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Research Article

Existence and Uniqueness of Very Singular Solution of a Degenerate Parabolic Equation with Nonlinear Convection

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We here investigate the existence and uniqueness of the nontrivial, nonnegative solutions of a nonlinear ordinary differential equation: $(|f'|^{p-2}f')' + \beta r f' + \alpha f + (f^q)' = 0$ satisfying a specific decay rate: $\lim_{r\to\infty} r^{\alpha/\beta} f(r) = 0$ with $\alpha := (p-1)/(pq-2p+2)$ and $\beta := (q-p+1)/(pq-2p+2)$. Here $p > 2$ and $q > p-1$. Such a solution arises naturally when we study a very singular self-similar solution for a degenerate parabolic equation with nonlinear convection term $u_t = (|u_x|^{p-2}u_x)_x + (u^q)_x$ defined on the half line $[0, +\infty)$.

1. Introduction

In this paper, we consider a quasilinear degenerate diffusion equation with nonlinear convection term defined on the half line as

$$u_t = (|u_x|^{p-2}u_x)_x + (u^q)_x, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.1}$$

with homogeneous *Neumann* boundary condition

$$u_x(0,t) = 0, \tag{1.2}$$

where $p > 2, q > p-1$.

Equation (1.1) (sometimes called the non-*Newtonian* filtration equation or p-Laplacian equation) arises in the study of the compressible fluid flows in a homogeneous isotropic rigid porous medium, combustion of solid fuels and has various other applications; see, [1, 2].

From a mathematical point of view, we note that (1.1) is a quasilinear equation which is nonuniform parabolic, and it *degenerates* on the set $\{u_x = 0\}$ (if $q = 1$, (1.1) reduces to the standard p-Laplacian by an easy change of variables).

We are mostly interested in nonnegative solutions of (1.1) having the form

$$u(x,t) = t^{-\alpha} f\left(xt^{-\beta}\right) := t^{-\alpha} f(r), \tag{1.3}$$

where α, β are positive numbers. We substitute (1.3) into (1.1) and find

$$\alpha := \frac{(p-1)}{(pq-2p+2)}, \qquad \beta := \frac{(q-p+1)}{(pq-2p+2)}, \tag{1.4}$$

and f, as a function of $r = xt^{-\beta}$, solves an ordinary differential equation

$$\left(|f'|^{p-2} f'\right)' + \beta r f' + \alpha f + (f^q)' = 0. \tag{1.5}$$

We observe that if $u(x,t)$ solves (1.1) then the *rescaled functions*

$$u_\rho(x,t) = \rho^{\alpha/\beta} u\left(\rho x, \rho^\beta t\right), \quad \rho > 0, \tag{1.6}$$

define a one parameter family of solutions to (1.1). A solution $u(x,t)$ is said to be *self-similar* when $u_\rho(x,t) = u(x,t)$, for every $\rho > 0$. It can be easily verified that $u(x,t)$ is a self-similar solution to (1.1) if and only if u has the form (1.3). We also remark that the self-similar solutions play an important role in the study of large time behaviors of general solutions (see [3–5]),and the evolution of interfaces of compactly supported solution of the diffusion-convection eqaution (see [6, 7]).

Every nonnegative, bounded solution of (1.5) has exactly one critical point and since we here apply the shooting method, led to solve a more general initial value problem,

$$\left(|f'|^{p-2} f'\right)' + \beta r f' + \alpha f + (f^q)' = 0, \tag{1.7}$$

for $r > 0$ with initial conditions

$$f'(0) = 0, \qquad f(0) = \lambda, \tag{1.8}$$

where λ may be any positive number.

Using *Shauder's* fixed point theorem or *Banach* contraction theorem, we find that initial value problem has a unique solution which we denote by $f(r;\lambda)$. In many cases, it turns out that the limit

$$L(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta} f(r), \tag{1.9}$$

exists and we distinguish between fast and slow orbits according to whether $L(\lambda) = 0$ or not, respectively. The fast orbit will bring out a very singular solution of (1.1). The very singular solution has a stronger singularity at the origin than the singular solution of that equation. By a *singular solution* we mean a nonnegative and nontrivial solution which satisfies the equation and vanishes outside any open neighborhood of the origin as $t \to 0$. A singular solution is called a *very singular solution* if the integral of $u(x,t)$ over any open neighborhood of the origin becomes unbounded as $t \to 0$, which is equivalent to, if u is given by (1.3),

$$\lim_{r\to\infty} r^{\alpha/\beta} f(r) = 0. \tag{1.10}$$

Furthermore, if $0 < \beta < \alpha$ and a solution f of (1.5) satisfies (1.10), then $u(x,t)$ given explicitly by (1.3) becomes a very singular self-similar solution of (1.1).

Our goal is to find values of q and initial data λ which insure that $f(\cdot,\lambda)$ is a fast decaying solution and to give an exact asymptotic behavior of solutions near infinity. More precisely, our main results include the following.

(i) If $\alpha \le \beta$ (*i.e.*, $q \ge 2(p-1)$), then there does not exist any fast orbit and indeed, only exists slow orbits for any $\lambda > 0$.

