, next we claim that $u_1(t) \in K$. Indeed, it is easy to check that

$$\begin{aligned}
\left|u_1(t)\right| &= \left|(Au_0)(t)\right| \geq C_1\max_{0\leq t\leq 1}\left|(Au_0)'(t)\right| = C_1\max_{0\leq t\leq 1}\left|u_1'(t)\right|, \\
\max_{0\leq t\leq 1}\left|u_1(t)\right| &= \max_{0\leq t\leq 1}\left|(Au_0)'(t)\right| \leq C_2\max_{0\leq t\leq 1}\left|(Au_0)'(t)\right| = C_2\max_{0\leq t\leq 1}\left|u_1'(t)\right|.
\end{aligned} \tag{3.10}$$

Using assumptions $S_1(L_1-L_2) > S_2(r_1-r_2)$ and $R_1-R_2 = 4\theta_1 C_2(r_1-r_2)$, we have $u_0(t)-u_0'(t) = L_1-L_2 \geq 8(r_1-r_2) \geq 4\theta_1 C_2(r_1-r_2) = R_1-R_2$, $t \in [\theta_1, 1-\theta_1]$, these imply that

$$\begin{aligned}
\|u_1\| = \|Au_0\| &= \max_{0\le t\le 1}\left[|(Au_0)(t)| + |(Au_0)'(t)|\right] \\
&\leq \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r) f(r, u_0(r), u_0'(r))\,dr\right) \\
&\leq \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r) f(r, L_1, L_2)\,dr\right) \\
&\leq \left(\frac{\beta}{\alpha}+2\right)\phi_q\left(\int_0^1 a(r)\,dr\right) S_1(L_1-L_2) M_1 \\
&\leq S_1(L_1-L_2), \\
\|u_1\| = \|Au_0\| &= \max_{0\le t\le 1}\left[|(Au_0)(t)| + |(Au_0)'(t)|\right] \\
&\geq \frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r) f(r, u_0(r), u_0'(r))\,dr\right) \\
&\geq \frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r) f(r, R_1, R_2)\,dr\right) \\
&\geq \frac{\beta}{\alpha}\phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r)\,dr\right) S_2(r_1-r_2) N_1 \\
&= S_2(r_1-r_2).
\end{aligned} \tag{3.11}$$

Therefore, $u_1(t) \in K$. We denote

$$u_{n+1}(t) := (Au_n)(t) = (A^{n+1}u_0)(t), \quad n = 0, 1, 2, \ldots. \tag{3.12}$$

Since $A(K) \subset K$, $u_n(t) \in K$, $n = 1, 2, \ldots$. From Lemma 2.4, A is compact, we assert that $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and there exists $u^*(t) \in K$, such that $u_{n_k}(t) \to u^*(t)$. Now, since $u_1(t) \in K \subset P$, for $t \in [0,1]$, we have

$$\begin{aligned}
u_1(t) - u_1'(t) &\leq (1 + C_1^{-1})|u_1(t)| \leq (1 + C_1^{-1})\|u_1\| \\
&\leq (1 + C_1^{-1}) S_1(L_1 - L_2) = L_1 - L_2 = u_0(t) - u_0'(t).
\end{aligned} \tag{3.13}$$

This combined with Lemma 2.5 gives

$$u_2(t) = (Au_1)(t) \leq (Au_0)(t) = u_1(t), \qquad u_2'(t) = (Au_1)'(t) \geq (Au_0)'(t) = u_1'(t), \tag{3.14}$$

so

$$u_2(t) - u_2'(t) \leq u_1(t) - u_1'(t), \quad 0 \leq t \leq 1. \tag{3.15}$$

By induction,

$$u_{n+1}(t) \le u_n(t), \quad u'_{n+1}(t) \ge u'_n(t), \quad 0 \le t \le 1, \ n = 1,2,\dots. \tag{3.16}$$

Hence, we assert that $u_n(t) \to u^*(t)$. Let $n \to \infty$ in (3.12) to obtain $(Au^*)(t) = u^*(t)$ since A is continuous. Since $\|u^*\| \ge S_2(r_1 - r_2) > 0$ and $u^*(t)$ is a nonnegative concave function on $[0,1]$, we conclude that $u^*(t) > 0$, $t \in (0,1)$. It is well known that the fixed point of operator A is the solution of the boundary value problem (1.1), (1.2). Therefore, $u^*(t)$ is a positive, nonincreasing solution of the boundary value problem (1.1), (1.2).

