MULTIPLICITY RESULTS FOR A CLASS OF ASYMMETRIC WEAKLY COUPLED SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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We prove the existence and multiplicity of solutions to a two-point boundary value problem associated to a weakly coupled system of asymmetric second-order equations. Applying a classical change of variables, we transform the initial problem into an equivalent problem whose solutions can be characterized by their nodal properties. The proof is developed in the framework of the shooting methods and it is based on some estimates on the rotation numbers associated to each component of the solutions to the equivalent system.

1. Introduction

This paper represents a first step in the direction of extending to systems some of the well-known results established over the last two decades on nonlinear equations with an asymmetric nonlinearity. Recall that we call a nonlinearity *asymmetric* if the limits $f'(+\infty)$ and $f'(-\infty)$ are different. The large literature on this type of nonlinear boundary value problem can be roughly summarized in the following statement: *in an asymmetric nonlinear boundary value problem with a large positive loading, the greater the asymmetry, the larger the number of multiple solutions.*

This principle applies in both the ordinary differential equation and partial differential equation setting, and has significant implications for vibrations in bridges and ships. To illustrate the principle, we consider the scalar problem

$$u'' + bu^+ = \sin(x), \qquad u(0) = u(\pi) = 0, \tag{1.1}$$

where we recall that $u^+ := \max\{u,0\}$, $u^- := \max\{-u,0\}$. A combination of the results of [8, 27] shows that if $n^2 < b < (n+1)^2$, the problem (1.1) has exactly $2n$ solutions. Thus the greater the difference between $f'(+\infty)$ (namely b) and $f'(-\infty)$ (namely 0), the larger the number of solutions (namely $2n$). We sometimes say that the nonlinearity crosses the first n eigenvalues.

Problem (1.1) has been widely studied in the literature. In addition to the papers [8, 27], other contributions in the scalar case have been provided by Hart et al. [22], Ruf [30, 31] and, more recently, by Sadyrbaev [33]. In these works, the nonlinearity is

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required to cross asymptotically fixed eigenvalues $\lambda_k = k^2$. García-Huidobro in [20] and Rynne in [32] generalize the classical multiplicity results achieved for second-order ODEs by studying mth-order problems. The list of results available in literature as far as nonlinearities crossing eigenvalues in the PDE's setting are concerned is very rich. In this direction, we refer to [6] by Castro and Gadam dealing with multiplicity of radial solutions. Other results can be found in [34].

Recently, however, especially in problems of vibrations in suspension bridges, it has become clear that there is a need to study, not just the single equation, but also coupled systems with nonlinearities that behave like u^+. This paper is the first to deal with the general program of creating a theory for asymptotically homogeneous systems analogous to the theory for the single equation [1, 2, 12, 13, 14, 15, 16, 17, 25, 28].

So as a first step in this direction, we consider the system

$$\begin{gathered}\begin{bmatrix} u_1''(t) \\ u_2''(t) \end{bmatrix} + \begin{bmatrix} b_1 & \varepsilon \\ \varepsilon & b_2 \end{bmatrix} \begin{bmatrix} u_1^+(t) \\ u_2^+(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}, \\ u_1(0) = u_2(0) = 0 = u_1(\pi) = u_2(\pi), \end{gathered} \tag{1.2}$$

where ε is suitably small and the positive numbers b_1, b_2 satisfy

$$h^2 < b_1 < (h+1)^2, \quad k^2 < b_2 < (k+1)^2 \quad \text{for some } h,k \in \mathbb{N}. \tag{1.3}$$

The ultimate goal is considerably more ambitious. Instead of a near-diagonal operator, we would hope at first to be able to replace the operator in (1.2) with a general $n \times n$ matrix, and make a connection between the eigenvalues of that matrix, the eigenvalues of the differential operator, and the multiplicity of the solutions. This paper is to be regarded as a first step in this program.

Following the scalar classical approach, we introduce the following change of variables:

$$v_i(t) = u_i(t) - \frac{\sin t}{b_i - 1}, \quad i = 1,2, \tag{1.4}$$

leading the given problem (1.2) into the Dirichlet problem of the form

$$\begin{gathered} v_1''(t) + b_1\left[\left(\frac{\sin t}{b_1 - 1} + v_1(t)\right)^+ - \frac{\sin t}{b_1 - 1}\right] + \varepsilon\left(\frac{\sin t}{b_2 - 1} + v_2(t)\right)^+ = 0, \\ v_2''(t) + b_2\left[\left(\frac{\sin t}{b_2 - 1} + v_2(t)\right)^+ - \frac{\sin t}{b_2 - 1}\right] + \varepsilon\left(\frac{\sin t}{b_1 - 1} + v_1(t)\right)^+ = 0, \\ v_1(0) = v_2(0) = 0 = v_1(\pi) = v_2(\pi). \end{gathered} \tag{1.5}$$

This paper is devoted to the study of problem (1.5), whose solutions are characterized by their nodal properties. In particular, if we define $\tau := \{(s_1,s_2) \in \mathbb{R}^2 : s_i = +1 \text{ or } s_i = -1\ \forall i = 1,2\}$, then the following theorem holds.

Theorem 1.1. *Assume that conditions (1.3) are satisfied. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0]$, for every $(n_1,n_2) \in \mathbb{N}^2 \setminus \{(1,1)\}$ with $n_1 \le h$, $n_2 \le k$, and for every $(s_1,s_2) \in \tau$ with $s_i = -1$ whenever $n_i = 1$, problem (1.5) has at least one solution $v = (v_1,v_2)$*

with $\operatorname{sgn}(v_i'(0)) = s_i$ *such that* v_i *has exactly* $n_i - 1$*(simple) zeros in* $(0,\pi)$ *for every* $i \in \{1,2\}$.

As an immediate corollary of Theorem 1.1, we obtain the required multiplicity result for the Dirichlet problem (1.2).

Corollary 1.2. *Assume that conditions (1.3) hold. Then there exists* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0,\varepsilon_0]$, *problem (1.2) has at least* $2(2hk - h - k) + 3$ *solutions.*

To prove this corollary, we note that if we apply Theorem 1.1 to every $(n_1,n_2) \in \mathbb{N}^2 \setminus \{(1,1)\}$ with $1 < n_1 \le h$ and $1 < n_2 \le k$, we are able to achieve the existence of at least $4(h-1)(k-1)$ solutions to problem (1.5). On the other hand, if we consider the case $n_i = 1$ for a fixed $i \in \{1,2\}$, then Theorem 1.1 guarantees the existence of at least $2(h-1+k-1)$ solutions. Three further solutions to problem (1.5) are represented by the vectors

$$\begin{aligned}
&v_1(t) = -\frac{b_1}{b_1-1}\sin t, \qquad v_2(t) = -\frac{b_2}{b_2-1}\sin t;\\
&v_1(t) = 0, \qquad v_2(t) = -\left(\frac{b_2}{b_2-1} - \frac{\varepsilon}{b_1-1}\right)\sin t;\\
&v_1(t) = -\left(\frac{b_1}{b_1-1} - \frac{\varepsilon}{b_2-1}\right)\sin t, \qquad v_2(t) = 0,
\end{aligned} \tag{1.6}$$

provided that we choose $\varepsilon \le \min\{b_1,b_2\} - 1$.

By adding up all these solutions, we complete the proof the corollary.

If we restrict ourselves to the uncoupled case by setting $\varepsilon = 0$, we know from the classical scalar results in the literature that problem (1.5) admits $4hk$ solutions. Observe that the uncoupled case has a greater number of solutions since in the corresponding setting also the vectors having a component which is identically zero can solve problem (1.5).