(ii) If $\alpha > \beta$ (*i.e.*, $p-1 < q < 2(p-1)$), then there exists λ_1 such that

 (i) $f(r;\lambda)$ is changing sign for $\lambda \in (0,\lambda_1)$;

 (ii) $f(r;\lambda)$ is a slow orbit having the behavior

$$f(r;\lambda) \sim L(\lambda) r^{-\alpha/\beta}, \tag{1.11}$$

 near infinity for $\lambda \in (\lambda_1, +\infty)$, with $L(\lambda) > 0$;

 (iii) $f(r;\lambda_1)$ is the only fast orbit having the compact support with interface relation

$$\lim_{r\to R^-} \left(f^{(p-2)/(p-1)}\right)'(r) = -(p-2)/(p-1)\beta^{1/(p-1)} R^{1/(p-1)}, \tag{1.12}$$

 for some $0 < R < \infty$.

There have been many works dealing with the existence, uniqueness,and qualitative behavior of self-similar solutions to a class of parabolic equations with absorption (or source, convection) term. For instance, it is thoroughly treated on the p-Laplacian equation with absorption term:

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - u^q, \quad (x,t) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^+, \tag{1.13}$$

with $p > 1, q > 1$. For $p = 2$ (linear diffusion case); see [8–10], for $p > 2$ (slow diffusion case); see [11] and for $1 < p < 2$ (fast diffusion case); see [3]. Recently some papers have studied for a class of heat equation with nonlinear convection term. They derived some estimates and used scaling, convergence of *rescaled* solutions to self-similar ones and thus obtained the asymptotic of general solutions; see [9, 12, 13] for details. Similar arguments have been used in the case of the porous medium equation; see [4]. In addition, classification of the singular self-similar solutions is found for the linear diffusion equation with convection on half line under the homogeneous *Neumann* boundary condition; see [14–17] and another important application aries in the problem about the evolution of interfaces of compactly supported solutions of the fast diffusion equation with absorption which motivated our investigation; see [6, 7].

The plan of the paper is the following. In Section 2, we derive basic properties of f which will be useful in the proof of the main results. In Section 3, we show that there does not exist any fast orbit and thus no very singular solution when $q \geq 2(p-1)$. In Section 4, we find a fast decaying solution when $p-1<q<2(p-1)$. In Section 5, we prove the uniqueness of the fast orbit.

2. Preliminary Results

In this section we will derive some properties of f which will be useful in the proof of the main results.

We first show that the sign of f' depends on the sign of α, and f decreases as long as it is positive.

Lemma 2.1. *Assume that $\alpha > 0$, $\beta > 0$, and $\lambda > 0$. Let f be a solution to* (1.5), (1.8) *on* $[0, R)$, *the maximal existence interval of positive solution with R possibly infinity. Then f decreases monotonically in* $(0, R)$.

Proof. By (1.5) and (1.8) we obtain $(|f'|^{p-2}f')(0) = -\alpha\lambda < 0$. Thus, the function f is strictly decreasing for r near 0. Suppose that there exists $r_0 < R$ such that $f'(r) < 0$ on $(0, r_0)$ and $f'(r_0) = 0$. From (1.5) one sees $(|f'|^{p-2}f')'(r_1) < 0$, which is impossible. □

By Lemma 2.1, $f'(r) < 0$ in $(0, R)$ for any $\lambda > 0$, and we find that if $R < \infty$, then $f(R) = 0$ and $f'(R^-) \leq 0$. We next show that if $f'(R^-) = 0$, then f vanishes identically after R.

Lemma 2.2. *Assume that $\alpha > 0$ and $\lambda > 0$. Let f be any solution of* (1.5) *with $f(R) = f'(R) = 0$ for $R > 0$. Then $f = 0$ for all $r \geq R$.*

Proof. By convention, (1.5) is rewritten as

$$\left(|f'|^{p-2}f'\right)' + \beta r f' + \alpha f + \left(|f|^{q-1}f\right)' = 0. \tag{2.1}$$

Thus, without loss of generality, we may assume that $f(r) > 0$ and $f'(r) > 0$ for r near R with $r > R$. For such r, we find easily from (2.1) that $(|f'|^{p-2}f')'(r) < 0$. Integrating over (R, r), we see that for $r > R$, $|f'|^{p-2}f'(r) < 0$, which contradicts to the assumption. □

Lemma 2.3. *Assume that $\alpha > 0$, $\beta > 0$, and $\lambda > 0$. Let f be a solution of* (1.5), (1.8) *for all $r > 0$. Then*

(i) $\lim_{r\to\infty} f(r) = 0$,

(ii) $\lim_{r\to\infty} f'(r) = 0$.

Proof. Since f is strictly decreasing and bounded below by 0, there exists

$$\lim_{r\to\infty} f(r) = l \in [0,\lambda). \tag{2.2}$$

If we define the energy function $E(r) = (p-1)/p|f'|^p + \alpha/2f^2$, then we obtain

$$\frac{d}{dr}E(r) = -(f')^2\left(\beta r + qf^{q-1}\right) \le 0, \tag{2.3}$$

for $r \ge 0$. Thus, $E(r)$ decreases monotonically to a limit and there also exists the limit

$$\lim_{r\to\infty} f'(r) = -l_1, \quad l_1 \in [0,\infty). \tag{2.4}$$

In particular l_1 must be zero. Otherwise f becomes negative for some positive r.