Step 3. Put $v_0(t) = (q(t)+1)(r_1 - r_2) + C_6 e^t$, C_6 is an arbitrary constant satisfying $-e^{-1}(r_1 - r_2) \le C_6 \le 0$, $t \in [0,1]$, then $v_0(t) \ge 0$, $v'_0(t) = -(r_1 - r_2) + C_6 e^t \le 0$, $v''_0(t) = C_6 e^t \le 0$, $t \in [0,1]$. Hence, $v_0(t) \in P$. From the definition of θ_1 and $R_1 - R_2$, we derive that

$$\begin{aligned} v_0(t) - v'_0(t) &= (q(t)+2)(r_1 - r_2) \ge (2+\theta_1)(r_1 - r_2) \\ &\ge 2(r_1 - r_2) \ge 4\theta_1 C_2 (r_1 - r_2) = R_1 - R_2, \quad t \in [\theta_1, 1-\theta_1], \\ v_0(t) - v'_0(t) &= (q(t)+2)(r_1 - r_2) \le 3(r_1 - r_2) \le L_1 - L_2, \quad t \in [0,1]. \end{aligned} \tag{3.17}$$

Setting $v_1(t) = (Av_0)(t)$, in what follows, we will prove that $v_1(t) \in K$. In fact, similar to (3.10)-(3.11), combined with the above inequalities, one has

$$\begin{aligned} |v_1(t)| &= |(Av_0)(t)| \ge C_1 \max_{0\le t\le 1} |(Av_0)'(t)| = C_1 \max_{0\le t\le 1} |v'_1(t)|, \\ \max_{0\le t\le 1}|v_1(t)| &= \max_{0\le t\le 1} |(Av_0)'(t)| \le C_2 \max_{0\le t\le 1} |(Av_0)'(t)| = C_2 \max_{0\le t\le 1} |u'_1(t)|, \\ \|v_1\| = \|Av_0\| &= \max_{0\le t\le 1} \left[|(Av_0)(t)| + |(Av_0)'(t)| \right] \\ &\le \left(\frac{\beta}{\alpha} + 2\right) \phi_q\left(\int_0^1 a(r) f(r, v_0(r), v'_0(r))\, dr\right) \\ &\le \left(\frac{\beta}{\alpha} + 2\right) \phi_q\left(\int_0^1 a(r) f(r, L_1, L_2)\, dr\right) \\ &\le S_1(L_1 - L_2), \\ \|v_1\| = \|Av_0\| &= \max_{0\le t\le 1} \left[|(Av_0)(t)| + |(Av_0)'(t)| \right] \\ &\ge \frac{\beta}{\alpha} \phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r) f(r, v_0(r), v'_0(r))\, dr\right) \\ &\ge \frac{\beta}{\alpha} \phi_q\left(\int_{\theta_1}^{1-\theta_1} a(r) f(r, R_1, R_2)\, dr\right) \\ &= S_2(r_1 - r_2). \end{aligned} \tag{3.18}$$

We deduce from (3.18) that $v_1(t) \in K$. Denote

$$v_{n+1}(t) := (Av_n)(t) = (A^{n+1} v_0)(t), \quad n = 0,1,2,\dots. \tag{3.19}$$