The next references we wish to quote rely on multiplicity results for systems of second-order ODEs. Interesting contributions in the periodic setting can be found in [19] by Fonda and Ortega providing multiplicity of forced periodic solutions to planar systems with nonlinearities crossing the two first eigenvalues of the differential operator and in the very recent work [18] by Fonda which is concerned with multiplicity results for planar Hamiltonian systems having periodic forcing terms. The paper [18] treats the case in which further interactions with the eigenvalues of the differential operator occur. We conclude the list of references by quoting [35] providing oscillating solutions, whose components have independent nodal properties, for a class of superlinear conservative ordinary differential systems and [3, 4, 5, 7, 11, 24] dealing with existence and multiplicity of solutions for different classes of weakly coupled systems in the framework of topological methods. In the literature, weakly coupled systems are usually studied by constructing a suitable homotopy which, by means of a continuation theorem, carries the initial problem into an autonomous one. In this way, the multiplicity results follow directly from the computation of the degree associated to suitable scalar equations.

The techniques used in the present paper do not require to follow the standard approach described above. Our proof is based on an application of the well-known Poincaré-Miranda theorem, ensuring the existence of solutions with prescribed nodal properties whenever some estimates on the rotation numbers of each component of the solutions to suitable Cauchy problems hold (see Theorem 2.2 for more details).

We point out that the methods adopted allow us to extend our results to the case of systems with a general number N of second-order ODEs, since the Poincaré-Miranda theorem generalizes the intermediate values theorem to N-dimensional vector fields.

The paper is organized as follows. In Section 2 we recall the statement of the Poincaré-Miranda theorem in the two-dimensional case and we present the general multiplicity theorem on which the proof of our main theorem is based. The concluding part of Section 2 is devoted to establish a relation between the initial data and the behaviour of the solutions to specific Cauchy problems associated to the system in (1.5). To this aim, we present some suitable versions of the elastic lemma.

In Section 3 we determine restrictions on the possible initial data of prescribed Cauchy problems. The bounds obtained will be crucial in order to prove some results concerning the simplicity of the zeros and to obtain the estimates on the rotation numbers needed to apply the multiplicity theorem stated in Section 2.

2. A shooting approach and the elastic lemmas

The first part of this section is devoted to present a multiplicity result (cf. Theorem 2.2 below) for a two-dimensional Dirichlet problem of second-order differential equations of the form

$$\begin{gathered} v''(t) = F(t, v(t)), \\ v(0) = v(\pi) = (0,0), \end{gathered} \tag{2.1}$$

where $F : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function, locally Lipschitz with respect to the second variable.

We first recall the statement of the Poincaré-Miranda theorem in the two-dimensional case (cf., e.g., [26, 29]).

Theorem 2.1 (Poincaré-Miranda theorem). *Let $g : [a,b] \times [c,d] \to \mathbb{R}^2$ be continuous and such that $g_1(a,y)g_1(b,y) < 0$ for every $y \in [c,d]$ and $g_2(x,c)g_2(x,d) < 0$ for every $x \in [a,b]$. Then, there exists $(\overline{x},\overline{y}) \in (a,b) \times (c,d)$ with $g(\overline{x},\overline{y}) = (0,0)$.*

Secondly, we introduce the notion of *rotation number*.

For every continuous curve $z = (x,y) : [0,\pi] \to \mathbb{R}^2 \setminus \{0\}$, consider a lifting $\tilde{z} : [0,\pi] \to \mathbb{R} \times \mathbb{R}_0^+$ to the polar coordinate covering space, given by $\tilde{z}(t) = (\vartheta_z(t), \rho_z(t))$, where $x = \rho_z \sin\vartheta_z$, $y = \rho_z \cos\vartheta_z$. Note that ϑ_z and ρ_z are continuous functions and, moreover, $\vartheta_z(t) - \vartheta_z(0)$ is independent on the lifting of z which has been considered. Hence, for each $t \in [0,\pi]$, we can define the rotation number

$$\operatorname{Rot}(t;z) := \frac{\vartheta_z(t) - \vartheta_z(0)}{\pi}. \tag{2.2}$$

Let $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, locally Lipschitz with respect to the second variable and let the curve $z^*(t) = (x(t), x'(t))$ represent a solution $x(\cdot)$ of

$$x''(t) + f(t, x(t)) = 0, \tag{2.3}$$

defined on $[0,\pi]$ and such that $x(t)^2 + x'(t)^2 > 0$ for every $t \in [0,\pi]$, then we can introduce the rotation number of z^*, whose expression can be written in the following form:

$$\operatorname{Rot}(t;z^*) = \frac{1}{\pi}\int_0^t \frac{x'(s)^2 + f(s,x(s))x(s)}{x'(s)^2 + x(s)^2}\,ds. \tag{2.4}$$

We are now in position to state the required multiplicity result. In particular, suitable estimates on the rotation numbers of each component of the solutions of prescribed Cauchy problems related to problem (2.1) will guarantee the existence of solutions to (2.1) characterized, component by component, by their nodal properties.

Theorem 2.2. *Consider a continuous function $F : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$, locally Lipschitz with respect to the second variable. Assume that there exist $\nu, \upsilon \in \{>,<\}$ and four positive numbers r_1, r_2, R_1, R_2 with $r_i < R_i$ for each $i \in \{1,2\}$ such that all the solutions $v = (v_1, v_2)$ of the problem*

$$\begin{gathered} v''(t) = F(t,v(t)), \qquad v(0) = (0,0), \\ v_1'(0)\,\nu\, 0, \qquad v_2'(0)\,\upsilon\, 0, \qquad r_i \le \left|v_i'(0)\right| \le R_i \quad \forall i \in \{1,2\}, \end{gathered} \tag{2.5}$$

satisfy $v_i(t)^2 + v_i'(t)^2 > 0$ for every $t \in [0,\pi]$ and for every $i \in \{1,2\}$.

Moreover, assume that there are $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} &\operatorname{Rot}\left(\pi;(v_1,v_1')\right) > n_1 \quad \text{for each solution } v \text{ of (2.5) with } \left|v_1'(0)\right| = r_1, \\ &\operatorname{Rot}\left(\pi;(v_1,v_1')\right) < n_1 \quad \text{for each solution } v \text{ of (2.5) with } \left|v_1'(0)\right| = R_1; \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\operatorname{Rot}\left(\pi;(v_2,v_2')\right) > n_2 \quad \text{for each solution } v \text{ of (2.5) with } \left|v_2'(0)\right| = r_2, \\ &\operatorname{Rot}\left(\pi;(v_2,v_2')\right) < n_2 \quad \text{for each solution } v \text{ of (2.5) with } \left|v_2'(0)\right| = R_2. \end{aligned} \tag{2.7}$$

Then, there is at least one solution v of the Dirichlet problem (2.1) with $v_1'(0)\,\nu\,0$ and $v_2'(0)\,\upsilon\,0$ such that v_i has exactly $n_i - 1$ zeros in $(0,\pi)$ for each $i \in \{1,2\}$.

We refer to [10, Theorem 3.1] for a scalar version of Theorem 2.2. Note that in the scalar case it is possible to deal with more general nonlinearities.

Proof. We consider four positive real numbers r_1, r_2, R_1, R_2 satisfying the given assumptions. Moreover, we define the constants $c_> := 1$ and $c_< := -1$. Let $n_1, n_2 \in \mathbb{N}$ and $\nu, \upsilon \in \{>,<\}$ be as in the statement of the theorem.

