Research Article

Existence of Positive, Negative, and Sign-Changing Solutions to Discrete Boundary Value Problems

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By using critical point theory, Lyapunov-Schmidt reduction method, and characterization of the Brouwer degree of critical points, sufficient conditions to guarantee the existence of five or six solutions together with their sign properties to discrete second-order two-point boundary value problem are obtained. An example is also given to demonstrate our main result.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a, b) = \{a, a + 1, \ldots, b\}$, when $a \leq b$. Δ is the forward difference operator defined by $\Delta u(n) = u(n + 1) - u(n)$, $\Delta^2 u(n) = \Delta(\Delta u(n))$.

Consider the following discrete second-order two-point boundary value problem (BVP for short):

$$\Delta^2 u(n - 1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1, T),$$

$$u(0) = 0 = u(T + 1), \tag{1.1}$$

where $V \in C^2(\mathbb{R}, \mathbb{R})$, $T \geq 1$ is a given integer.

By a solution u to the BVP (1.1), we mean a real sequence $\{u(n)\}_{n=0}^{T+1} (= (u(0), u(1), \ldots, u(T + 1)))$ satisfying (1.1). For $u = \{u(n)\}_{n=0}^{T+1}$ with $u(0) = 0 = u(T + 1)$, we say that $u \neq 0$ if there exists at least one $n \in \mathbb{Z}(1, T)$ such that $u(n) \neq 0$. We say that u is *positive* (and write $u > 0$) if for all $n \in \mathbb{Z}(1, T)$, $u(n) \geq 0$, and $\{n \in \mathbb{Z}(1, T) : u(n) > 0\} \neq \emptyset$, and similarly,

u is *negative* $(u < 0)$ if for all $n \in \mathbb{Z}(1,T)$, $u(n) \leq 0$, and $\{n \in \mathbb{Z}(1,T) : u(n) < 0\} \neq \emptyset$. We say that u is *sign-changing* if u is neither positive nor negative. Under convenient assumptions, we will prove the existence of five or six solutions to (1.1), which include positive, negative, and sign-changing solutions.

Difference BVP has widely occurred as the mathematical models describing real-life situations in mathematical physics, finite elasticity, combinatorial analysis, and so forth; for example, see [1, 2]. And many scholars have investigated difference BVP independently mainly for two reasons. The first one is that the behavior of discrete systems is sometimes sharply different from the behavior of the corresponding continuous systems. For example, every solution of logistic equation $x'(t) = ax(t)(1 - x(t)/k)$ is monotone, but its discrete analogue $\Delta x(n) = ax(n)(1 - x(n)/k)$ has chaotic solutions; see [3] for details. The second one is that there is a fundamental relationship between solutions to continuous systems and the corresponding discrete systems by employing discrete variable methods [1]. The classical results on difference BVP employs numerical analysis and features from the linear and nonlinear operator theory, such as fixed point theorems. We remark that, usually, the application of the fixed point theorems yields existence results only.

Recently, however, a few scholars have used critical point theory to deal with the existence of multiple solutions to difference BVP. For example, in 2004, Agarwal et al. [4] employed the mountain pass lemma to study (1.1) with $V'(u(n)) = f(n, u(n))$ and obtained the existence of multiple solutions. Very recently, Zheng and Zhang [5] obtained the existence of exactly three solutions to (1.1) by making use of three-critical-point theorem and analytic techniques. We also refer to [6–9] for more results on the difference BVP by using critical point theory. The application of critical point theory to difference BVP represents an important advance as it allows to prove multiplicity results as well.

Here, by using critical point theory again, as well as Lyapunov-Schmidt reduction method and degree theory, a sharp condition to guarantee the existence of five or six solutions together with their sign properties to (1.1) is obtained. And this paper offers, to the best of our knowledge, a new method to deal with the sign of solutions in the discrete case.

Here, we assume that $V'(0) = 0$ and

$$V''(\infty) = \lim_{|t| \to \infty} \frac{V'(t)}{t} \in \mathbb{R}. \tag{1.2}$$

Hence, V' grows asymptotically linear at infinity.

The solvability of (1.1) depends on the properties of V' both at zero and at infinity. If

$$V''(\infty) = \lambda_l, \qquad \left(\text{or } V''(0) = \lim_{|t| \to 0} \frac{V'(t)}{t} = \lambda_l \right), \tag{1.3}$$

where λ_l is one of the eigenvalues of the eigenvalue problem

$$\Delta^2 u(n-1) + \lambda u(n) = 0, \quad n \in \mathbb{Z}(1,T),$$

$$u(0) = 0 = u(T+1), \tag{1.4}$$

then we say that (1.1) is resonant at infinity (or at zero); otherwise, we say that (1.1) is nonresonant at infinity (or at zero). On the eigenvalue problem (1.4), the following results hold (see [1] for details).

Proposition 1.1. *For the eigenvalue problem* (1.4)*, the eigenvalues are*

$$\lambda = \lambda_l = 4 \sin^2 \frac{l\pi}{2(T+1)}, \quad l = 1, 2, \ldots, T, \tag{1.5}$$

and the corresponding eigenfunctions with λ_l are $\phi_l(n) = \sin(l\pi n/(T+1))$, $l = 1, 2, \ldots, T$.

Remark 1.2. (i) The set of functions $\{\phi_l(n), l = 1, 2, \ldots, T\}$ is orthogonal on $\mathbb{Z}(1, T)$ with respect to the weight function $r(n) \equiv 1$; that is,

$$\sum_{n=1}^{T} (\phi_l(n), \phi_j(n)) = 0, \quad \forall l \neq j. \tag{1.6}$$

Moreover, for each $l \in \mathbb{Z}(1, T)$, $\sum_{n=1}^{T} \sin^2(l\pi n/(T+1)) = (T+1)/2$.

(ii) It is easy to see that ϕ_l is positive and ϕ_l changes sign for each $l \in \mathbb{Z}(2, T)$; that is, $\{n : \phi_l(n) > 0\} \neq \emptyset$ and $\{n : \phi_l(n) < 0\} \neq \emptyset$ for $l \in \mathbb{Z}(2, T)$.

The main result of this paper is as follows.