Since $v_1(t) \in K \subset P$, from Lemma 2.4, we have

$$
\begin{aligned}
v_1(t) - v_1'(t) &\geq \frac{1}{2}|v_1(t)| + \frac{1}{2}|v_1(t)| \\
&\geq \frac{1}{2}q(t)\max_{0\leq t\leq 1}|v_1(t)| + \frac{1}{2}C_1\max_{0\leq t\leq 1}|v_1'(t)| \\
&\geq \frac{1}{4}q(t)\left[\max_{0\leq t\leq 1}|v_1(t)| + C_1\max_{0\leq t\leq 1}|v_1'(t)|\right] \\
&\quad + \frac{1}{4}C_1\left[C_2^{-1}\max_{0\leq t\leq 1}|v_1(t)| + \max_{0\leq t\leq 1}|v_1'(t)|\right] \\
&\geq \frac{1}{4}C_1 q(t)\|v_1\| + \frac{1}{4}C_1C_2^{-1}\|v_1\| \\
&\geq \frac{1}{4}C_1 q(t)S_2(r_1 - r_2) + \frac{1}{4}C_1C_2^{-1}S_2(r_1 - r_2) \\
&\geq (q(t) + 2)(r_1 - r_2) \\
&= v_0(t) - v_0'(t), \quad t \in [0,1].
\end{aligned}
\tag{3.20}
$$

By Lemma 2.5, we get

$$
v_2(t) = (Av_1)(t) \geq (Av_0)(t) = v_1(t), \qquad v_2'(t) = (Av_1)'(t) \leq (Av_0)'(t) = v_1'(t), \tag{3.21}
$$

so $v_2(t) - v_2'(t) \geq v_1(t) - v_1'(t)$, $0 \leq t \leq 1$. By induction, $v_{n+1}(t) \geq v_n(t)$, $v_{n+1}'(t) \leq v_n'(t)$, $0 \leq t \leq 1$, $n = 1, 2, \ldots$. Hence, we assert that $v_n(t) \to v^*(t)$, and $v^*(t) > 0$, $t \in (0,1)$. Therefore, $v^*(t)$ is a positive, nonincreasing solution of the boundary value problem (1.1), (1.2). □

Remark 3.2. (i) We can easily get that $u^*(t)$ and $v^*(t)$ are the maximal and minimal solutions of the boundary value problem (1.1), (1.2) in K. Of course u^* and v^* may coincide and then the boundary value problem (1.1), (1.2) has only one solution in K.

(ii) It is worth pointing out that $u_0(t), v_0(t) \notin K$. In fact, $u_0(t) \in G_1$, $v_0(t) \in G_2$, which are determined with some linear ordinary equations and the cone P, this is different from the results in [6, 10], where

$$
\begin{aligned}
G_1 &:= \{u_0(t) \in P : u_0(t) - u_0'(t) = L_1 - L_2\}, \\
G_2 &:= \{v_0(t) \in P : v_0(t) - v_0'(t) = (q(t) + 2)(r_1 - r_2)\}.
\end{aligned}
\tag{3.22}
$$

Corollary 3.3. *Assume (H$_1$), (H$_2$) and (H$_3$) hold, suppose that*

(H$_6$) $f_0 := \limsup_{|u(t)|+|v(t)|\to 0} \min_{\theta_1\leq t\leq 1-\theta_1}(f(t,u(t),v(t))/(|u(t)|+|v(t)|)^{p-1}) > \phi_p(2N_1/\theta_1 C_1)$;

(H$_7$) $f_\infty := \liminf_{|u(t)|+|v(t)|\to\infty} \max_{0\leq t\leq 1}(f(t,u(t),v(t))/(|u(t)|+|v(t)|)^{p-1}) < \phi_p(M_1)$.