Fixed $z_0 \in \mathcal{R} := [c_\nu r_1, c_\nu R_1] \times [c_\upsilon r_2, c_\upsilon R_2]$, we denote by $v(\cdot;z_0) = (v_1(\cdot;z_0), v_2(\cdot;z_0))$ the unique solution of the initial value problem

$$v''(t) = F(t,v(t)) \qquad v(0) = (0,0), \qquad v'(0) = z_0. \tag{2.8}$$

According to this notation, we define the function $g : \mathcal{R} \to \mathbb{R}^2$ by setting

$$g_i(z_0) = \operatorname{Rot}\left(\pi;\left(v_i(\cdot;z_0), v_i'(\cdot;z_0)\right)\right) - n_i. \tag{2.9}$$

The Lipschitz assumption on the nonlinearity guarantees the continuity of the function g. Moreover, from conditions (2.6) and (2.7) we, respectively, get

$$\begin{aligned} g_1(c_\nu r_1, y_0)g_1(c_\nu R_1, y_0) < 0 \quad \forall y_0 \in [c_\upsilon r_2, c_\upsilon R_2], \\ g_2(x_0, c_\upsilon r_2)g_2(x_0, c_\upsilon R_2) < 0 \quad \forall x_0 \in [c_\nu r_1, c_\nu R_1]. \end{aligned} \tag{2.10}$$

By applying the Poincaré-Miranda theorem, we infer the existence of a vector $\overline{z_0} \in \mathcal{R}$ such that $g(\overline{z_0}) = (0,0)$. Recalling the definition of the rotation number, we can finally conclude that there exists a solution v of the Dirichlet problem (2.1) with $v'(0) = \overline{z_0}$, $v_1'(0)\nu 0$, and $v_2'(0)\upsilon 0$ such that v_i has exactly $n_i - 1$ zeros in $(0,\pi)$ for each $i \in \{1,2\}$. □

Remark 2.3. Theorem 2.2 holds true if we invert both the inequalities in (2.6) and/or both the inequalities in (2.7).

As a particular case, we will apply Theorem 2.2 to the given system

$$\begin{aligned} v_1''(t) + b_1\left[\left(\frac{\sin t}{b_1 - 1} + v_1(t)\right)^+ - \frac{\sin t}{b_1 - 1}\right] + \varepsilon\left(\frac{\sin t}{b_2 - 1} + v_2(t)\right)^+ = 0, \\ v_2''(t) + b_2\left[\left(\frac{\sin t}{b_2 - 1} + v_2(t)\right)^+ - \frac{\sin t}{b_2 - 1}\right] + \varepsilon\left(\frac{\sin t}{b_1 - 1} + v_1(t)\right)^+ = 0, \end{aligned} \tag{2.11}$$

admitting a unique solution $v = (v_1, v_2)$ such that $(v(0), v'(0)) = z_0$ for a fixed $z_0 \in \mathbb{R}^4$.

The next part of this section is devoted to present some versions of the well-known "elastic lemma." By following a classical procedure (cf., e.g., [21]), it is possible to estimate the C^1-norm of every solution of system (2.11) having bounded initial conditions.

Lemma 2.4. *Suppose that $b_i > 1$ for every $i \in \{1,2\}$ and consider $c, d \in [0,\pi]$ with $c < d$. Then, for every $R_1 > 0$ there exist $R_2 = R_2(R_1, b_1, b_2) > R_1$ and $\hat{\eta} = \hat{\eta}(R_1, b_1, b_2) > 0$ such that for every $\varepsilon \in (0, \hat{\eta}]$ and for every solution v of (2.11) with*

$$\min_{t \in [c,d]} |(v(t), v'(t))| \le R_1, \tag{2.12}$$

it follows that

$$|(v(t), v'(t))| \le R_2 \quad \forall t \in [c,d]. \tag{2.13}$$

Proof. Fix $R_1 > 0$ and an arbitrarily small $\mu > 0$. Then, there exists a positive constant $\hat{\eta}$ satisfying

$$\left[2R_1 + 2\hat{\eta}\left(\frac{1}{b_1 - 1} + \frac{1}{b_2 - 1}\right)\right]e^{2\pi(\max\{b_1, b_2\} + \hat{\eta})} \le 2R_1 e^{2\pi \max\{b_1, b_2\}} + \mu. \tag{2.14}$$

For every $\varepsilon \le \hat{\eta}$ we take a solution $v = (v_1, v_2)$ of (2.11) such that $|(v(t_0), v'(t_0))| \le R_1$, with $t_0 \in [c,d]$ fixed. Then, the explicit expression of R_2 is given by $R_2 := 2R_1 e^{2\pi \max\{b_1, b_2\}} + \mu$.

The nonlinear terms in system (2.11) can be easily estimated. More precisely, for each $i, j \in \{1,2\}$ with $i \neq j$ and for every $t \in [0,\pi]$ the following inequality holds:

$$\left| b_i\left[\left(\frac{\sin t}{b_i-1}+v_i(t)\right)^+ - \frac{\sin t}{b_i-1}\right] + \varepsilon\left(\frac{\sin t}{b_j-1}+v_j(t)\right)^+\right| \le b_i\,|\,v_i(t)\,| + \varepsilon\left(\frac{\sin t}{b_j-1}+|\,v_j(t)\,|\right). \tag{2.15}$$

Hence, we obtain that for every $t \in [t_0, d]$,

$$\begin{aligned} r(t) &:= |\,(v(t), v'(t))\,| = \sqrt{v_1(t)^2 + v_2(t)^2 + v_1'(t)^2 + v_2'(t)^2} \\ &\le 2R_1 + \varepsilon\|\sin t\|_1\left(\frac{1}{b_1-1}+\frac{1}{b_2-1}\right) \\ &\quad + \int_{t_0}^{t}\left(|\,v_1'(s)\,| + |\,v_2'(s)\,| + (b_1+\varepsilon)\,|\,v_1(s)\,| + (b_2+\varepsilon)\,|\,v_2(s)\,|\right)ds. \end{aligned} \tag{2.16}$$

It immediately follows that for every $t \in [t_0, d]$,

$$r(t) \le 2R_1 + 2\varepsilon\left(\frac{1}{b_1-1}+\frac{1}{b_2-1}\right) + 2\left(\max\{b_1,b_2\}+\varepsilon\right)\int_{t_0}^{t} r(s)ds. \tag{2.17}$$

By applying Gronwall's lemma and recalling the definition of R_2 and the inequality (2.14) satisfied by $\hat{\eta}$, we can conclude that

$$r(t) \le \left[2R_1 + 2\varepsilon\left(\frac{1}{b_1-1}+\frac{1}{b_2-1}\right)\right]e^{2(\max\{b_1,b_2\}+\varepsilon)(d-t_0)} \le R_2 \quad \forall t \in [t_0,d]. \tag{2.18}$$

Arguing as above in the left interval of t_0 given by $[c, t_0]$, we can extend inequality (2.18) to the whole interval $[c,d]$, achieving the thesis. □

A dual situation with respect to Lemma 2.4 occurs on each component of a solution to system (2.11) under suitable assumptions. It can be expressed by the following lemma.