Theorem 1.3. *If $V''(0) < \lambda_1$, $V''(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \in \mathbb{Z}(2, T-1)$, and $0 < V''(t) \leq \gamma < \lambda_{k+1}$, then* (1.1) *has at least five solutions. Moreover, one of the following cases occurs:*

(i) *k is even and* (1.1) *has two sign-changing solutions,*

(ii) *k is even and* (1.1) *has six solutions, three of which are of the same sign,*

(iii) *k is odd and* (1.1) *has two sigh-changing solutions,*

(iv) *k is odd and* (1.1) *has three solutions of the same sign.*

Remark 1.4. The assumption $V''(0) < \lambda_1$ in Theorem 1.3 is sharp in the sense that when $\lambda_{k-1} < V''(0) < \lambda_k$, $\lambda_k < V''(\infty) < \lambda_{k+1}$ for $k \in \mathbb{Z}(2, T-1)$, Theorem 1.4 of [5] gives sufficient conditions for (1.1) to have exactly three solutions with some restrictive conditions.

Example 1.5. Consider the BVP

$$\Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1, 5),$$

$$u(0) = 0 = u(6), \tag{1.7}$$

where $V \in C^2(\mathbb{R}, \mathbb{R})$ is defined as follows:

$$
V'(t) = \begin{cases}
\arctan t - \dfrac{4t}{5}, & |t| \le \dfrac{1}{3}, \\[2mm]
\text{a strictly increasing function satisfying } \dfrac{1}{10} \le V''(t) \le \dfrac{49}{20}, & \dfrac{1}{3} \le |t| \le 1, \\[2mm]
\dfrac{(\arctan t)}{10} + \dfrac{12t}{5}, & |t| \ge 1.
\end{cases} \tag{1.8}
$$

It is easy to verify that $V'(0) = 0$, $V''(0) = 1/5 < \lambda_1 = 2 - \sqrt{3}$, $V''(\infty) = 12/5 \in (2,3) = (\lambda_3, \lambda_4)$, and $0 < V''(t) \le 49/20 < 3 = \lambda_4$. So, all the conditions in Theorem 1.3 are satisfied with $k = 3$. And hence (1.7) has at least five solutions, among which two sign-changing solutions or three solutions of the same sign.

By the computation of critical groups, for $k = 1$, we have the following.

Corollary 1.6 (see Remark 3.7 below). *If $V''(0) < \lambda_1$, $V''(\infty) \in (\lambda_1, \lambda_2)$, and $0 < V''(t) \le \gamma < \lambda_2$, then* (1.1) *has at least one positive solution and one negative solution.*

2. Preliminaries

Let

$$
E = \{u : \mathbb{Z}(0, T + 1) \longrightarrow \mathbb{R}, \ u(0) = 0 = u(T + 1)\}. \tag{2.1}
$$

Then, E is a T-dimensional Hilbert space with inner product

$$
(u, v) = \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)), \quad u, v \in E, \tag{2.2}
$$

by which the norm $\| \cdot \|$ can be induced by

$$
\|u\| = \left(\sum_{n=0}^{T} |\Delta u(n)|^2 \right)^{1/2}, \quad u \in E. \tag{2.3}
$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$, and $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}$.
Define

$$
J(u) = \frac{1}{2} \sum_{n=0}^{T} |\Delta u(n)|^2 - \sum_{n=1}^{T} V(u(n)), \quad u \in E. \tag{2.4}
$$

Then, the functional J is of class C^2 with

$$\langle J'(u), v \rangle = \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)) - \sum_{n=1}^{T} (V'(u(n)), v(n))$$

$$= -\sum_{n=1}^{T} \left(\Delta^2 u(n-1) + V'(u(n)), v(n) \right), \quad u, v \in E. \tag{2.5}$$

So, solutions to (1.1) are precisely the critical points of J in E.

As we have mentioned, we will use critical point theory, Lyapunov-Schmidt reduction method, and degree theory to prove our result. Let us collect some results that will be used below. One can refer to [10–12] for more details.

Let E be a Hilbert space and $J \in C^1(E, \mathbb{R})$. Denote

$$J^c = \{u \in E : J(u) \le c\}, \qquad \mathcal{K} = \{u \in E : J'(u) = 0\}, \qquad \mathcal{K}_c = \{u \in \mathcal{K} : J(u) = c\}, \tag{2.6}$$

for $c \in \mathbb{R}$. The following is the definition of the Palais-Smale (PS) compactness condition.

Definition 2.1. The functional J satisfies the (PS) condition if any sequence $\{u_m\} \subset E$ such that $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.

In [13], Cerami introduced a weak version of the (PS) condition as follows.

Definition 2.2. The functional J satisfies the Cerami (C) condition if any sequence $\{u_m\} \subset E$ such that $J(u_m)$ is bounded and $(1 + \|u_m\|)\|J'(u_m)\| \to 0$, as $m \to \infty$ has a convergent subsequence.

If J satisfies the (PS) condition or the (C) condition, then J satisfies the following deformation condition which is essential in critical point theory (cf. [14, 15]).

Definition 2.3. The functional J satisfies the (D_c) condition at the level $c \in \mathbb{R}$ if for any $\bar{\epsilon} > 0$ and any neighborhood $\mathcal{N}$ of $\mathcal{K}_c$, there are $\epsilon > 0$ and a continuous deformation $\eta : [0,1] \times E \to E$ such that

    (i) $\eta(0, u) = u$ for all $u \in E$,

    (ii) $\eta(t, u) = u$ for all $u \notin J^{-1}([c - \bar{\epsilon}, c + \bar{\epsilon}])$,

    (iii) $J(\eta(t, u))$ is non-increasing in t for any $u \in E$,

    (iv) $\eta(1, J^{c+\epsilon} \setminus \mathcal{N}) \subset J^{c-\epsilon}$.

J satisfies the (D) condition if J satisfies the (D_c) condition for all $c \in \mathbb{R}$.

Let H_* denote singular homology with coefficients in a field $\mathbb{F}$. If $u \in E$ is a critical point of J with critical level $c = J(u)$, then the critical groups of u are defined by

$$C_q(J, u) = H_q(J^c, J^c \setminus \{u\}), \quad q \in \mathbb{Z}(0). \tag{2.7}$$

Suppose that $J(\mathcal{K})$ is strictly bounded from below by $a \in \mathbb{R}$ and that J satisfies (D_c) for all $c \leq a$. Then, the qth critical group at infinity of J is defined in [16] as

$$C_q(J, \infty) = H_q(E, J^a), \quad q \in \mathbb{Z}(0). \tag{2.8}$$

Due to the condition (D_c), these groups are not dependent on the choice of a.