(Particularly, $f_0 = +\infty$, $f_\infty = 0$). Then there exist four constants r_i, L_i, $i = 1,2$, with $r_1, L_1 > 0$, $r_2, L_2 < 0$, $S_1(L_1 - L_2) > S_2(r_1 - r_2)$, such that the boundary value problem (1.1), (1.2) *has at least two nonincreasing positive solutions $u^*(t)$ and $v^*(t) \in P$ with*

$$S_2(r_1 - r_2) \le \|u^*\| \le S_1(L_1 - L_2), \qquad S_2(r_1 - r_2) \le \|v^*\| \le S_1(L_1 - L_2), \tag{3.23}$$

and $\lim_{n\to\infty}(A^n u_0)(t) = u^*(t)$, $\lim_{n\to\infty}(A^n v_0)(t) = v^*(t)$, where

$$u_0(t) = L_1 - L_2 + C_5 e^t, \quad v_0(t) = (q(t) + 1)(r_1 - r_2) + C_6 e^t, \quad t \in [0,1], \tag{3.24}$$

C_5 and C_6 are arbitrary constants which satisfy $-e^{-1}(L_1 - L_2) \le C_5 \le 0$, $-e^{-1}(r_1 - r_2) \le C_6 \le 0$.

Proof. It is very easy to verify the conditions (H_4) and (H_5) can be obtained from (H_6) and (H_7), so we omit the proof. □

Obviously, though the similar arguments of Theorem 3.1, we could get the following theorem.

Theorem 3.4. *Assume (H_1^*), (H_2^*) and (H_3^*) hold, suppose that there exist six positive constants R_j, r_j, L_j, $j = 3,4$, with $S_3(L_3 + L_4) > S_4(r_3 + r_4)$ and $R_3 + R_4 := 4\theta_2 C_4(r_3 + r_4)$, such that*

(H_8) $\max_{0\le t\le 1} f(t, L_3, L_4) \le \phi_p[(L_3 + L_4)S_3 M_2]$;

(H_9) $\min_{\theta_2\le t\le 1-\theta_2} f(t, R_3, R_4) \ge \phi_p[(r_3 + r_4)S_4 N_2]$.

Then the boundary value problem (1.1), (1.3) *has at least two nondecreasing positive solutions $\overline{u}^*(t)$ and $\overline{v}^*(t) \in \overline{P}$ with*

$$S_4(r_3 + r_4) \le \|\overline{u}^*\| \le S_3(L_3 + L_4), \qquad S_4(r_3 + r_4) \le \|\overline{v}^*\| \le S_3(L_3 + L_4), \tag{3.25}$$

and $\lim_{n\to\infty}(\overline{A}^n \overline{u}_0)(t) = \overline{u}^*(t)$, $\lim_{n\to\infty}(\overline{A}^n \overline{v}_0)(t) = \overline{v}^*(t)$, where

$$\overline{P} := \{u(t) \in X : u(t) \ge 0,\ u(t) \text{ is concave and nondecreasing on } [0,1]\}.$$

$$(\overline{A}u)(t) := \frac{\delta}{\gamma}\phi_q\left(\int_0^1 a(r) f(r, u(r), u'(r))dr\right) + \int_0^t \phi_q\left(\int_s^1 a(r) f(r, u(r), u'(r))dr\right)ds, \quad t \in [0,1],$$

$$\overline{u}_0(t) = L_3 + L_4 + C_7 e^{-t}, \quad \overline{v}_0(t) = (t+1)(r_3 + r_4) + C_8 e^{-t}, \quad t \in [0,1]. \tag{3.26}$$

C_7 and C_8 are arbitrary constants which satisfy $-(L_3 + L_4) \le C_7 \le 0$, $-(r_3 + r_4) \le C_8 \le 0$.

Remark 3.5. In [11], Liu and Zhang studied the following boundary value problem:

$$\begin{gathered}(\phi(u'(t)))' + a(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) - a_1 u'(0) = 0, \qquad u(1) + a_2 u'(1) = 0,\end{gathered} \tag{3.27}$$

where $a_1, a_2 \geq 0$ and ϕ is an increasing positive homomorphism and homeomorphism with $\phi(0) = 0$ (for more details, see [11–13]). In that paper, the authors claimed that $u \in C[0,1] \cap C^1(0,1)$ is a solution if and only if $u \in C[0,1]$ is a solution of the following integral equation:

$$u(t) = \begin{cases} a_1\phi^{-1}\left(\int_0^\tau a(s)f(u(s))ds\right) + \int_0^t \phi^{-1}\left(\int_s^\tau a(r)f(u(r))dr\right)ds, & 0 \leq t \leq \tau, \\ a_2\phi^{-1}\left(\int_\tau^1 a(s)f(u(s))ds\right) + \int_t^1 \phi^{-1}\left(\int_\tau^s a(r)f(u(r))dr\right)ds, & \tau \leq t \leq 1, \end{cases} \tag{3.28}$$

where $\tau = 0$, if $u'(0) = 0$; $\tau = 1$, if $u'(1) = 0$, otherwise τ is a solution of the equation

$$g_1(t) = g_2(t), \tag{3.29}$$

where

$$\begin{aligned} g_1(t) &= a_1\phi^{-1}\left(\int_0^t a(s)f(u(s))ds\right) + \int_0^t \phi^{-1}\left(\int_s^t a(r)f(u(r))dr\right)ds, \quad 0 \leq t < 1, \\ g_2(t) &= a_2\phi^{-1}\left(\int_t^1 a(s)f(u(s))ds\right) + \int_t^1 \phi^{-1}\left(\int_t^s a(r)f(u(r))dr\right)ds, \quad 0 < t \leq 1. \end{aligned} \tag{3.30}$$

Unfortunately, such a claim is incorrect since

$$\phi(-u) \neq -\phi(u). \tag{3.31}$$

Similar reason, the results in [12, 13] are also incorrect, as they deal with the nonlinear systems. Under the boundary condition (1.3), we could avoid (3.31) to occur. With the similar argument of Theorem 3.4, we could have the similar theorem to Theorem 3.4 for the following boundary value problem:

$$\begin{gathered} (\phi(u'(t)))' + a(t)f(t,u(t),u'(t)) = 0, \quad t \in (0,1), \\ \gamma u(0) - \delta u'(0) = 0, \qquad u'(1) = 0. \end{gathered} \tag{3.32}$$

Here, we omit the proofs.

4. Example

In this section, we will give an example to illustrate our results.

Example 4.1. Consider the boundary value problem

$$\begin{gathered} (\phi_p(u'(t)))' + a(t)f(t,u(t),u'(t)) = 0, \quad t \in (0,1), \\ u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 0, \end{gathered} \tag{4.1}$$

where $\phi_p(s) = |s|^{p-2}s$ with $p > 1$, take $p = 3$, $\alpha = 1$, $\beta = 1$, $a(t) = 1$, $t \in [0,1]$,

$$f(t,u,v) = t + u - v, \quad u \geq 0, \ v \leq 0, \ t \in [0,1], \tag{4.2}$$

we have $q = 3/2$, $C_1 = 1$, $C_2 = 2$, $S_1 = 1/2$ and $S_2 = 16$. Take $\theta_1 = 1/8$, we get $M_1 = 1/3$, $N_1 = 2\sqrt{3}/3$. Choose $L_1 = 100$, $L_2 = -100$, $r_1 = 0.01$, $r_2 = -0.01$, $R_1 = 0.01$, and $R_2 = -0.01$, so $f(t,u(t),v(t))$ satisfies the following:

(1) $f : [0,1] \times [0,+\infty) \times (-\infty,0] \to [0,+\infty)$ is continuous;

(2) $f(t,u,v)$ is nondecreasing in $u - v$ for all $t \in [0,1]$;

(3) $\max_{0\le t\le 1} f(t,L_1,L_2) = \max_{0\le t\le 1} f(t,100,-100) = 201 \le \phi_3[(L_1 - L_2)S_1M_1] = [(100+|-100|) \times 1/2 \times 1/3]^2 \approx 1111.1$;

(4) $\min_{\theta_1\le t\le 1-\theta_1} f(t,R_1,R_2) = \min_{\theta_1\le t\le 1-\theta_1} f(t,0.01,-0.01) = 0.145 \ge \phi_3[(r_1 - r_2)S_2N_1] = [(0.01 - 0.01) \times 16 \times 2\sqrt{3}/3]^2 \approx 0.1365$.