Lemma 2.5. *Fix $i, j \in \{1,2\}$ with $i \neq j$ and suppose that $b_l > 1$ for every $l \in \{1,2\}$. Assume that there exist three positive constants $\overline{\eta}$, ρ_1, L such that for every $\varepsilon \in (0,\overline{\eta}]$ and for every solution $v = (v_1, v_2)$ of system (2.11),*

$$\max_{t\in[0,\pi]} |\,(v_i(t), v_i'(t))\,| > \rho_1, \quad v_j(t) \le L \quad \forall t \in [0,\pi]. \tag{2.19}$$

Then, there exist $\rho_2 = \rho_2(\rho_1, b_i) \in (0,\rho_1)$ and $\eta_i^* = \eta_i^*(b_1,b_2,L,\rho_1,\overline{\eta}) \in (0,\overline{\eta}]$ such that for every $\varepsilon \in (0,\eta_i^*]$ and for every solution $v = (v_1,v_2)$ of (2.11) it holds that

$$|\,(v_i(t), v_i'(t))\,| > \rho_2 \quad \forall t \in [0,\pi]. \tag{2.20}$$

Proof. Consider three constants $\overline{\eta}, \rho_1, L > 0$ satisfying the assumptions of this lemma. Moreover, we choose an arbitrarily small constant $\mu = \mu(\rho_1) > 0$ such that $\mu < \rho_1/2$. We are now in position to write the explicit expressions of the positive constants ρ_2 and η_i^*, which are given, respectively, by

$$\rho_2 := \frac{\sqrt{2}}{2}(\rho_1 - 2\mu)\mathrm{e}^{-\sqrt{2}\pi b_i}, \qquad \eta_i^* := \min\left\{\frac{\mu(b_j-1)}{2+\pi L(b_j-1)}\mathrm{e}^{-\sqrt{2}\pi b_i}, \overline{\eta}\right\}. \tag{2.21}$$

We take $\varepsilon \in (0, \eta_i^*]$ and consider a solution $v = (v_1, v_2)$ of (2.11).

The thesis is easily achieved by following the same steps of [9, Lemma 2.4.2]. More precisely, arguing by contradiction and taking into account the first inequality in (2.19), we can find an interval $I = [c,d]$, where $|(v_i(\bar{t}), v_i'(\bar{t}))| = \rho_2$ and $|(v_i(t_0), v_i'(t_0))| = \rho_1$ for some $\bar{t}, t_0 \in [c,d]$.

We are now interested in estimating $|(v_i, v_i')|$ along the whole interval $[c,d]$.

To this aim we recall that the function v_i solves the equation

$$v_i''(t) + b_i\left[\left(\frac{\sin t}{b_i - 1} + v_i(t)\right)^+ - \frac{\sin t}{b_i - 1}\right] + \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+ = 0, \tag{2.22}$$

whose nonlinear term satisfies the following inequality:

$$\left|b_i\left[\left(\frac{\sin t}{b_i - 1} + v_i(t)\right)^+ - \frac{\sin t}{b_i - 1}\right] + \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+\right| \le b_i\,|\,v_i(t)\,| + \varepsilon\left(\frac{\sin t}{b_j - 1} + L\right). \tag{2.23}$$

We are now ready to apply the classic elastic lemma (cf. [21, Lemma 2.1] and [9, Lemma 2.4.1]). Since $\min_{t\in[c,d]} |(v_i(t), v_i'(t))| \le \rho_2$, we infer that

$$|\,(v_i(t), v_i'(t))\,| \le \left(\sqrt{2}\rho_2 + \varepsilon\left\|\frac{\sin t}{b_j - 1} + L\right\|_1\right)\mathrm{e}^{\sqrt{2}b_i(d-c)} \quad \forall t \in [c,d]. \tag{2.24}$$

Taking into account the definition of ρ_2, we obtain that

$$|\,(v_i(t), v_i'(t))\,| \le \sqrt{2}\mathrm{e}^{\sqrt{2}\pi b_i}\rho_2 + \varepsilon\left(\frac{2}{b_j - 1} + L\pi\right)\mathrm{e}^{\sqrt{2}\pi b_i} \le (\rho_1 - 2\mu) + \mu < \rho_1 \quad \forall t \in [c,d]; \tag{2.25}$$

a contradiction with the fact that $|(v_i(t_0), v_i'(t_0))| = \rho_1$ for $t_0 \in [c,d]$. □

We have written Lemma 2.5 in components since we are interested in proving that all the zeros of every component of the solutions to system (2.11) are simple (cf. Proposition 3.11), provided that we choose a sufficiently small ε and suitable initial conditions.

We also remark that, in general, it is not possible to prove the existence of two positive constants ρ_2, η_i^* such that for every $\varepsilon \in (0, \eta_i^*]$ and for every solution to problem (2.11) the relation $|(v_i(0), v_i'(0))| \neq 0$ implies $|(v_i(t), v_i'(t))| > \rho_2$ for every $t \in [0,\pi]$.

Indeed, the choice of η_i^* in Lemma 2.5 depends on the particular ρ_1 satisfying (2.19) which has been considered. For this reason, to get the simplicity of the zeros of each component of the solutions to system (2.11), we need to consider solutions v having $v_i'(0)$ bounded in modulus from below for each $i \in \{1,2\}$. Proposition 3.10 will provide the required lower estimates on $|v_i'(0)|$ for every solution v to the system (2.11) with ε small enough.

3. The main result

The first part of this section is devoted to establish a relation between the ith initial slope $v_i'(0)$ and the number of zeros in $(0,\pi]$ of v_i, $v = (v_1, v_2)$ being a solution to system (2.11) satisfying the initial conditions

$$v_1(0) = v_2(0) = 0. \tag{3.1}$$

This relation will provide the estimates on the rotation numbers required by Theorem 2.2 in order to get the multiplicity results.

Using techniques similar to the one adopted in the classical work [8], we can state a first proposition providing a relation between the negative value of the ith initial slope $v_i'(0)$ and the absence of zeros of v_i, when $v = (v_1, v_2)$ solves problem (2.11). More precisely, the following holds.

Proposition 3.1. *Fix $i \in \{1,2\}$ and assume that $b_i > 1$. Then, for every $\varepsilon > 0$ and for every solution $v = (v_1, v_2)$ to system (2.11) satisfying (3.1)*

$$v_i'(0) < -\frac{b_i}{b_i - i} \Longrightarrow v_i \text{ has no zeros in } (0,\pi]. \tag{3.2}$$

Proof. We fix $i, j \in \{1,2\}$ with $i \neq j$ and define $\hat{v}_i(t) := -b_i \sin t/(b_i - 1)$. Consider a solution $v = (v_1, v_2)$ to problem (2.11) and (3.1) satisfying $v_i'(0) < -b_i/(b_i - 1) = \hat{v}_i'(0)$. This means that there exists a right neighbourhood of 0 in which $v_i < \hat{v}_i$, since $v_i(0) = \hat{v}_i(0) = 0$.

We argue by contradiction, assuming that there exists at least a zero of v_i in $(0,\pi]$ and denote by c the first zero of v_i in $(0,\pi]$. In particular, $v_i(c) = 0 \geq \hat{v}_i(c)$. Hence, we can deduce the existence of $b \in (0,c]$ such that

$$v_i(t) < \hat{v}_i(t) \quad \forall t \in (0,b), \qquad v_i(b) = \hat{v}_i(b). \tag{3.3}$$

Since $\hat{v}_i(t) \leq -\sin t/(b_i - 1)$ for every $t \in [0,b]$, the function v_i solves the following equation:

$$v_i''(t) - \frac{b_i}{b_i - 1}\sin t + \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+ = 0 \quad \forall t \in [0,b]. \tag{3.4}$$

It immediately follows that

$$v_i'(t) - \hat{v}_i'(t) = \left(v_i'(0) + \frac{b_i}{b_i - 1}\right) - \varepsilon\int_0^t \left(\frac{\sin s}{b_j - 1} + v_j(s)\right)^+ ds < 0 \quad \forall t \in [0,b], \tag{3.5}$$

whence $v_i(b) < \hat{v}_i(b)$, a contradiction. □

Before exhibiting further estimates on the number of zeros of v_i, we need some preliminary lemmas.

First, note that a nontrivial component v_i of a solution v of the system (2.11) is strictly concave at a positive bump, since from (2.11) we get

$$v_i''(t) = -b_i v_i(t) - \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+ \quad \text{if } i \neq j \in \{1,2\},\ v_i(t) \geq -\frac{\sin t}{b_i - 1}. \tag{3.6}$$

The following lemma allows to estimate the length of the positive bumps of v_i.