We now prove that $l = 0$. Suppose to the contrary that $l > 0$. We find that $\alpha f \to \alpha l$ and $(f^q)' \to 0$. From (1.5), one gets

$$\left(|f'|^{p-2}f'\right)' + \beta r f' \le -\frac{\alpha}{2}l \tag{2.5}$$

at near infinity. Let $w = |f'|^{p-2}f'$, then $f' = -|w|^{1/(p-1)}$ and we have from (2.5) that

$$w' + \beta r|w|^{(2-p)/(p-1)}w \le -\frac{\alpha}{2}l. \tag{2.6}$$

Multiplying this by an integrating factor

$$\rho(x) = e^{\int_0^r \beta s|w(s)|^{(2-p)/(p-1)}ds}, \tag{2.7}$$

and integrating from 0 to r, we obtain

$$(rf')^{p-1} \le -\frac{\alpha}{2}l\frac{r^{p-1}\int_0^r \rho(r)d\tau}{\rho(r)}, \tag{2.8}$$

which in turn implies with the use of L'Hopital theorem

$$\limsup_{r\to\infty} rf' \le -\frac{(p-1)\alpha}{2\beta}l. \tag{2.9}$$

This leads to a contradiction. □

3. The case $\beta \geq \alpha$ $(q \geq 2(p-1))$

In this section, we show that there does not exist any fast orbit for the problem (1.5) and (1.8) and thus there is no very singular solution for (1.1) when $0 < \alpha \leq \beta$.

Theorem 3.1. *Assume that $\beta \geq \alpha$ $(q \geq 2(p-1))$. For each $\lambda > 0$, let $f(r;\lambda)$ be the solution of* (1.5), (1.8). *Then $R = \infty$, and* $\liminf_{r\to\infty} r^{\alpha/\beta} f(r;\lambda) > 0$.

Proof. We assume that $R < \infty$, on the contrary, and integrate (1.5) over $(0, R)$ to get

$$|f'|^{p-2} f'(R) + (\alpha - \beta)\int_0^R f(r)dr - \lambda^q = 0, \tag{3.1}$$

which is impossible. Thus f is positive for all $r \geq 0$ and $R = \infty$. □

Moreover, we have for $r > 0$,

$$\left\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f\right\}' = r^{\alpha/\beta-1}\left\{\left(|f'|^{p-2}f'\right)' + \frac{\alpha/\beta-1}{r}|f'|^{p-2}f' + \alpha f + \beta r f'\right\}. \tag{3.2}$$

By (1.5), we get

$$\left\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f\right\}' = r^{\alpha/\beta-1}\left\{\frac{\alpha/\beta-1}{r}|f'|^{p-2}f' - (f^q)'\right\} > 0, \tag{3.3}$$

by the condition $\beta \geq \alpha$ and $f' < 0$. If we define the function $F(r) := r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f$, then we see that $F(0) = 0$ and $F(r)$ is strictly increasing for all $r > 0$. Since f is a decreasing function, one must have $\liminf_{r\to\infty} r^{\alpha/\beta} f(r;\lambda) > 0$.

We will see later that the limit $\lim_{r\to\infty} r^{\alpha/\beta} f(r;\lambda)$ exists for each $\lambda > 0$. Thus we may conclude together with Theorem 3.1 that there exist slow orbits only.

4. The Case $\alpha > \beta$ $(p-1 < q < 2(p-1))$

In this section, we first show that the solution changes sign for small λ and we next show that the solution becomes a slow orbit for suitably large λ. We then find a fast orbit between them. The slow orbits will be shown to be ordered, and the minimal one becomes the fast orbit as we have seen in many cases; see [10, 16].

Define the following three sets for any initial value $\lambda > 0$,

$$\begin{aligned}
\mathcal{S}_2 &= \{\lambda > 0;\ R < \infty,\ f'(R^-,\lambda) < 0\},\\
\mathcal{S}_2 &= \{\lambda > 0;\ R < \infty,\ f'(R^-,\lambda) = 0\},\\
\mathcal{S}_3 &= \{\lambda > 0;\ R = \infty,\ f(r,\lambda) > 0\}.
\end{aligned} \tag{4.1}$$

Obviously, these sets are disjoint and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = (0,\infty)$.

We first show that the problem (1.5), (1.8) has changed sign for "small" $\lambda > 0$.

Theorem 4.1. *The set* $\mathcal{S}_1 \neq \emptyset$ *and open.*

Proof. By integrating (1.5), one has

$$|f'|^{p-2} f' + \beta r f = \phi(r) := -(\alpha - \beta)\int_0^r f\,dr - f^q + \lambda^q. \tag{4.2}$$

One easily finds that $\phi(0) = 0$, $\phi'(r) = -(\alpha - \beta) f - q f^{q-1} f'$, and $\phi'(0) = -(\alpha - \beta)\lambda < 0$.