Assume that $\#\mathcal{K} < \infty$ and J satisfies the (D) condition. The Morse-type numbers of the pair (E, J^a) are defined by $M_q = M_q(E, J^a) = \sum_{u \in \mathcal{K}} \dim C_q(J, u)$, and the Betti numbers of the pair (E, J^a) are defined by $\beta_q = \dim C_q(J, \infty)$. By Morse theory [10, 11], the following relations hold:

$$\sum_{j=0}^{q}(-1)^{q-j}M_j \geq \sum_{j=0}^{q}(-1)^{q-j}\beta_j, \quad q \in \mathbb{Z}(0), \tag{2.9}$$

$$\sum_{q=0}^{\infty}(-1)^q M_q = \sum_{q=0}^{\infty}(-1)^q \beta_q. \tag{2.10}$$

It follows that $M_q \geq \beta_q$ for all $q \in \mathbb{Z}(0)$. If $\mathcal{K} = \emptyset$, then $\beta_q = 0$ for all $q \in \mathbb{Z}(0)$. Thus, when $\beta_q \neq 0$ for some $q \in \mathbb{Z}(0)$, J must have a critical point u with $C_q(J, u) \ncong 0$.

The critical groups of J at an isolated critical point u describe the local behavior of J near u, while the critical groups of J at infinity describe the global property of J. In most applications, unknown critical points will be found from (2.9) or (2.10) if we can compute both the critical groups at known critical points and the critical groups at infinity. Thus, the computation of the critical groups is very important. Now, we collect some useful results on computation of critical groups which will be employed in our discussion.

Proposition 2.4 (see [16]). *Let E be a real Hilbert space and $J \in C^1(E, \mathbb{R})$. Suppose that E splits as $E = X \oplus Y$ such that J is bounded from below on Y and $J(x) \to -\infty$ for $x \in X$ as $\|x\| \to \infty$. Then $C_k(J, \infty) \neq 0$ for $k = \dim X < \infty$.*

Proposition 2.5 (see [17]). *Let E be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, X, Y closed subspaces of E such that $E = X \oplus Y$. Assume that $J \in C^1(E, \mathbb{R})$ satisfies the (PS) condition and the critical values of J are bounded from below. If there is a real number $m > 0$ such that for all $v \in X$ and $w_1, w_2 \in Y$, there holds*

$$\langle \nabla J(v + w_1) - \nabla J(v + w_2), w_1 - w_2 \rangle \geq m\|w_1 - w_2\|^2, \tag{2.11}$$

then there exists a C^1-functional $\varphi : X \to \mathbb{R}$ such that

$$C_q(J, \infty) \cong C_q(\varphi, \infty), \quad q \in \mathbb{Z}(0). \tag{2.12}$$

Moreover, if $k = \dim X < \infty$ and $C_k(J, \infty) \neq 0$, then $C_q(J, \infty) \cong \delta_{q,k}\mathbb{F}$.

Let B_r denote the open ball in E about 0 of the radius r, and let ∂B_r denote its boundary.

Lemma 2.6 (Mountain Pass Lemma [10, 11]). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfying the (PS) condition. Suppose that $J(0) = 0$ and*

(J1) *there are constants $\rho > 0$, $a > 0$ such that $J|_{\partial B_\rho} \geq a > 0$, and*

(J2) *there is a $u_0 \in E \setminus B_\rho$ such that $J(u_0) \leq 0$.*

Then, J possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)), \tag{2.13}$$

where

$$\Gamma = \{h \in C([0,1], E) h(0) = 0, \ h(1) = u_0\}. \tag{2.14}$$

Definition 2.7 (Mountain pass point). An isolated critical point u of J is called a mountain pass point if $C_1(J, u) \not\cong 0$.

To compute the critical groups of a mountain pass point, we have the following result.

Proposition 2.8 (see [11]). *Let E be a real Hilbert space. Suppose that $J \in C^2(E, \mathbb{R})$ has a mountain pass point u and that $J''(u)$ is a Fredholm operator with finite Morse index satisfying*

$$J''(u) \geq 0, \quad 0 \in \sigma(J''(u)) \Longrightarrow \dim \ker(J''(u)) = 1. \tag{2.15}$$

Then,

$$C_q(J, u) \cong \delta_{q,1}\mathbb{F}, \quad q \in \mathbb{Z}(0). \tag{2.16}$$

The following theorem gives a relation between the Leray-Schauder degree and the critical groups.

Theorem 2.9 (see [10, 11]). *Let E be a real Hilbert space, and let $J \in C^2(E, \mathbb{R})$ be a function satisfying the (PS) condition. Assume that $J'(x) = x - Ax$, where $A : E \to E$ is a completely continuous operator. If u is an isolated critical point of J, that is, there exists a neighborhood U of u, such that u is the only critical point of J in $\overline{U}$, then*

$$d(I - A, U, 0) = \sum_{q=0}^{\infty} (-1)^q \dim C_q(J, u), \tag{2.17}$$

where d denotes the Leray-Schauder degree.

Finally, we state a global version of the Lyapunov-Schmidt reduction method.

Lemma 2.10 (see [18]). *Let E be a real separable Hilbert space. Let X and Y be closed subspaces of E such that $E = X \oplus Y$ and $J \in C^1(E, \mathbb{R})$. If there are $m > 0$, $\alpha > 1$ such that for all $x \in X$, $y, y_1 \in Y$,*

$$\langle J'(x + y) - J'(x + y_1), y - y_1 \rangle \geq m \|y - y_1\|^{\alpha}, \tag{2.18}$$

then the following results hold.

(i) *There exists a continuous function* $\psi : X \to Y$ *such that*

$$J(x + \psi(x)) = \min_{y \in Y} J(x + y). \tag{2.19}$$

Moreover, $\psi(x)$ *is the unique member of* Y *such that*

$$\langle J'(x + \psi(x)), y \rangle = 0, \quad \forall y \in Y. \tag{2.20}$$

(ii) *The function* $\overline{J} : X \to \mathbb{R}$ *defined by* $\overline{J}(x) = J(x + \psi(x))$ *is of class* C^1, *and*

$$\left\langle \overline{J}'(x), x_1 \right\rangle = \langle J'(x + \psi(x)), x_1 \rangle, \quad \forall x, x_1 \in X. \tag{2.21}$$

(iii) *An element* $x \in X$ *is a critical point of* $\overline{J}$ *if and only if* $x + \psi(x)$ *is a critical point of* J.