Lemma 3.2. *Fix $i \in \{1,2\}$ and assume that $b_i > 1$. For every $\varepsilon \geq 0$ and for every solution $v = (v_1, v_2)$ to system (2.11), denote by $\alpha, \beta \in \mathbb{R}$ two consecutive zeros of v_i such that $v_i > 0$ for every $t \in (\alpha, \beta)$. Then,*

$$\beta - \alpha \leq \frac{\pi}{\sqrt{b_i}}. \tag{3.7}$$

Proof. Fix $i \in \{1,2\}$ and take a solution $v = (v_1, v_2)$ to system (2.11). Consider $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and

$$v_i(t) > 0 \quad \forall t \in (\alpha, \beta), \qquad v_i(\alpha) = 0 = v_i(\beta). \tag{3.8}$$

By following a standard procedure (cf. [8]), we define $\xi(t) := \sin(\pi(t-\alpha)/(\beta-\alpha))$. By definition, $\xi(t) > 0$ for every $t \in (\alpha, \beta)$, $\xi(\alpha) = 0 = \xi(\beta)$. Since $v = (v_1, v_2)$ is a solution to system (2.11), we know that

$$v_i''(t) = -b_i v_i(t) - \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+, \quad \xi''(t) = -\left(\frac{\pi}{\beta - \alpha}\right)^2 \xi(t) \quad \forall t \in [\alpha, \beta]. \tag{3.9}$$

Thus, we obtain that $\int_\alpha^\beta (v_i''(s)\xi(s) - v_i(s)\xi''(s))ds = 0$, from which follows

$$\left(\frac{\pi^2}{(\beta-\alpha)^2} - b_i\right)\int_\alpha^\beta v_i(s)\xi(s)ds = \varepsilon\int_\alpha^\beta \xi(s)\left(\frac{\sin s}{b_j - 1} + v_j(s)\right)^+ ds \geq 0. \tag{3.10}$$

We can finally conclude that $\beta - \alpha \leq \pi/\sqrt{b_i}$. □

A lemma analogous to Lemma 3.2 can be stated to establish a lower bound on the distance between two zeros of v_i when v_i is negative between the two zeros. More precisely, the following holds.

Lemma 3.3. *Fix $i \in \{1,2\}$ and assume that $b_i > 1$. For every $\varepsilon \geq 0$ and for every solution $v = (v_1, v_2)$ to system (2.11), denote by $\alpha_*, \beta_* \in \mathbb{R}$ two zeros of v_i such that $v_i(t) < 0$ for every $t \in (\alpha_*, \beta_*)$. Then,*

$$\beta_* - \alpha_* \geq \frac{\pi}{\sqrt{b_i}}. \tag{3.11}$$

Proof. The proof of this lemma is similar to the one of Lemma 3.2. We argue exactly as before, by introducing $\xi_*(t) := \sin(\pi(t-\alpha_*)/(\beta_*-\alpha_*))$ and integrating the function $v_i''\xi_* - v_i\xi_*''$. The only difference consists in the sign of v_i on (α_*,β_*) and in the presence of one more negative addendum in the expression of v_i''. Indeed, from system (2.11) we immediately obtain that $v_i''(t) = -b_iv_i(t) - b_i(\sin t/(b_i-1)+v_i(t))^- - \varepsilon(\sin t/(b_j-1)+v_j(t))^+$ when $j \in \{1,2\}$, $j \neq i$. □

The statements of the previous lemmas include also the case in which $\varepsilon = 0$, since Lemma 3.3 will be applied in a scalar context (cf. Lemma 3.7 and the corresponding proof).

Henceforth, we concentrate our attention on the solutions $v = (v_1,v_2)$ to problem (2.11) and (3.1) verifying

$$|v_i'(0)| \leq 2\pi^2\frac{b_i}{b_i-1}+\sigma \quad \forall i \in \{1,2\}, \tag{3.12}$$

for a fixed $\sigma > 0$. We observe that imposing this condition will not affect the number of solutions to the Dirichlet problem (1.5). Indeed, in Remark 3.6 we will ensure that every solution to problem (1.5) satisfies condition (3.12). We also point out that the choice of the constant in (3.12) depends on the fact that v_i is characterized by particular nodal properties whenever $v = (v_1,v_2)$ solves (2.11), (3.1) and verifies $|v_i'(0)| = 2\pi^2(b_i/(b_i-1))+\sigma$ (we refer to Proposition 3.5 for more details).

Let us first show that the C^1-norm of the solutions to problem (2.11) and (3.1) satisfying (3.12) is bounded, provided that we choose a sufficiently small constant ε.

Proposition 3.4. *Assume that $b_i > 1$ for every $i \in \{1,2\}$ and fix $\sigma > 0$. Then, there exist two positive constants $M = M(b_1,b_2,\sigma)$ and $\varepsilon^* = \varepsilon^*(b_1,b_2,\sigma)$ such that for every $\varepsilon \in (0,\varepsilon^*]$ and for every solution $v = (v_1,v_2)$ to problem (2.11) satisfying (3.1) and (3.12) it follows that*

$$|v_i(t)| \leq |(v(t),v'(t))| \leq M \quad \forall t \in [0,\pi],\ \forall i \in \{1,2\}. \tag{3.13}$$

Proof. Conditions (3.1) and (3.12) ensure that

$$\min_{t\in[0,\pi]}|(v(t),v'(t))| \leq |(v_1'(0),v_2'(0))| \leq 2\pi^2\left(\frac{b_1}{b_1-1}+\frac{b_2}{b_2-1}\right)+2\sigma. \tag{3.14}$$

Thus, Lemma 2.4 guarantees the existence of $M = M(b_1,b_2,\sigma) > 0$ and of $\varepsilon^* = \varepsilon^*(b_1,b_2,\sigma) > 0$ such that for every $\varepsilon \in (0,\varepsilon^*]$ and for every solution $v = (v_1,v_2)$ to problem (2.11) satisfying (3.1) and (3.12) it follows that $|v_i(t)| \leq |(v(t),v'(t))| \leq M$ for every $t \in [0,\pi]$ and $i \in \{1,2\}$. This completes the proof. □

In the same context of the above proposition, we are able to provide a relation between the ith initial slope $v_i'(0)$ of a solution v to system (2.11) and the exact number of zeros of v_i in $(0,\pi]$. More precisely, the following proposition holds.

PROPOSITION 3.5. *Assume that $b_l > 1$ for every $l \in \{1,2\}$. Fix $\sigma > 0$ and $i \in \{1,2\}$. Then, there exists $\hat{\varepsilon}_i = \hat{\varepsilon}_i(b_1,b_2,\sigma) > 0$ such that for every $\varepsilon \in (0,\hat{\varepsilon}_i]$ and for every solution $v = (v_1,v_2)$ to system (2.11) satisfying (3.1) and (3.12),*

$$v_i'(0) = 2\pi^2\frac{b_i}{b_i-1} + \sigma \Longrightarrow v_i \text{ has one and only one simple zero in } (0,\pi]. \tag{3.15}$$

We point out that if the assumptions of Proposition 3.5 hold, then v_i has one zero in $(0,\pi]$. From Lemma 3.2, this zero is less than or equal to $\pi/\sqrt{b_i}$ and, consequently, it cannot coincide with π.