Suppose that $\phi(r) < 0$ and thus

$$|f'|^{p-2} f' + \beta r f < 0, \quad 0 < r < r_0, \tag{4.3}$$

for some r_0 to be determined later. An integration of (4.3) yields

$$f^{(p-2)/(p-1)}(r) < \lambda^{(p-2)/(p-1)} - \frac{(p-1)^2 \beta^{1/(p-1)}}{p(p-2)} r^{p/(p-1)}. \tag{4.4}$$

Thus if $r_0 > R_0 := \left((p(p-2)/(p-1)^2 \beta^{1/(p-1)})\lambda^{(p-2)/(p-1)}\right)^{(p-1)/p}$, then f must change sign and we are done. Otherwise, we may assume that $\phi(r_0) = 0$ for some $r_0 \leq R_0$. From definition, we obtain $f'(r_0) = -\beta^{1/(p-1)} {r_0}^{1/(p-1)} f^{1/(p-1)}(r_0)$ and

$$\phi'(r_0) = -(\alpha - \beta) f(r_0) - q f^{q-1} f'(r_0) \geq 0. \tag{4.5}$$

Combining these, we have

$$0 < \alpha - \beta \leq q \beta^{1/(p-1)} r_0^{1/(p-1)} f^{q-1-(p-2)/(p-1)}. \tag{4.6}$$

Since f is a decreasing solution, we also have $f(r_0) \leq \lambda$ and

$$\alpha - \beta \leq q \beta^{1/(p-1)} \left(\frac{p(p-2)}{(p-1)^2 \beta^{1/(p-1)}}\right)^{1/p} \lambda^{(p-2)/[p(p-1)] + q - 1 - (p-2)/(p-1)}. \tag{4.7}$$

The inequality (4.7) does not hold for all sufficiently small λ, which proves the first part of the theorem. The continuous dependence of solutions on the initial values implies that $\mathcal{S}_1$ is an open set. □

We next prove that the problem (1.5), (1.8) has a global positive decaying solution for all suitably large λ.

Lemma 4.2. *Let* $\alpha > \beta$, *then for any* R_0 *there exists* λ_0 *such that* $f(r) = f(r, \lambda) > 0$ *for* $0 < r < R_0$ *and* $f(R_0) + |f'(R_0)|^{p-2} f'(R_0) > 0$ *for all* $\lambda \geq \lambda_0$.

Proof. We define $f_\lambda(t) = (1/\lambda)f(r,\lambda), t = r\lambda^\delta$ with $\delta = (q-p+1)/(p-1) > 0$. Then f_λ satisfies $f'_\lambda(0) = 0, f_\lambda(0) = 1$, and the following equation:

$$\left(|f'_\lambda|^{p-2}f'_\lambda\right)' + \lambda^{-qp-2(p-1)/p-1}(\beta t f'_\lambda + \alpha f_\lambda) + \left(f_\lambda^q\right)' = 0. \tag{4.8}$$

By integrating the above equality over $(0,t)$, we obtain

$$|f'_\lambda|^{p-2}f'_\lambda + \lambda^{-(qp-2(p-1))/(p-1)}(\alpha-\beta)\int_0^t f_\lambda d\tau + \lambda^{-(qp-2(p-1))/(p-1)}\beta t f_\lambda + \left(f_\lambda^q - 1\right) = 0. \tag{4.9}$$

Since f_λ is bounded by 1, for any $\epsilon > 0$ there is λ_0 such that whenever $\lambda \geq \lambda_0$,

$$1-\epsilon < |f'_\lambda|^{p-2}f'_\lambda + f_\lambda^q < 1+\epsilon, \tag{4.10}$$

for $t \in [0, [qp-2(p-1)]/(p-1)-\epsilon]$, which implies the lemma. □

We also prove the next key observation.

Proposition 4.3. *Assume that $\alpha > 0, \beta > 0, \mu > 0$ and f be any globally positive solution to* (1.5), (1.8). *Consider the function $E_c(r) := cf + rf'$ for $c > 0$. Then*

(i) *if $c > \alpha/\beta$, then $E_c(r)$ is eventually positive;*

(ii) *if $c < \alpha/\beta$, then $E_c(r)$ is eventually negative.*

Proof. By direct calculation and (1.5), we obtain

$$(p-1)|f'|^{p-2}\, E'_c(r) = (c+1)(p-1)|f'|^{p-2}f' - \beta r^2 f' - \alpha r f - qrf^{q-1}f', \tag{4.11}$$

and at any $r = r_0$ for which $E_c(r_0) = 0$, we have

$$(p-1)|f'|^{p-2}\, E'_c(r_0) = -(c+1)(p-1)c^{p-1}(f/r_0)^{p-1} + (\beta c-\alpha)r_0 f + qcf^q. \tag{4.12}$$

Since the middle term on the right-hand side of (4.12) dominates the others for all sufficiently large r_0, the sign of $E'_c(r_0)$ is only decided by the sign of $c\beta - \alpha$ and thus $E_c(r)$ becomes of the same sign eventually.

In order to prove (i), we suppose that there exists r_1 such that $E_c(r) < 0$ for all $r \geq r_1$. From (1.5) and Lemma 2.1 we deduce that

$$\left(|f'|^{p-2}f'\right)' - (\beta c-\alpha)f = -\beta E_c(r) - (f^q)' > 0, \tag{4.13}$$

for $r \geq r_1$. Multiplying the previous inequality by f' and integrating from r to τ with $r_1 \leq r \leq \tau$, we have

$$\frac{(p-1)}{p|f'|^p(\tau)} - c_1 f^2(\tau) \leq \frac{(p-1)}{p|f'|^p(r)} - c_1 f^2(r), \tag{4.14}$$

where $c_1 := (\beta c - \alpha)/2$. Letting $\tau \to \infty$ and using Lemmas 2.1 and 2.3, we get the following inequality:

$$-f' f^{-2/p} \geq c_2 > 0, \quad r \geq r_1. \tag{4.15}$$

Integrating the previous inequality from r_1 to $r \geq r_1$, we obtain

$$p/(p-2) f^{(p-2)/p}(r_1) - p/(p-2) f^{(p-2)/p}(r) \geq c_2(r - r_1). \tag{4.16}$$

Letting $r \to \infty$, we get a contradiction.