(iv) *Let* $\dim X < \infty$ *and* P *be the projection onto* X *across* Y. *Let* $S \subset X$ *and* $\Sigma \subset E$ *be open bounded regions such that*

$$\{x + \psi(x) : x \in S\} = \Sigma \cap \{x + \psi(x) : x \in X\}. \tag{2.22}$$

If $\overline{J}'(x) \neq 0$ *for* $x \in \partial S$, *then*

$$d\left(\overline{J}', S, 0\right) = d(J', \Sigma, 0), \tag{2.23}$$

where d *denotes the Leray-Schauder degree.*

(v) *If* $u = x + \psi(x)$ *is a critical point of mountain pass type of* J, *then* x *is a critical point of mountain pass type of* $\overline{J}$.

3. Proof of Theorem 1.3

In this section, firstly, we obtain a positive solution u^+ and a negative solution u^- with $C_q(J, u^+) \cong C_q(J, u^-) \cong \delta_{q,1}\mathbb{F}$ to (1.1) by using cutoff technique and the mountain pass lemma. Then, we give a precise computation of $C_q(J, 0)$. And we remark that under the assumptions of Theorem 1.3, $C_q(J, \infty)$ can be completely computed by using Propositions 2.4 and 2.5. Based on these results, four nontrivial solutions $\{u^+, u^-, u_0, u_1\}$ to (1.1) can be obtained by (2.9) or (2.10). However, it seems difficult to obtain the sign property of u_0 and u_1 through their depiction of critical groups. To conquer this difficulty, we compute the Brouwer degree of the sets of positive solutions and negative solutions to (1.1). Finally, the third nontrivial solution to (1.1) is obtained by Lyapunov-Schmidt reduction method, and its characterization of the local degree results in one or two more nontrivial solutions to (1.1) together with their sign property.

Let

$$V'^{+}(t) = \begin{cases} V'(t), & t \geq 0, \\ V''(0)t, & t < 0, \end{cases} \qquad V'^{-}(t) = \begin{cases} V'(t), & t \leq 0, \\ V''(0)t, & t > 0, \end{cases} \tag{3.1}$$

and $V^{\pm}(x) = \int_0^x V'^{\pm}(s)ds$. The functionals $J^{\pm} : E \to \mathbb{R}$ are defined as

$$J^{\pm}(u) = \frac{1}{2}\|u\|^2 - \sum_{n=1}^{T} V^{\pm}(u(n)). \tag{3.2}$$

Remark 3.1. From the definitions of $V^{\pm}$ and $V''(0) < \lambda_1$, it is easy to see that if $u \in E$ is a critical point of J^+ (or J^-), then $u > 0$ (or $u < 0$).

Lemma 3.2. *The functionals $J^{\pm}$ satisfy the (PS) condition; that is, every sequence $\{u_m\}$ in E such that $J^{\pm}(u_m)$ is bounded, and $J^{\pm}(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.*

Proof. We only prove the case of J^+. The case of J^- is completely similar. Since E is finite dimensional, it suffices to show that $\{u_m\}$ is bounded. Suppose that $\{u_m\}$ is unbounded. Passing to a subsequence, we may assume that $\|u_m\| \to \infty$ and for each n, either $|u_m(n)| \to \infty$ or $\{u_m(n)\}$ is bounded.

Set $w_m = u_m/\|u_m\| \in E$. For a subsequence, w_m converges to some w with $\|w\| = 1$. Since for all $\varphi \in E$, we have

$$\langle J'^{+}(u_m), \varphi \rangle = \sum_{n=0}^{T}(\Delta u_m(n), \Delta\varphi(n)) - \sum_{n=1}^{T}(V'^{+}(u_m(n)), \varphi(n)). \tag{3.3}$$

Hence,

$$\frac{\langle J'^{+}(u_m), \varphi \rangle}{\|u_m\|} = \sum_{n=0}^{T}(\Delta w_m(n), \Delta\varphi(n)) - \sum_{n=1}^{T}\left(\frac{V'^{+}(u_m(n))}{\|u_m\|}, \varphi(n)\right). \tag{3.4}$$

If $|u_m(n)| \to \infty$, then

$$\lim_{m \to \infty} \frac{V'^{+}(u_m(n))}{\|u_m\|} = \lim_{m \to \infty} \frac{V'^{+}(u_m(n))}{u_m(n)}w_m(n) = V''(\infty)w^{+}(n) + V''(0)w^{-}(n), \tag{3.5}$$

where $w^{+}(n) = \max\{w(n), 0\}$, $w^{-}(n) = \min\{w(n), 0\}$. If $\{u_m(n)\}$ is bounded, then

$$\lim_{m \to \infty} \frac{V'^{+}(u_m(n))}{\|u_m\|} = 0, \qquad w(n) = 0. \tag{3.6}$$

Letting $m \to \infty$ in (3.4), we have

$$\sum_{n=0}^{T} \big(\Delta w(n), \Delta \varphi(n)\big) - \sum_{n=1}^{T} \big(V''(\infty)w^{+}(n) + V''(0)w^{-}(n), \varphi(n)\big) = 0, \qquad (3.7)$$

which implies that $w(n)$ satisfies

$$\Delta^{2} w(n-1) + V''(\infty)w^{+}(n) + V''(0)w^{-}(n) = 0, \quad n \in \mathbb{Z}(1, T),$$

$$w(0) = 0 = w(T+1). \qquad (3.8)$$

Because $V''(0) < \lambda_{1}$, we see that if $w \neq 0$ is a solution to (3.8), then u is positive. Since this contradicts $V''(\infty) \in (\lambda_{k}, \lambda_{k+1})$, we conclude that $w \equiv 0$ is the only solution to (3.8). A contradiction to $\|w\| = 1$. □

Lemma 3.3. *Under the conditions of Theorem 1.3, J^{+} has a positive mountain pass-type critical point u^{+} with $C_{q}(J^{+}, u^{+}) \cong C_{q}(J, u^{+}) \cong \delta_{q,1}\mathbb{F}$; J^{-} has a negative mountain pass-type critical point u^{-} with $C_{q}(J^{-}, u^{-}) \cong C_{q}(J, u^{-}) \cong \delta_{q,1}\mathbb{F}$.*

Proof. We only prove the case of J^{+}. Firstly, we will prove that J^{+} satisfies all the conditions in Lemma 2.6. And hence, J^{+} has at least one nonzero critical point u^{+}. In fact, $J^{+} \in C^{1}(E, \mathbb{R})$, and J^{+} satisfies the (PS) condition by Lemma 3.2. Clearly, $J^{+}(0) = 0$. Thus, we still have to show that J^{+} satisfies (J1), (J2). To verify (J1), set $\alpha := V''(0) < \lambda_{1}$, then for any $\epsilon > 0$, there exists $\rho_{1} > 0$, such that

$$V''(t) \leq V''(0) + \epsilon = \alpha + \epsilon, \quad \text{for } |t| \leq \rho_{1}. \qquad (3.9)$$