Proof. Fix $\sigma > 0$ and $i, j \in \{1,2\}$ with $i \neq j$. By applying Proposition 3.4, we immediately infer the existence of two positive constants $M = M(b_1,b_2,\sigma)$ and $\varepsilon^* = \varepsilon^*(b_1,b_2,\sigma)$ such that for every $\varepsilon \in (0,\varepsilon^*]$ and for every solution $v = (v_1,v_2)$ to problem (2.11) satisfying (3.1) and (3.12) it follows that

$$|v_j(t)| \leq |(v(t),v'(t))| \leq M \quad \forall t \in [0,\pi]. \tag{3.16}$$

We define $\hat{\varepsilon}_i = \hat{\varepsilon}_i(b_1,b_2,\sigma)$ by setting

$$\hat{\varepsilon}_i := \min\left\{\varepsilon^*, \sigma\frac{\sqrt{b_i}(b_j-1)}{\pi(Mb_j - M + 1)}\right\}. \tag{3.17}$$

For every $\varepsilon \in (0,\hat{\varepsilon}_i]$ we consider a solution $v = (v_1,v_2)$ to problem (2.11) and (3.1) such that $v_i'(0) = 2\pi^2(b_i/(b_i-1)) + \sigma$. Denote by τ the first zero of v_i in $(0,\pi]$. Since v_i is strictly concave at a positive bump, we know that $v_i > 0$ in $(0,\tau)$ and that $v_i'(\tau) < 0$.

We claim that

$$v_i'(\tau) \leq -2\frac{b_i}{b_i-1}. \tag{3.18}$$

We suppose by contradiction that $v_i'(\tau) > -2(b_i/(b_i-1))$. Consider now $\gamma \in (0,\tau)$ such that $v_i(\gamma) := \max_{t\in(0,\tau)} v_i(t)$. As a consequence of the strict concavity of the positive bumps, we can deduce that

$$v_i'(t) \geq v_i'(\tau) > -2\frac{b_i}{b_i-1} \quad \forall t \in [\gamma,\tau]. \tag{3.19}$$

This implies that

$$0 = v_i(\tau) = v_i(\gamma) + \int_\gamma^\tau v_i'(s)ds > v_i(\gamma) - 2\frac{b_i}{b_i-1}(\tau-\gamma). \tag{3.20}$$

Our aim consists in getting the contradiction by providing suitable upper bounds on $v_i'(0)$. According to (3.6), we obtain

$$2\pi^2\frac{b_i}{b_i-1} + \sigma = v_i'(0) = b_i\int_0^\gamma v_i(s)ds + \varepsilon\int_0^\gamma\left(\frac{\sin s}{b_j-1} + v_j(s)\right)^+ ds. \tag{3.21}$$

Taking into account the definition of γ, the inequality (3.20), and Lemma 3.2, we are able to estimate from above the first addendum

$$b_i \int_0^\gamma v_i(s)ds \le b_i v_i(\gamma)\gamma < b_i 2\frac{b_i}{b_i-1}(\tau-\gamma)\gamma \le 2\pi^2\frac{b_i}{b_i-1}. \tag{3.22}$$

According to the definition of $\hat{\varepsilon}_i$ given in (3.17) and to the estimates provided by (3.16) and by Lemma 3.2, we finally infer that

$$v_i'(0) < 2\pi^2\frac{b_i}{b_i-1} + \varepsilon\frac{\pi}{\sqrt{b_i}}\left(\frac{1}{b_j-1}+M\right) \le 2\pi^2\frac{b_i}{b_i-1}+\sigma, \tag{3.23}$$

contradicting (3.21). Hence, the claim is proved and (3.18) holds.

Moreover, for every $t \ge \tau$ we get

$$v_i'(t) = v_i'(\tau) + \frac{b_i}{b_i-1}(\cos\tau - \cos t) - b_i\int_\tau^t \left(\frac{\sin s}{b_i-1}+v_i(s)\right)^+ ds - \varepsilon\int_\tau^t\left(\frac{\sin s}{b_j-1}+v_j(s)\right)^+ ds. \tag{3.24}$$

From (3.18) we obtain $v_i'(\tau) \le -2(b_i/(b_i-1)) < (b_i/(b_i-1))(\cos t - \cos\tau)$. Hence, $v_i'(t) < 0$ for every $t \ge \tau$. This implies that there does not exist any other zero of v_i in the interval $(\tau, \pi]$. The thesis follows. □

Remark 3.6. It is possible to prove that if $b_l > 1$ for every $l \in \{1,2\}$, then for every $\sigma > 0$ there exists $\bar{\varepsilon} = \bar{\varepsilon}(\sigma, b_1, b_2) > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon}]$ and for every solution $v = (v_1, v_2)$ to the Dirichlet problem (1.5),

$$-\frac{b_i}{b_i-1} \le v_i'(0) \le \frac{2\pi^2 b_i}{b_i-1}+\sigma \quad \forall i \in \{1,2\}. \tag{3.25}$$

The validity of the lower estimate on $v_i'(0)$ follows immediately from an application of Proposition 3.1.

The estimate from above can be easily obtained by refining the arguments used to prove Proposition 3.5 and by taking into account that both v_1 and v_2 satisfy the Dirichlet boundary conditions. For this reason we have chosen to restrict ourselves to the study of solutions to problem (2.11) and (3.1) whose ith initial slope is bounded in modulus by the constant $2\pi^2 b_i/(b_i-1)+\sigma$ for a fixed $\sigma > 0$ and for every $i \in \{1,2\}$ (as assumed in condition (3.12)).

Propositions 3.1 and 3.5 provide an upper bound on the rotation number of v_i calculated in π when $|v_i'(0)| = 2\pi^2 b_i/(b_i-1)+\sigma$. In order to get the required multiplicity result by an application of Theorem 2.2, we will exhibit a lower bound on the rotation number of v_i for a suitable ith initial slope.

To this aim, we will compare the behaviour of the ith component of the solutions $v = (v_1, v_2)$ to (2.11) and (3.1) with the solutions u to the scalar problem

$$u''(t) + b_i\left[\left(\frac{\sin t}{b_i-1}+u(t)\right)^+ - \frac{\sin t}{b_i-1}\right] = 0. \tag{3.26}$$

As a first step, we recall some properties of u.

LEMMA 3.7. *Fix $i \in \{1,2\}$ and assume that $b_i > 1$. Consider the constants $k < 0$, $\varepsilon \geq 0$, and $a \in (0,\pi)$. Denote by ξ the solution of the linear Cauchy problem*

$$\begin{gathered} \xi''(t) + b_i\xi(t) = 0, \\ \xi(a) = 0, \qquad \xi'(a) = k, \end{gathered} \tag{3.27}$$

and denote by u the solution of the scalar Cauchy problem

$$\begin{gathered} u''(t) + b_i\left[\left(\frac{\sin t}{b_i - 1} + u(t)\right)^{+} - \frac{\sin t}{b_i - 1}\right] = 0, \\ u(a) = 0, \qquad u'(a) = k. \end{gathered} \tag{3.28}$$

If there exists $\gamma \in (a, a + \pi/\sqrt{b_i})$ such that $\gamma < \pi$ and $\xi(\gamma) = -\sin\gamma/(b_i - 1)$ and $\xi'(\gamma) < -\cos\gamma/(b_i - 1)$, then

$$\xi(t) = u(t) \quad \forall t \in [a,\gamma], \qquad \xi(t) > u(t) \quad \forall t \in \left(\gamma, a + \frac{\pi}{\sqrt{b_i}}\right]. \tag{3.29}$$

Proof. Fix $i \in \{1,2\}$. Consider the solutions ξ and u to problems (3.27) and (3.28), respectively. We define $b := a + \pi/\sqrt{b_i}$. Clearly, $\xi(b) = 0$ and ξ is negative in (a,b). Moreover, Lemma 3.3 guarantees that $u(b) < 0 = \xi(b)$, since $b - a = \pi/\sqrt{b_i}$.