We prove (ii) similarly. Suppose that there exists r_2 such that $E_c(r) > 0$ for all $r \geq r_2$. From (1.5) and assumption,

$$\left(|f'|^{p-2} f'\right)' + \alpha f = -\beta r f' + \alpha f - q f^{q-1} f' \leq \beta c f + qc/r f^q. \tag{4.17}$$

Since f decreases, we may rewrite this as

$$\left(|f'|^{p-2} f'\right)' \leq -c_2 f < c_2 \frac{r f'}{c}, \tag{4.18}$$

where we define $c_2 = \alpha - c\beta + (cq/r_2)\lambda^{q-1}$ and assume it to be positive by retaking r_2. The inequality (4.18) is rewritten as $(p-1)/(p-2)(|f'|^{p-2} f')' \leq -c_3 r$ for some positive constant c_3 and an integration from $r = r_2$ to $r = \infty$ yields a contradiction, which completes the proof. We rewrite the problem (1.5), (1.8) as the following system:

$$\begin{aligned} f' &= |h|^{-(p-2)/(p-1)} h, \\ h' &= -\beta r |h|^{-(p-2)/(p-1)} h - \alpha f - q f^{q-1} |h|^{-(p-2)/(p-1)} h. \end{aligned} \tag{4.19}$$

Given any $\delta > 0$, we denote

$$\mathcal{L}_\delta := \{(f, h) : 0 < f \leq 1, \ 0 > h > -\delta f\}, \tag{4.20}$$

then we obtain the following lemma. □

Lemma 4.4. *For given $\delta > 0$ there exists a $r_\delta := [\delta + \alpha \delta^{-1/(p-1)}]/\beta$ such that $\mathcal{L}_\delta$ is positively invariant for $r \geq r_\delta$. That is, $(f(r_0), h(r_0)) \in \mathcal{L}_\delta$ for $r_0 \geq r_\delta$ implies that the orbit $(f(r), h(r))$ of* (4.19) *remains in triangle region $\mathcal{L}_\delta$ for all $r \geq r_\delta$.*

Proof. We will show that given $\delta > 0$ there exists $r_\delta > 0$ such that if $r \geq r_\delta$ then the vector field determined by (4.19) points into $\mathcal{L}_\delta$, except at the critical point $(0, 0)$. It is easy to see this fact on the top $h = 0$ and the line $f = 1$, and it is enough to verify this only on the line $h = -\delta f$. By the system (4.19), we have

$$\begin{aligned}\frac{h'}{f'} &= \frac{-\beta r|h|^{-(p-2)/(p-1)}h - \alpha f - qf^{q-1}|h|^{-(p-2)/(p-1)}h}{|h|^{-(p-2)/(p-1)}h}\\ &= -\beta r + \frac{\alpha}{\delta}|h|^{(p-2)/(p-1)} - qf^{q-1} < -\beta r + \alpha\delta^{-1/(p-1)}f^{(p-2)/(p-1)} \leq -\delta,\end{aligned} \tag{4.21}$$

if $r \geq r_\delta := [\delta + \alpha\delta^{-1/(p-1)}]/\beta$. □

As a consequence, we can prove the existence of globally positive solutions.

Theorem 4.5. *The set* $\mathcal{S}_3 \neq \emptyset$ *and open.*

Proof. From Lemma 4.2, we can find r_0 such that $f > 0$ for $0 \leq r \leq r_0$ and $f(r_0) + |f'(r_0)|^{p-2}f'(r_0) > 0$ for all sufficiently large λ. Thus $(f(r_0), |f'|^{p-2}f'(r_0)) \in \mathcal{L}_1$ and by Lemma 4.4, f is positive for all $r > 0$, which proves the first part of the theorem.

We next prove that $\mathcal{S}_3$ is an open set. Set $\lambda_0 \in \mathcal{S}_3$ and then by Proposition 4.3, $E_1(r) = f + rf'$ becomes positive for all large r. Thus there exists sufficiently large r_0 such that $(f(r_0), |f'|^{p-2}f'(r_0)) \in \mathcal{L}_1$. Then by continuous dependence of solutions on the initial value there is a neighborhood N of λ_0 such that $f(r;\lambda) > 0$ and $(f(r_0;\lambda), |f'|^{p-2}f'(r_0;\lambda)) \in \mathcal{L}_1$ for any $(r,\lambda) \in [0, r_0] \times N$. By Lemma 4.4, we deduce that the orbits remain in $\mathcal{L}_1$ for any $r > r_0$, which implies in particular that $f(r,\lambda) > 0$ for any $r > r_0$ and $\lambda \in N$. Therefore, $f(r;\lambda) > 0$ for any $r > 0$ and $\lambda \in N$ and $\mathcal{S}_3$ is open. □

We are now going to find exact decay-rates for globally positive solutions.

Theorem 4.6. *For any given* $\lambda > 0$, *let* f *be any solution to* (1.5), (1.8) *such that* $f > 0$ *for any* $r > 0$. *Then* $\lim_{r\to\infty} r^{\alpha/\beta} f(r;\lambda) = L(\lambda) > 0$ *exists.*

Proof.