So, by Taylor series expansion,

$$V(t) \leq \frac{1}{2}(\alpha + \epsilon)t^{2}, \quad \text{for } |t| \leq \rho_{1}. \qquad (3.10)$$

Take $\epsilon = (\lambda_{1} - \alpha)/2 > 0$, then $\alpha + \epsilon = (\lambda_{1} + \alpha)/2 \in (\alpha, \lambda_{1})$. If we set $\rho_{2} = (\lambda_{1} + \alpha)/2$, then

$$V(t) \leq \frac{1}{2}\rho_{2}t^{2}, \quad \text{for } |t| \leq \rho_{1}. \qquad (3.11)$$

Since for all $u \in E$, if $\|u\| \leq \sqrt{\lambda_1}\rho_1$, then $|u(n)| \leq \rho_1$ for every $n \in \mathbb{Z}(1,T)$ and hence

$$
\begin{aligned}
J^+(u) &= \frac{1}{2}\|u\|^2 - \sum_{n=1}^{T} V^+(u(n)) \\
&= \frac{1}{2}\|u\|^2 - \sum_{n \in N_1} V(u(n)) - \frac{1}{2}\sum_{n \in N_2} \alpha(u(n), u(n)) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_2 \sum_{n \in N_1} (u(n), u(n)) - \frac{1}{2}\alpha \sum_{n \in N_2} (u(n), u(n)) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_2 \sum_{n=1}^{T} (u(n), u(n)) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\frac{\rho_2}{\lambda_1}\|u\|^2,
\end{aligned}
\tag{3.12}
$$

where $N_1 = \{n \in \mathbb{Z}(1,T) \mid u(n) \geq 0\}$, $N_2 = \{n \in \mathbb{Z}(1,T) \mid u(n) < 0\}$. If we take

$$
\rho = \sqrt{\lambda_1}\rho_1, \qquad a = \frac{1}{2}\left(1 - \frac{\rho_2}{\lambda_1}\right)\rho^2,
\tag{3.13}
$$

then $J^+(u)|_{\partial B_\rho} \geq a > 0$. And hence, (J1) holds.

To verify (J2), note that $V''(\infty) \in (\lambda_k, \lambda_{k+1})$ implies that there exist $\overline{\gamma} > \lambda_k (> \lambda_1)$ and $b \in \mathbb{R}$, such that

$$
V(t) \geq \frac{\overline{\gamma}}{2}t^2 + b, \quad \text{for } t \in \mathbb{R}.
\tag{3.14}
$$

So, if we take $\phi_1(n) > 0$ with $\|\phi_1\| = 1$, then

$$
\begin{aligned}
J^+(t\phi_1) &= \frac{t^2}{2} - \sum_{n=1}^{T} V(t\phi_1(n)) \\
&\leq \frac{t^2}{2} - \frac{\overline{\gamma}t^2}{2}(\phi_1, \phi_1) - bT \\
&= \frac{t^2}{2} - \frac{\overline{\gamma}t^2}{2\lambda_1} - bT \longrightarrow -\infty, \quad 0 < t \longrightarrow +\infty.
\end{aligned}
\tag{3.15}
$$

So, if we take t sufficiently large such that $t > \rho$ and for $u_0 = te \in E$, $J^+(u_0) \leq 0$, then (J2) holds.

Now, by Lemma 2.6, J^+ has at least a nonzero critical point u^+. And for all $n \in \mathbb{Z}(1,T)$, we claim that $u^+(n) \geq 0$. If not, set $A_1 = \{n \in \mathbb{Z}(1,T) \mid u^+(n) < 0\}$, then for all $n \in A_1$, $\Delta^2 u^+(n-1) + V''(0)u^+(n) = 0$. By $V''(0) < \lambda_1$, $u^+(n) \equiv 0$ for all $n \in A_1$. Hence, $A_1 = \emptyset$.

In the following, we will compute the critical groups $C_q(J^+, u^+)$ by using Proposition 2.8.

Assume that

$$\langle J''(u^+)v, v\rangle = \langle v, v\rangle - \sum_{n=1}^{T}\left(V''(u^+(n))v(n), v(n)\right) \geq 0, \quad \forall v \in E, \tag{3.16}$$

and that there exists $v_0 \not\equiv 0$ such that

$$\langle J''(u^+)v_0, v\rangle = 0, \quad \forall v \in E. \tag{3.17}$$

This implies that v_0 satisfies

$$\Delta^2 v_0(n-1) + V''(u^+(n))v_0(n) = 0, \quad n \in \mathbb{Z}(1, T),$$
$$v_0(0) = v_0(T+1) = 0. \tag{3.18}$$

Hence, the eigenvalue problem

$$\Delta^2 v(n-1) + \lambda V''(u^+(n))v(n) = 0, \quad n \in \mathbb{Z}(1, T),$$
$$v(0) = v(T+1) = 0 \tag{3.19}$$

has an eigenvalue $\lambda = 1$. Condition $V''(t) > 0$ implies that 1 must be a simple eigenvalue; see [1]. So, $\dim \ker(J''(u^+)) = 1$. Since E is finite dimensional, the Morse index of u^+ must be finite and $J''(u^+)$ must be a Fredholm operator. By Proposition 2.8, $C_q(J, u^+) \cong \delta_{q,1}\mathbb{F}$. Finally, choose the neighborhood U^+ of u^+ such that $u > 0$ for all $u \in U^+$, then

$$C_q(J, u^+) \cong C_q(J^+, u^+) \cong \delta_{q,1}\mathbb{F}. \tag{3.20}$$

The proof is complete.   □

Lemma 3.4. *By* $V''(0) < \lambda_1$, *one has*

$$C_q(J, 0) \cong C_q(J^+, 0) \cong \delta_{q,0}\mathbb{F}. \tag{3.21}$$

Proof. By assumption, we have $J'^+(0) = J'(0) = 0$ and for all $u \in E \setminus \{0\}$,

$$\langle J''^+(0)u, u\rangle = \langle J''(0)u, u\rangle = \|u\|^2 - V''(0)\sum_{n=1}^{T}|u(n)|^2 \geq \left(1 - \frac{V''(0)}{\lambda_1}\right)\|u\|^2 > 0, \tag{3.22}$$

which implies that 0 is a local minimizer of both J^+ and J. Hence, (3.21) holds.   □