Therefore, by Theorem 3.1, the boundary value problem (4.1) has at least two nonincreasing positive solutions $u^*(t)$ and $v^*(t)$, such that

$$0.32 \le \|u^*\| \le 100, \qquad 0.32 \le \|v^*\| \le 100, \tag{4.3}$$

and $\lim_{n\to\infty}(A^n u_0)(t) = u^*(t)$, $\lim_{n\to\infty}(A^n v_0)(t) = v^*(t)$, where

$$u_0(t) = 200 + C_5 e^t, \quad v_0(t) = 0.02(2-t) + C_6 e^t, \quad t \in [0,1]. \tag{4.4}$$

C_5 and C_6 are arbitrary constants which satisfy $-200/e \le C_5 \le 0$, $-0.02/e \le C_6 \le 0$.

For $n = 1,2,\ldots$, the two iterative schemes are $u_0(t) = 200 + C_5 e^t$, $t \in [0,1]$, C_5 is arbitrary constant with $-200/e \le C_5 \le 0$,

$$\begin{aligned} u_{n+1}(t) = (Au_n)(t) &= \left[\int_0^1 \left(r + u_n(r) - u_n'(r)\right) dr\right]^{1/2} \\ &\quad + \int_t^1 \left(\int_0^s \left(r + u_n(r) - u_n'(r)\right) dr\right)^{1/2} ds, \quad t \in [0,1], \end{aligned} \tag{4.5}$$

$v_0(t) = 0.02(2-t) + C_6 e^t$, $t \in [0,1]$, C_6 is arbitrary constant with $-0.02/e \le C_6 \le 0$,

$$\begin{aligned} v_{n+1}(t) = (Av_n)(t) &= \left[\int_0^1 \left(r + v_n(r) - v_n'(r)\right) dr\right]^{1/2} \\ &\quad + \int_t^1 \left(\int_0^s \left(r + v_n(r) - v_n'(r)\right) dr\right)^{1/2} ds, \quad t \in [0,1]. \end{aligned} \tag{4.6}$$

Remark 4.2. The nonlinear term $f(t,u,v)$ in v is nonincreasing, so the results in [10] do not hold.

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References

[1] D. O'Regan, "Some general existence principles and results for $(\phi_p(y'))' = q(t)f(t, y, y')$, $0 < t < 1$," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 3, pp. 648–668, 1993.

[2] D. Jiang and W. Gao, "Upper and lower solution method and a singular boundary value problem for the one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 631–648, 2000.

[3] J. Wang, "The existence of positive solutions for the one-dimensional p-Laplacian," *Proceedings of the American Mathematical Society*, vol. 125, no. 8, pp. 2275–2283, 1997.

[4] Y. Guo and W. Ge, "Three positive solutions for the one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 491–508, 2003.

[5] L. Kong and J. Wang, "Multiple positive solutions for the one-dimensional p-Laplacian," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 8, pp. 1327–1333, 2000.

[6] D.-X. Ma, Z.-J. Du, and W.-G. Ge, "Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator," *Computers & Mathematics with Applications*, vol. 50, no. 5-6, pp. 729–739, 2005.

[7] Z. Bai, Z. Gui, and W. Ge, "Multiple positive solutions for some p-Laplacian boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 477–490, 2004.

[8] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 313–322, 2001.

[9] Z. Wang and J. Zhang, "Positive solutions for one-dimensional p-Laplacian boundary value problems with dependence on the first order derivative," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 618–630, 2006.

[10] B. Sun and W. Ge, "Existence and iteration of positive solutions for some p-Laplacian boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 6, pp. 1820–1830, 2007.

[11] B. Liu and J. Zhang, "The existence of positive solutions for some nonlinear boundary value problems with linear mixed boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 505–516, 2005.

[12] B. Liu and J. Zhang, "The existence of positive solutions for some nonlinear equation systems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 970–981, 2006.

[13] D. Ji, H. Feng, and W. Ge, "The existence of symmetric positive solutions for some nonlinear equation systems," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 51–59, 2008.