Step 1. By Lemmas 2.1 and 2.3 we know that $f'(r) < 0$ for $r > 0$ and $\lim_{r\to\infty} f(r) = 0, \lim_{r\to\infty} f'(r) = 0$. Moreover,we have seen that if $c < \beta/\alpha$, then $E_c(r) = cf + rf' < 0$ for all sufficiently large r; say, $r > r_0$. We easily find that

$$f(r) \leq f(r_0)r^{-c}, \quad r > r_0. \tag{4.22}$$

We also recall that if $d > \alpha/\beta$, then $E_d(r) = df + rf' > 0$ and thus

$$-f'(r) < \frac{df(r)}{r}, \quad r > r_1, \tag{4.23}$$

for some $r_1 > 0$.

Step 2. From (1.5), we get

$$\left\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f\right\}' = r^{\alpha/\beta-1}\left\{(\alpha/\beta-1)/r|f'|^{p-2}f' - (f^q)'\right\}, \tag{4.24}$$

and integrating over $(0, r)$, we see that

$$r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f = (\alpha/\beta-1)\int_0^r |f'|^{p-2}f's^{\alpha/\beta-2}ds + q\int_0^r f^{q-1}|f'|s^{\alpha/\beta-1}ds. \tag{4.25}$$

Using (4.22) and (4.23), we find that two integrals of the right hand side of (4.25) converge and $\lim_{r\to\infty} r^{\alpha/\beta-1}|f'|^{p-2}f' = 0$. Therefore, the limit $\mathrm{L}(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta}f(r,\lambda)$ exists and finite.

Step 3. We now show that $\mathrm{L}(\lambda) > 0$. Assume that $\mathrm{L}(\lambda) = 0$. Integrating (4.24) over (r, ∞), we have

$$\begin{aligned} &r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f \\ &\quad = (1-\alpha/\beta)\int_r^\infty |f'|^{p-2}f's^{\alpha/\beta-2}ds - q\int_r^\infty f^{q-1}|f'|s^{\alpha/\beta-1}ds. \end{aligned} \tag{4.26}$$

Again using (4.23), we see that $\mathrm{L}(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta}f(r,\lambda)$ exists and finite. On the other hand, by (4.23),

$$f(r) \geq f(r_1)r^{-d}, \qquad r > r_1. \tag{4.27}$$

These conflictions implies that $\mathrm{L}(\lambda) > 0$.

□

Remark 4.7. Obviously, the limit value $\mathrm{L}(\lambda) = 0$ is achieved only when f has the compact support and Proposition 4.3 and Theorem 4.6 remain true for the case $\alpha \leq \beta$.

We finally show that there exists a fast orbit.

Theorem 4.8. *The set $\mathcal{S}_2 \neq \emptyset$ and closed. Moreover, the interface relation holds*

$$\lim_{r\to R^-}\left(f^{(p-2)/(p-1)}\right)'(r) = -\frac{(p-2)}{(p-1)}\beta^{1/(p-1)}R^{1/(p-1)}, \tag{4.28}$$

for any $\lambda \in \mathcal{S}_2$.

Proof. By Theorems 4.1 and 4.5, we immediately see that $\mathcal{S}_2$ is nonempty and closed set. From Lemma 2.2, any solution $f = f(r,\lambda)$ with $\lambda \in \mathcal{S}_2$ has a compact support; say, $[0, R]$ and f satisfies condition $f(R) = 0, f'(R) = 0$. Integrating the equation (1.5) from r to R, we get

$$|f'|^{p-2}f'(r) = -\beta r f(r) + (\alpha-\beta)\int_r^R f(s)ds - f^q(r). \tag{4.29}$$

Dividing by f, we have

$$\left|f'\right|^{p-2}f'(r)/f(r) = -\beta r + (\alpha-\beta)\int_r^R f(s)ds/f(r) - f^{q-1}(r). \tag{4.30}$$

Since f is strictly decreasing, we find that

$$0 \le \int_r^R f(s)ds \le f(r)(R-r). \tag{4.31}$$

Hence

$$\lim_{r\to R^-}\int_r^R f(s)ds/f(r) = 0. \tag{4.32}$$

Letting $r \to R^-$ in (4.30), then we obtain

$$\lim_{r\to R^-}\left|f'\right|^{p-2}f'(r)/f(r) = -\beta R, \tag{4.33}$$

and which is equivalent to the second result of the theorem. □

In addition, we show the monotonicity of the solutions of the problem (1.5), (1.8) with respect to λ in the sense that two positive orbits do not intersect each other.

Theorem 4.9. *Assume that $\alpha > 0$, $\beta > 0$ and f_i are solutions of problem* (1.5), (1.8) *on* $[0, R_i)$ *with initial data $f_i(0) = \lambda_i > 0, i = 1, 2$; where $[0, R_i)$ denote the maximal existence interval of f_i and R_i are possible infinity. Then*

$$\lambda_2 > \lambda_1 \Longrightarrow f_2(r) > f_1(r), \quad \forall 0 \le r \le R := \min\{R_1, R_2\}. \tag{4.34}$$

Proof. Suppose contrarily that there exists $R_0 \in [0, R]$ such that $f_1(r) < f_2(r)$ for $r \in [0, R_0)$ and $f_1(R_0) = f_2(R_0)$. We define

$$g_k(r) := k^{-p/(p-2)}f_1(kr), \quad r \in [0, R_1/k) \tag{4.35}$$

for $k > 0$ and then $g_k(r)$ solves

$$\left(\left|g_k'\right|^{p-2}g_k'\right)' + \beta r g_k' + \alpha g_k + k^{[pq-2(p-1)]/(p-2)}\left(g_k^q\right)' = 0. \tag{4.36}$$