Remark 3.5. Under the conditions of Theorem 1.3, we have

$$C_q(J, \infty) \cong \delta_{q,k}\mathbb{F}. \tag{3.23}$$

We will use Propositions 2.4 and 2.5 to prove (3.23). Very similar to the proof of Lemma 3.2, we can prove that J satisfies the (PS) condition. And it is easy to prove that J satisfies (2.11). In fact, let

$$X = \mathrm{span}\{\phi_1, \phi_2, \ldots, \phi_k\}, \qquad Y = \mathrm{span}\{\phi_{k+1}, \ldots, \phi_T\}. \tag{3.24}$$

By $V''(t) \le \gamma < \lambda_{k+1}$, for all $x \in X$ and $y, y_1 \in Y$, we have

$$
\begin{aligned}
\langle J'(x+y) - J'(x+y_1), y - y_1 \rangle &= \|y - y_1\|^2 - \sum_{n=1}^{T} V''(\xi) \left[y(n) - y_1(n) \right]^2 \\
&\ge \|y - y_1\|^2 - \gamma \sum_{n=1}^{T} \left[y(n) - y_1(n) \right]^2 \\
&\ge \|y - y_1\|^2 - \frac{\gamma}{\lambda_{k+1}} \|y - y_1\|^2.
\end{aligned} \tag{3.25}
$$

Hence, if we set $m = 1 - \gamma/\lambda_{k+1}$, then (2.11) holds.

Now, noticing that $V''(\infty) \in (\lambda_k, \lambda_{k+1})$ implies that there exist $\overline{\gamma} > \lambda_k$, $\overline{\gamma}_1 < \lambda_{k+1}$ and $b \in \mathbb{R}$ such that

$$\frac{\overline{\gamma}}{2} t^2 + b \le V(t) \le \frac{\overline{\gamma}_1}{2} t^2 + b, \quad \text{for } t \in \mathbb{R}. \tag{3.26}$$

Hence, we have

$$J(u) \longrightarrow -\infty, \quad \text{as } u \in X, \ \|u\| \longrightarrow \infty, \tag{3.27}$$

$$J(u) \longrightarrow +\infty, \quad \text{as } u \in Y, \ \|u\| \longrightarrow \infty. \tag{3.28}$$

Then, (3.23) is proved by Propositions 2.4 and 2.5.

Remark 3.6. Following the proof of Theorem 3.1 in [17], (3.23) implies that there must exist a critical point $u_0 \ne 0$ of J satisfying

$$C_q(J, u_0) \cong \delta_{q,k} \mathbb{F}. \tag{3.29}$$

It is known that the critical groups are useful in distinguishing critical points. So far, we have obtained four critical points 0, u^+, u^-, and u_0 together with their characterization of critical groups. Assume that 0, u^+, u^-, and u_0 are the only critical points of J. Then, the Morse inequality (2.10) becomes

$$(-1)^0 + (-1)^1 \times 2 + (-1)^k = (-1)^k. \tag{3.30}$$

This is impossible. Thus, J must have at least one more critical point u_1. Hence, (1.1) has at least five solutions. However, it seems difficult to obtain the sign property of u_0 and u_1.

To obtain more refined results, we seek the third nontrivial solution u_0 to (1.1) by Lyapunov-Schmidt reduction method and then its characterization of the local degree results in one or two more nontrivial solutions to (1.1) together with their sign property.

Remark 3.7. The condition $k \geq 2$ in Theorem 1.3 is necessary to obtain three or more nontrivial solutions to (1.1). In fact, if $k = 1$, then we have

$$C_q(J, u_0) \cong C_q(J, u^+) \cong C_q(J, u^-) \cong \delta_{q,1}\mathbb{F}. \tag{3.31}$$

Hence, u_0 may coincide with u^+ or u^- which becomes an obstacle to seek other critical points by using Morse inequality. If $k = 0$, then

$$C_q(J, u_0) \cong C_q(J, 0) \cong \delta_{q,0}\mathbb{F}. \tag{3.32}$$

Hence, one cannot exclude the possibility of $u_0 = 0$.

To compute the degree of the set of positive (or negative) solutions to (1.1), we need the following lemma.

Lemma 3.8. *There exists $\rho > 0$ large enough, such that*

$$d(J'^+, B_\rho, 0) = d\left(J'^-, B_\rho, 0\right) = 0. \tag{3.33}$$

Proof. We only prove the case of J^+. For any $\gamma_1 > \lambda_1$, define $p(\gamma_1) := p$ as

$$p(t) = \begin{cases} \gamma_1 t, & t \geq 0, \\ V''(0)t, & t < 0. \end{cases} \tag{3.34}$$

Let $P(t) = \int_0^t p(s)ds$. The functional $Q: E \to \mathbb{R}$ is defined as

$$Q(u) = \frac{1}{2}\|u\|^2 - \sum_{n=1}^{T} P(u(n)). \tag{3.35}$$

It is obvious that Q is of class C^1 and its critical points are precisely solutions to

$$\Delta^2 u(n-1) + p(u(n)) = 0, \quad n \in \mathbb{Z}(1, T),$$
$$u(0) = 0 = u(T+1). \tag{3.36}$$

Since $V''(0) < \lambda_1$, we see that if $u \neq 0$ is a solution to (3.36), then u is positive. Because this contradicts $\gamma > \lambda_1$, we conclude that $u \equiv 0$ is the only critical point of Q.

We claim that if B is a ball in E containing zero, then $d(Q', B, 0) = 0$. In fact, since $\gamma_1 > \lambda_1 > V''(0)$ and $h(t) := p(t) - \lambda_1 t > 0$ for $t \neq 0$. Hence, for $u \in \partial B$, we have

$$
\begin{aligned}
\langle Q'(u), \phi_1 \rangle &= \sum_{n=0}^{T} (\Delta u(n), \Delta \phi_1(n)) - \sum_{n=1}^{T} \lambda_1 (u(n), \phi_1(n)) - \sum_{n=1}^{T} (h(u(n)), \phi_1(n)) \\
&= -\sum_{n=1}^{T} \left(\Delta^2 \phi_1(n-1) + \lambda_1 \phi_1(n), u(n) \right) - \sum_{n=1}^{T} (h(u(n)), \phi_1(n)) \\
&= -\sum_{n=1}^{T} (h(u(n)), \phi_1(n)) < 0,
\end{aligned}
\tag{3.37}
$$

where we have used the fact that ϕ_1 is positive on $\mathbb{Z}(1, T)$. Then, for each $s \in [0, 1]$ and $u \in E \cap \partial B$, we have

$$
\langle sQ'(u) + (1-s)(-\phi_1), \phi_1 \rangle < 0.
\tag{3.38}
$$

Hence, by invariance under homotopy of Brouwer degree, we have

$$
d(Q', B, 0) = d(K, B, 0) = 0,
\tag{3.39}
$$

where $K(u) = -\phi_1$.