By the initial conditions, we have $\xi(a) + \sin a/(b_i - 1) = u(a) + \sin a/(b_i - 1) = \sin a/(b_i - 1) > 0$, since $a \in (0,\pi)$. We denote by c the first zero (greater than a) of $\psi(t) := \xi(t) + \sin t/(b_i - 1)$. It is easy to show that c represents also the first zero of $\widetilde{\psi}(t) := u(t) + \sin t/(b_i - 1)$. Indeed, $\xi \equiv u$ in $[a,c]$, being ξ and u solutions of the linear Cauchy problem (3.27) in the interval $[a,c]$. Moreover, by the definition of γ, $c \leq \gamma$.

We now claim that $c = \gamma$.

In order to prove this assertion, suppose by contradiction that $c < \gamma$. Observe that ψ solves the problem

$$\psi''(t) = -b_i\psi(t) + \sin t, \qquad \psi(c) = 0 = \psi(\gamma). \tag{3.30}$$

Furthermore, consider $\phi(t) := \sin(\sqrt{b_i}(t - c))$. By definition, $\phi(t) > 0$ for every $t \in (c, c + \pi/\sqrt{b_i})$ and $\phi(c) = 0$. In particular, $\psi''(t)\phi(t) - \psi(t)\phi''(t) = \sin t\phi(t) > 0$ for every $t \in (c,\gamma)$. Hence, we obtain

$$0 < \int_c^\gamma \left(\psi''(s)\phi(s) - \phi(s)\psi''(s)\right)ds = \int_c^\gamma \left(\psi'(s)\phi(s) - \psi(s)\phi'(s)\right)' ds = \psi'(\gamma)\phi(\gamma). \tag{3.31}$$

Since, by assumption, $\psi'(\gamma) < 0$, we deduce that $\phi(\gamma) < 0$, a contradiction. We have proved that $c = \gamma$.

In particular, $\psi'(c) = \widetilde{\psi}'(c) < 0$ and, consequently, there exists $d \in (c,b)$ such that ψ and $\widetilde{\psi}$ are both negative in (c,d). This implies that

$$\xi'(t) - u'(t) = -b_i\int_c^t \left(\xi(s) + \frac{\sin s}{b_i - 1}\right)ds = -b_i\int_c^t \psi(s)\,ds > 0 \quad \forall t \in (c,d]. \tag{3.32}$$

From the equality $u(c) = \xi(c)$, we immediately obtain that $\xi(t) > u(t)$ for every $t \in (c,d]$.

It remains to extend the inequality $\xi > u$ to the whole interval $(c,b]$.

We suppose by contradiction that there exists $\beta \in (d,b]$ such that

$$u(\beta) = \xi(\beta), \qquad \xi(t) > u(t) \quad \forall t \in (c,\beta). \tag{3.33}$$

Note that $u-\xi$ solves the equation $(u-\xi)''(t) = -b_i(u-\xi)(t) - b_i(\sin t/(b_i-1)+u(t))^-$. With arguments similar to the ones used in the proof of Lemma 3.3, one can prove that

$$\beta - c > \frac{\pi}{\sqrt{b_i}}, \tag{3.34}$$

a contradiction with the fact that $\beta - c \le b - c = a + \pi/\sqrt{b_i} - c < \pi/\sqrt{b_i}$. □

It is well known that the solutions to problem (3.28) are characterized by their nodal properties. In particular, the following lemma is satisfied.

LEMMA 3.8. *Fix $i \in \{1,2\}$ and assume that b_i satisfies the condition $n^2 < b_i < (n+1)^2$ for some $n \in \mathbb{N}$. Consider $k \in \mathbb{R} \setminus \{0\}$ such that $|k| < 1/(b_i-1)$ and denote by u the solution to problem (3.28) where $a = 0$. Then,*

$$u(t) = \frac{k}{\sqrt{b_i}} \sin\left(\sqrt{b_i}t\right) > -\frac{\sin t}{b_i - 1} \quad \forall t \in \left(0, \max\left\{\pi - \frac{\pi}{2\sqrt{b_i}}, \frac{n\pi}{\sqrt{b_i}}\right\}\right]. \tag{3.35}$$

Moreover,

$$u \text{ has exactly } n \text{ (simple) zeros in } (0,\pi], \qquad |u(\pi)| \ge \frac{|k|}{\sqrt{b_i}} \left|\sin\left(\sqrt{b_i}\pi\right)\right|. \tag{3.36}$$

Notice that the n zeros of u in $(0,\pi]$ are the points $t_h = h\pi/\sqrt{b_i}$, where $1 \le h \le n$.

Proof. Consider $k \in \mathbb{R}\setminus\{0\}$ such that $|k| < 1/(b_i-1)$ and define $B_n := \max\{\pi - \pi/(2\sqrt{b_i}), n\pi/\sqrt{b_i}\}$. Fixed $a = 0$ in (3.28), we denote by u the solution of the Cauchy problem (3.28) defined on $[0,\pi]$.

As a first step, we claim that

$$\frac{\sin t}{b_i - 1} > \frac{|k|}{\sqrt{b_i}} \left|\sin\left(\sqrt{b_i}t\right)\right| \ge -\frac{k}{\sqrt{b_i}} \sin\left(\sqrt{b_i}t\right) \quad \forall t \in (0, B_n]. \tag{3.37}$$

We first show that inequality (3.37) is satisfied in the interval $(0, \pi/(2\sqrt{b_i})]$, that is,

$$f(t) := -\frac{|k|}{\sqrt{b_i}} \sin\left(\sqrt{b_i}t\right) + \frac{\sin t}{b_i - 1} > 0 \quad \forall t \in \left(0, \frac{\pi}{2\sqrt{b_i}}\right]. \tag{3.38}$$

By deriving f, we obtain

$$f'(t) = -|k| \cos\left(\sqrt{b_i}t\right) + \frac{\cos t}{b_i - 1} > \frac{1}{b_i - 1}\left(-\cos\left(\sqrt{b_i}t\right) + \cos t\right) \quad \forall t \in \left(0, \frac{\pi}{2\sqrt{b_i}}\right). \tag{3.39}$$

Since $(d/ds)\cos st = -t\sin st < 0$ when $t \in (0, \pi/(2\sqrt{b_i}))$ and $s \in [1, \sqrt{b_i}]$, we can easily conclude that $f'(t) > 0$ for every $t \in (0, \pi/(2\sqrt{b_i}))$. From the equality $f(0) = 0$, we achieve (3.38).

Moreover taking into account inequality (3.38), we get

$$\frac{\sin t}{b_i-1} \geq \frac{1}{b_i-1}\sin\left(\frac{\pi}{2\sqrt{b_i}}\right) > \frac{|k|}{\sqrt{b_i}}\sin\frac{\pi}{2} \geq \frac{|k|}{\sqrt{b_i}}\left|\sin\left(\sqrt{b_i}t\right)\right| \quad \forall t \in \left[\frac{\pi}{2\sqrt{b_i}}, \pi - \frac{\pi}{2\sqrt{b_i}}\right], \tag{3.40}$$

which combined with (3.38) extends the validity of (3.37) to the whole interval $(0, \pi - \pi/(2\sqrt{b_i})]$.

According to (3.38), we also obtain that

$$\frac{\sin t}{b_i-1} > \frac{|k|}{\sqrt{b_i}}\left|\sin\left(\sqrt{b_i}(\pi - t)\right)\right| = \frac{|k|}{\sqrt{b_i}}\sin\left(\sqrt{b_i}(\pi - t)\right) \quad \forall t \in \left[\pi - \frac{\pi}{2\sqrt{b_i}}, \pi\right). \tag{3.41}$$

To prove the claim, it remains to show that

$$\sin\left(\sqrt{b_i}(\pi - t)\right) \geq \left|\sin\left(\sqrt{b_i}t\right)\right| \quad \forall t \in \left[\pi - \frac{\pi}{2\sqrt{b_i}}, \frac{n\pi}{\sqrt{b_i}}\right] \tag{3.42}$$

when $\pi - \pi/(2\sqrt{b_i}) < n\pi/\sqrt{b_i}$ (or, equivalently, when $\sqrt{b_i} < n + 1/2$).