By Lemma 2.1 we know that f_1 is strictly decreasing on $[0, R_1)$ and so g_k is strictly decreasing with respect to k. In particular, $\lim_{k\to 0} g_k(r) = +\infty$ for any $r \in [0, R]$. Thus there exists a small $k_0 > 0$ such that

$$g_k(r) > f_2(r), \quad \text{for } r \in [0, R], k \in [0, k_0], \tag{4.37}$$

and the set

$$I := \{k \in (0, k_0); g_k(r) > f_2(r), \text{ for } r \in [0, R_0]\} \tag{4.38}$$

is nonempty and open. Setting $l := \sup I$, we see that $l < 1$, $l \notin I$ and there exists $r_0 \in [0, R_0]$ such that $g_l(r_0) = f_2(r_0)$. □

If $r_0 = R_0$, then $g_l(R_0) = l^{-p/(p-2)} f_1(lR_0) = f_2(R_0)$. Since $f_1(R_0) = f_2(R_0)$ and g_l is strictly decreasing with respect to l, we conclude that $l = 1$ and which contradicts to the hypothesis. If $r_0 \in (0, R_0)$, then g_l must touch f_2 at $r = r_0$ from the above. But in this case we deduce from (1.5) that

$$\left(|g_l'|^{p-2} g_l'\right)''(r_0) - \left(|f_2'|^{p-2} f_2'\right)''(r_0) = \left(1 - l^{[pq-2(p-1)]/(p-2)}\right)\left(f_2^q\right)'(r_0) < 0, \tag{4.39}$$

which obviously violates the strong maximum principle. Thus g_l must touch f_2 at $r = 0$ from the above. But also from (1.5), we find $(|f_2'|^{p-2} f_2')'(0) = -\alpha\lambda_2$ and $(|f_2'|^{p-2} f_2')''(0) = -(f_2^q)''(0) = -q\lambda_2^{q-1} f_2''(0)$. Similarly for g_l, we obtain,

$$\left(|g_l'|^{p-2} g_l'\right)''(0) - \left(|f_2'|^{p-2} f_2'\right)''(0) = \left(l^{[pq-2(p-1)]/(p-2)} - 1\right) q\lambda_2^{q-1} f_2''(0) < 0, \tag{4.40}$$

which leads to another contradiction and completes all the proofs.

5. Uniqueness

In this section, we show that there exists only one fast decaying solution for the problem (1.5), (1.8).

Recall that such a solution has compact support $[0, R]$ and has an interface relation

$$\lim_{r \to R^-} \left(f^{(p-2)/(p-1)}\right)'(r) = -\frac{(p-2)}{(p-1)} \beta^{1/(p-1)} R^{1/(p-1)}, \tag{5.1}$$

by Theorem 4.8.

Theorem 5.1. *The set $\mathcal{S}_2$ consists of only one element.*

Proof. Let F and f be any two fast orbits with compact supports $[0, R_i]$, for $i = 1, 2$, respectively and satisfy $F(0) > f(0)$. We define

$$f_k(r) = k f\left(k^{-\gamma} r\right), \quad \gamma = (p-2)/p, \tag{5.2}$$

and then f_k will be larger than F on $[0, R_2]$ for sufficiently large k. We now define

$$\tau = \min\{k \geq 1; f_k(r) \geq F(r), 0 \leq r \leq R_2\}. \tag{5.3}$$

The uniqueness proof is now reduced to showing that τ is not greater than 1. Suppose that $\tau > 1$, on the contrary. We will show that there exists an $\epsilon > 0$ such that $f_{\tau-\epsilon}(r) \geq F(r)$ for every $r \in [0, R_2]$. Indeed, we are going to show that $f_\tau(r)$ does not touch $F(r)$ in compact support $[0, R_2]$ by dividing into three cases: □

(i) in the interior of the support;

(ii) at the origin;

(iii) at R_2.

In fact, $f_\tau(r)$ solves

$$\left(|f'_\tau|^{p-2} f'_\tau\right)' + \beta r f'_\tau + \alpha f_\tau + \left(f_\tau^q\right)' = -\tau\left(1 - \tau^{q-\gamma-1}\right)(f^q)'. \tag{5.4}$$

(i) If f_τ touches F at $r_0 \in (0, R_2)$, then $f_\tau(r_0) = F(r_0), f'_\tau(r_0) = F'(r_0) < 0$ and

$$\left(|f'_\tau|^{p-2} f'_\tau\right)'(r_0) < \left(|F'|^{p-2} F'\right)'(r_0), \tag{5.5}$$

but $f_\tau(r) \geq F(r)$ near $r = r_0$, which obviously violates the strong maximum principle.