Now, let $\gamma_1 = V''(\infty)$. We claim that for $\rho > 0$ large enough and for all $s \in [0, 1]$, the function $sJ'^+ + (1-s)Q'$ has no zero on ∂B_ρ.

In fact, we have proved that for all $\rho > 0$ and for all $u \in \partial B_\rho$, we have

$$
\langle Q'(u), \phi_1 \rangle < 0.
\tag{3.40}
$$

On the other hand, by the definition of $V''(\infty)$, for all $\epsilon > 0$, there exists $\rho > 0$ large enough such that $V''(\infty) - \epsilon < V'(t)/t < V''(\infty) + \epsilon$ for $|t| \geq \rho$. Since $V''(\infty) \in (\lambda_k, \lambda_{k+1})$, for $t \geq \rho$, take $\epsilon_1 = (V''(\infty) - \lambda_k)/2$, then

$$
\frac{V'(t)}{t} > \frac{V''(\infty) + \lambda_k}{2} > \lambda_k.
\tag{3.41}
$$

For $t \leq -\rho$, take $\epsilon_2 = (\lambda_{k+1} - V''(\infty))/2$, then

$$
\frac{V'(t)}{t} < \frac{V''(\infty) + \lambda_{k+1}}{2} < \lambda_{k+1}.
\tag{3.42}
$$

Hence, if we take $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then for $t \geq \rho$, we have $V'(t) > \lambda_k t > \lambda_1 t$, and for $t \leq -\rho$, we have $V'(t) > \lambda_{k+1} t$. So, if we let

$$
q(t) := V'^+(t) - \lambda_1 t = \begin{cases} V'(t) - \lambda_1 t, & t \geq 0, \\ V''(0)t - \lambda_1 t, & t < 0, \end{cases}
\tag{3.43}
$$

then $q(t) > 0$ for all $|t| \geq \rho$. And for all $u \in \partial B_\rho$, we have

$$
\begin{aligned}
\langle J'^{+}(u), \phi_1 \rangle &= \sum_{n=0}^{T} (\Delta u(n), \Delta \phi_1(n)) - \sum_{n=1}^{T} (V'^{+}(u(n)), \phi_1(n)) \\
&= -\sum_{n=1}^{T} \left(\Delta^2 \phi_1(n-1), u(n) \right) - \sum_{n=1}^{T} (V'^{+}(u(n)), \phi_1(n)) \\
&= \sum_{n=1}^{T} (\lambda_1 \phi_1(n), u(n)) - \sum_{n=1}^{T} (V'^{+}(u(n)), \phi_1(n)) \\
&= -\sum_{n=1}^{T} (q(u(n)), \phi_1(n)) < 0.
\end{aligned}
\tag{3.44}
$$

So far, we have proved that for $\rho > 0$ large enough, $sJ'^{+} + (1-s)Q'$ has no zero point on ∂B_ρ for each $s \in [0, 1]$. Hence, by invariance under homotopy of Brouwer degree, we obtain

$$
d(J'^{+}, B_\rho, 0) = d(Q', B_\rho, 0) = 0.
\tag{3.45}
$$

This completes the proof. $\square$

Remark 3.9. By Theorem 2.9 and the above results, we have the following characterization of degree of critical points.

   (i) If U^{+} (U^{-}) is a neighborhood of u^{+} (u^{-}) containing no other critical points, then

$$
\begin{aligned}
d(J'^{+}, U^{+}, 0) &= d(J', U^{+}, 0) = -1, \\
d(J'^{+}, U^{-}, 0) &= d(J', U^{-}, 0) = -1.
\end{aligned}
\tag{3.46}
$$

   (ii) Assume that B is a ball centered at zero containing on other critical points, then

$$
d(J'^{+}, B, 0) = 1 = d(J', B, 0).
\tag{3.47}
$$

Hence, if Σ is a bounded region containing the positive critical points and no other critical points, then by (3.33) we have

$$
\begin{aligned}
d(J', \Sigma, 0) = d(J'^{+}, \Sigma, 0) &= d\left(J'^{+}, B_\rho - \overline{B}, 0 \right) \\
&= d(J'^{+}, B_\rho, 0) - d(J'^{+}, B, 0) = -1.
\end{aligned}
\tag{3.48}
$$

Similarly, we see that if Σ_1 is a bounded region containing the negative critical points and no other critical points, then

$$
d(J', \Sigma_1, 0) = -1.
\tag{3.49}
$$

Now, we can give the proof of Theorem 1.3.

Proof of Theorem 1.3. The functional J satisfies (2.18) in Lemma 2.10 due to the fact that J satisfies (2.11). Hence, by Lemma 2.10, there exists $\psi : X \to \mathbb{R}$ such that

$$J(x + \psi(x)) = \min_{y \in Y} J(x + y). \tag{3.50}$$

Moreover, $\psi(x)$ is the unique member of Y such that

$$\langle J'(x + \psi(x)), y \rangle = 0, \quad \forall y \in Y. \tag{3.51}$$

The function $\overline{J} : X \to \mathbb{R}$ defined by $\overline{J}(x) = J(x + \psi(x))$ is of class C^1. Because $\overline{J}(x) \leq J(x)$, (3.27) implies that $\overline{J}(x) \to -\infty$ as $\|x\| \to \infty$. Since $\dim X < \infty$, there must exist $x_0 \in X$ such that $\overline{J}(x_0) = \max_{x \in X} J(x + \psi(x))$. Take $u_0 = x_0 + \psi(x_0)$, then $J'(u_0) = 0$ by (iii) of Lemma 2.10. If V is a neighborhood of u_0 containing no other critical points of J, taking $W = \{x \in X : x + \psi(x) \in V\}$, then $d(\overline{J}', W, 0) = (-1)^k$. Then, by part (iv) of Lemma 2.10, we have

$$d(J', V, 0) = (-1)^k. \tag{3.52}$$

Suppose that k Is Even

Let R_1 be large enough so that if $\overline{J}'(x) = 0$, then $\|x\| < R_1$. Because $\dim X < \infty$ and $\psi(x)$ is of class C^1, there exists $R_2 > 0$ such that $\|\psi(x)\| < R_2$ for $\|x\| < R_1$. Because $-\overline{J}$ is coercive, $d(\overline{J}', B_{R_1}, 0) = (-1)^k$. Hence, if we set $C = \{x + y : \|x\| < R_1, \|y\| < R_2\}$, then by (iv) of Lemma 2.10, we have

$$d(J', C, 0) = d\left(\overline{J}', B_{R_1}, 0\right) = (-1)^k = 1. \tag{3.53}$$

Suppose that $\mathcal{K} = \{u \in E \mid J'(u) = 0\}$ is finite. Let S_1, S_2, and S_3 be disjoint open bounded regions in E such that $\overline{S}_1 \cap \mathcal{K} = \{0\}$, $\overline{S}_2 \cap \mathcal{K}$ is the set of positive critical points of J, and $\overline{S}_3 \cap \mathcal{K}$ is the set of negative critical points of J. So far, we have proved that

$$d(J', S_1, 0) = 1, \qquad d(J', S_2, 0) = d(J', S_3, 0) = -1. \tag{3.54}$$

(i) If $u_0 = x_0 + \psi(x_0) \notin S_2 \cup S_3$, then u_0 is sign changing. Let S_4 denote an open bounded region disjoint from $\overline{S}_1 \cup S_2 \cup S_3$ such that $\overline{S}_4 \cap \mathcal{K} = \{u_0\}$. By the excision property of Brouwer degree, we have

$$-1 = d(J', C, 0) = d(J', S_1, 0) + d(J', S_2, 0) + d(J', S_3, 0) + d(J', S_4, 0)$$
$$+ d\left(J', C - \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0\right) \tag{3.55}$$
$$= 1 - 1 - 1 + 1 + d\left(J', C - \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0\right),$$

Thus, by Kronecker existence property of Brouwer degree, we see that there must exist $u_1 \in C - \overline{S_1 \cup S_2 \cup S_3 \cup S_4}$ such that $J'(u_1) = 0$, which proves that (1.1) has at least five solutions. In this case, both u_0 and u_1 change sign.

(ii) Suppose now that $u_0 \in S_2 \cup S_3$. Without loss of generality, we may assume that $u_0 \in S_2$. Let $S_{4,2}$ be a neighborhood of u_0 such that $\overline{S}_{4,2} \cap \mathcal{K} = \{u_0\}$. By Lemma 3.3, there exists a critical point of mountain pass type $u^+ \in S_2$ such that if S_5 is a neighborhood of u^+ such that $S_5 \cap \mathcal{K} = \{u^+\}$, then $d(J', S_5, 0) = -1$. Thus,

$$
\begin{aligned}
-1 = d(J', S_2, 0) &= d(J', S_{4,2}, 0) + d(J', S_5, 0) + d\left(J', S_2 - \overline{S_{4,2} \cup S_5}, 0\right) \\
&= 1 - 1 + d\left(J', S_2 - \overline{S_{4,2} \cup S_5}, 0\right).
\end{aligned}
\tag{3.56}
$$

Thus, by Kronecker existence property of Brouwer degree, there exists $u_1 \in S_2 - \overline{S_{4,2} \cup S_5}$ such that $J'(u_1) = 0$. Finally,

$$
\begin{aligned}
1 = d(J', C, 0) &= d(J', S_1, 0) + d(J', S_2, 0) + d(J', S_3, 0) \\
&\quad + d\left(J', C - \overline{S_1 \cup S_2 \cup S_3}, 0\right) \\
&= 1 - 1 - 1 + d\left(J', C - \overline{S_1 \cup S_2 \cup S_3}, 0\right).
\end{aligned}
\tag{3.57}
$$

Thus, there must exist $u_2 \in C - \overline{S_1 \cup S_2 \cup S_3}$ such that $J'(u_2) = 0$. Thus, the set $\{u^+, u_0, u_1, u_2\}$ together with a critical point u^- of J in S_3 shows that (1.1) has five nontrivial solutions. Since $u_2 \notin S_2 \cup S_3$ and $u^+, u_0, u_1 \in S_2$, u_2 is a sign-changing solution, and u^+, u_0, and u_1 have the same sign. This completes the proof of Theorem 1.3, when k is even.

Suppose that k Is Odd

(iii) Let S_1, S_2, and S_3 be as above. If $u_0 \notin S_2 \cup S_3$, the proof follows very closely that of the case (i).

(iv) Suppose that $u_0 = x_0 + \psi(x_0) \in S_2 \cup S_3$, hence $u_0 \in S_2$. Because $u_0 > 0$, there exists $\epsilon > 0$ such that $x + \psi(x) > 0$ if $\|x - x_0\| < \epsilon$. So, if $x \in \{x : \|x - x_0\| < \epsilon\}$, then $\overline{J} = \overline{J}^+$ and x_0 is a local maximum of $\overline{J}^+$. Since we are assuming (1.1) to have only finitely many solutions, x_0 is a strictly local maximum of $\overline{J}^+$. Let $\delta > 0$ be such that $\overline{J}^+(x) < \overline{J}^+(x_0)$ if $0 < \|x - x_0\| < \delta$. Since $k \geq 2$, $\{x : 0 < \|x - x_0\| < \epsilon\}$ is path connected. Thus, x_0 is not a critical point of mountain pass type. By Lemma 3.3, J^+ has a critical point of mountain pass type $u^+ = x^+ + \psi(x^+)$. By (v) of Lemma 2.10, $x_0 \neq x^+$, and hence $u_0 \neq u^+$. Let V_0, V_1 be neighborhoods of u_0 and u^+, respectively, such that $\overline{V}_0 \cap \mathcal{K} = \{u_0\}$ and $\overline{V}_1 \cap \mathcal{K} = \{u^+\}$. Thus,

$$
\begin{aligned}
-1 = d(J'^+, S_2, 0) &= d(J'^+, V_0, 0) + d(J'^+, V_1, 0) + d\left(J'^+, S_2 - \overline{V_0 \cup V_1}, 0\right) \\
&= -2 + d\left(J'^+, S_2 - \overline{V_0 \cup V_1}, 0\right).
\end{aligned}
\tag{3.58}
$$

Thus, by Kronecker existence property of Brouwer degree, there exists a third positive solution $u_1 \in S_2 - \overline{V_0 \cup V_1}$. So far, we have proved that (1.1) has at least four nontrivial solutions $\{u^-, u^+, u_0, u_1\}$ and that $u^+, u_0, u_1 \in S_2$ have the same sign. This proves Theorem 1.3. □

Acknowledgments

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