(ii) If f_τ touches F at $r_0 = 0$, then $f_\tau(0) = F(0) > 0, f'_\tau(0) = F'(0) = 0$ and $(|f'_\tau|^{p-2} f'_\tau)' = -\alpha f_\tau(0) = (|F'|^{p-2} F')'(0) < 0$. Differentiating the equations (1.5) and (5.4), we reduce that

$$\left(|f'_\tau|^{p-2} f'_\tau\right)''(0) - \left(|F'|^{p-2} F'\right)''(0) = -\tau\left(1 - \tau^{q-\gamma-1}\right)(f^q)''(0) < 0. \tag{5.6}$$

Thus, we have

$$\left(|f'_\tau|^{p-2} f'_\tau\right)''(r) - \left(|F'|^{p-2} F'\right)''(r) \leq 0 \tag{5.7}$$

near $r_0 = 0$, which leads to a contradiction.

(iii) For the final case, we define the functions u, U_τ corresponding to F and f_τ by

$$\begin{aligned} u(x,t) &=: t^{-\alpha} F(r), \\ U_\tau(x,t) &=: t^{-\alpha} f_\tau(r) =: \tau t^{-\alpha} f\left(\tau^{-\gamma} r\right), \end{aligned} \tag{5.8}$$

where $\gamma = (p-2)/p, r = rt^{-\beta}$ as defined before. Then $u(x,t)$ is a solution of (1.1) and U_τ is a supersolution . Indeed, a straightforward computation shows that

$$(U_\tau)_t - \left(|(U_\tau)_x|_x^{p-2}\right)(U_\tau)_x - \left((U_\tau)^q\right)_x = \tau\left(\tau^{q-\gamma-1} - 1\right)\left|(f^q)'\right| \geq 0, \quad \text{for} \quad \tau > 1. \tag{5.9}$$

Following directly the proof of Lemma 10 in [18], we can show that for fixed $t > 0$ and all sufficiently small $\delta' > 0$, there exists a $\theta = \theta(\delta') \in (0,1)$ such that $U_\tau(x,t) \le U_\tau(x,t+\delta')$, if x satisfies $\theta R_2 \le x t^{-\beta}\tau^{-\gamma} \le R_2$ and $\lim_{\delta' \downarrow 0} \theta(\delta') = \theta_0 \in (0,1)$. In the proof, we use the interface relation (5.1) crucially (see [18] for details). In particular, we have

$$U_\tau(x,1) \le U_\tau(x,1+\delta'), \tag{5.10}$$

for $\theta R_2 \tau^\gamma \le x < R_2 \tau^\gamma (1+\delta')^\beta$. In other words, we found a separation near the right end $r = R_2$.

On the other hand, as previously proved, f_τ cannot touch F at $r_0 \in [0, R_2)$, which implies for any $\epsilon_1 > 0$, there exists $\kappa = \kappa(\epsilon_1) \in (0,1)$ such that $F(x) \le \kappa f_\tau(x)$, that is,

$$u(x,1) \le \kappa U_\tau(x,1). \tag{5.11}$$

We choose $\epsilon_1 > 0$ so that $0 < \epsilon_1 < 1 - \theta_0$ and find $\delta_0 = \delta_0(\epsilon_1)$ such that

$$1 - \epsilon_1 > \theta(\delta'), \tag{5.12}$$

for $\delta' \in (0, \delta_0)$. By continuity of U_τ, there exists $\delta_1 = \delta_1(\epsilon_1) \in (0, \delta_0)$ such that

$$\kappa U_\tau(x,1) \le U_\tau(x,1+\delta'), \tag{5.13}$$

for any $\delta' \in (0,\delta_1)$ and $0 \le x < (1-\epsilon_1) R_2 \tau^\gamma$. Combining (5.10), (5.11), and (5.13) and using again the continuity of U_τ, we deduce that for $\delta \in (\delta', \delta_1)$, which $\delta - \delta'$ small enough, we have

$$F(x) < U_\tau(x,1+\delta) = \tau(1+\delta)^{-\alpha} f\left(x(1+\delta)^{-\beta}\tau^{-\gamma}\right), \tag{5.14}$$

for any $x \ge 0$. Furthermore, from the continuity with respect to τ, there exists $\tau_1 \in (0,\tau)$ such that

$$u(x,1) = F(x) \le \tau_1 (1+\delta)^{-\alpha} f\left(x(1+\delta)^{-\beta}\tau_1^{-\gamma}\right) = U_{\tau_1}(x,1+\delta), \tag{5.15}$$

for any $x \ge 0$. By parabolic maximum principle, we have $u(x,t) \le U_{\tau_1}(x,t+\delta)$, that is,

$$t^{-\alpha} F\left(x t^{-\beta}\right) \le \tau_1 (t+\delta)^{-\alpha} f\left(x(t+\delta)^{-\beta}\tau_1^{-\gamma}\right), \tag{5.16}$$

for any $t \ge 1$ and $x \ge 0$. Rewriting (5.16) of the form

$$F(r) \le \tau_1 [t/(t+\delta)]^{-\alpha} f\left(r[t/(t+\delta)]^{-\beta}\tau_1^{-\gamma}\right), \tag{5.17}$$

and letting $t \to \infty$, we find that

$$F(r) \leq \tau_1 f\left(r\tau_1^{-\gamma}\right), \tag{5.18}$$

which contradicts the fact that τ is the smallest constant with that property. Thus f_τ does not meet at $r_0 = R_2$.

Hence we may find $\epsilon > 0$ so that

$$f_{\tau-\epsilon}(r) \geq F(r), \quad \text{for every } r \in [0, R_2], \tag{5.19}$$

which means that we can slightly reduce the factor τ. Hence we may conclude that $\tau = 1$ but it is obviously impossible.

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