Since by assumption $\sqrt{b_i} > n$, it easily follows that

$$0 \leq -\sqrt{b_i}t + n\pi < \sqrt{b_i}(\pi - t) \leq \frac{\pi}{2} \quad \forall t \in \left[\pi - \frac{\pi}{2\sqrt{b_i}}, \frac{n\pi}{\sqrt{b_i}}\right]. \tag{3.43}$$

Since the function $t \mapsto \sin t$ is nonnegative and strictly increasing in $[0, \pi/2]$, we can finally deduce that inequality (3.42) holds. Indeed,

$$\sin\left(\sqrt{b_i}(\pi - t)\right) > \sin\left(-\sqrt{b_i}t + n\pi\right) = \left|\sin\left(\sqrt{b_i}t\right)\right| \quad \forall t \in \left[\pi - \frac{\pi}{2\sqrt{b_i}}, \frac{n\pi}{\sqrt{b_i}}\right]. \tag{3.44}$$

This completes the proof of the claim.

We have so proved that inequality (3.37) holds. Since u is a solution of (3.28), it is easy to deduce that $u(t) = (k/\sqrt{b_i})\sin(\sqrt{b_i}t)$ for every $t \in [0, B_n]$.

Moreover since, by assumptions, $B_n < \pi < (n+1)\pi/\sqrt{b_i}$, we can conclude that u has exactly n zeros in $(0, B_n]$. It remains to prove that u has no other zeros in $(B_n, \pi]$.

If $\sin t/(b_i - 1) + (k/\sqrt{b_i})\sin(\sqrt{b_i}t) \geq 0$ for every $t \in (B_n, \pi]$, then $u(t) = (k/\sqrt{b_i})\sin(\sqrt{b_i}t)$ for every $t \in [0, \pi]$ and it has no zeros in $(B_n, \pi]$. Furthermore, $u(\pi) = (k/\sqrt{b_i})\sin(\sqrt{b_i}\pi)$.

Otherwise, assume the existence of $\gamma \in (B_n, \pi) \subset (n\pi/\sqrt{b_i}, (n+1)\pi/\sqrt{b_i})$ such that $\sin\gamma/(b_i - 1) + (k/\sqrt{b_i})\sin(\sqrt{b_i}\gamma) = 0$ and $\cos\gamma/(b_i - 1) + k\cos(\sqrt{b_i}\gamma) < 0$. First observe that

$$u\left(\frac{n\pi}{\sqrt{b_i}}\right) = \frac{k}{\sqrt{b_i}}\sin(n\pi), = 0, \qquad u'\left(\frac{n\pi}{\sqrt{b_i}}\right) = k\cos(n\pi) < 0. \tag{3.45}$$

The assumptions of Lemma 3.7 are satisfied. Hence, by applying this lemma we can conclude that

$$0 > \frac{k}{\sqrt{b_i}} \sin\left(\sqrt{b_i}t\right) = u(t) \quad \forall t \in \left(\frac{n\pi}{\sqrt{b_i}}, \gamma\right], \qquad 0 > \frac{k}{\sqrt{b_i}} \sin\left(\sqrt{b_i}t\right) > u(t) \quad \forall t \in (\gamma, \pi]. \tag{3.46}$$

In particular, also in this case u has no zeros in $(B_n, \pi]$ and

$$|u(\pi)| = -u(\pi) > -\frac{k}{\sqrt{b_i}} \sin\left(\sqrt{b_i}\pi\right) = \frac{|k|}{\sqrt{b_i}} \left|\sin\left(\sqrt{b_i}\pi\right)\right|. \tag{3.47}$$

The thesis of the lemma is achieved. □

We now establish a relation between the behaviour of the solutions of (3.28) and the behaviour of the ith component of the solutions to system (2.11).

Lemma 3.9. *Suppose that $b_l > 1$ for every $l \in \{1,2\}$ and fix $i \in \{1,2\}$. Consider $k \in \mathbb{R}$ and $\sigma > 0$. Denote by u the solution of (3.28) where $a = 0$. Then, for every $\mu > 0$ there exists $\underline{\varepsilon}_i > 0$ such that for every $\varepsilon \in (0, \underline{\varepsilon}_i]$ and for every solution $v = (v_1, v_2)$ of (2.11) with $v_i'(0) = k$ satisfying (3.1) and (3.12), it follows that*

$$|(v_i(t) - u(t), v_i'(t) - u'(t))| \leq \mu \quad \forall t \in [0, \pi]. \tag{3.48}$$

Proof. Fix $\sigma > 0$ and $i, j \in \{1,2\}$ with $i \neq j$. By applying Proposition 3.4, we immediately infer the existence of two positive constants $M = M(b_1, b_2, \sigma)$ and $\varepsilon^* = \varepsilon^*(b_1, b_2, \sigma)$ such that for every $\varepsilon \in (0, \varepsilon^*]$ and for every solution $v = (v_1, v_2)$ to problem (2.11) satisfying (3.1) and (3.12), it follows that $|v_j(t)| \leq M$ for every $t \in [0, \pi]$. We define $\underline{\varepsilon}_i = \underline{\varepsilon}_i(b_1, b_2, \sigma, \mu)$ by setting

$$\underline{\varepsilon}_i := \min\left\{\varepsilon^*, \frac{\mu(b_j - 1)}{2 + \pi M(b_j - 1)} e^{-\sqrt{2}b_i\pi}\right\}. \tag{3.49}$$

For every $\varepsilon \in (0, \underline{\varepsilon}_i]$ we consider a solution $v = (v_1, v_2)$ to problem (2.11) with $v_i'(0) = k$ satisfying (3.1) and (3.12). Denoting by u the solution to (3.28) with $a = 0$, we obtain that for every $t \in [0, \pi]$,

$$v_i''(t) - u''(t) = b_i\left[-\left(\frac{\sin t}{b_i - 1} + v_i(t)\right)^+ + \left(\frac{\sin t}{b_i - 1} + u(t)\right)^+\right] - \varepsilon\left(\frac{\sin t}{b_j - 1} + v_j(t)\right)^+. \tag{3.50}$$

Hence, we conclude that for every $t \in [0, \pi]$,

$$|v_i''(t) - u''(t)| \leq b_i |v_i(t) - u(t)| + \varepsilon\left(\frac{\sin t}{b_j - 1} + M\right). \tag{3.51}$$

From the classic elastic lemma (cf. [21, Lemma 2.1] and [9, Lemma 2.4.1]), it follows that

$$|(v_i(t) - u(t), v_i'(t) - u'(t))| \leq \varepsilon\left\|\frac{\sin t}{b_j - 1} + M\right\|_1 e^{\sqrt{2}b_i\pi} \quad \forall t \in [0, \pi]. \tag{3.52}$$

Taking into account the definition of $\underline{\varepsilon}_i$ given in (3.49), we finally infer

$$\left|\left(v_i(t)-u(t),v_i'(t)-u'(t)\right)\right| \leq \varepsilon\left(\frac{2}{b_j-1}+M\pi\right)e^{\sqrt{2b_i}\pi} \leq \mu \quad \forall t\in[0,\pi], \tag{3.53}$$

which completes the proof. □

By applying the previous lemmas, we are able to exhibit a lower bound on the number of zeros of the ith component of the solutions of suitable Cauchy problems associated to system (2.11).

Proposition 3.10. *Fix $i,j\in\{1,2\}$ with $i\neq j$ and $\sigma>0$. Assume that $b_j>1$ and b_i satisfies the condition $n^2<b_i<(n+1)^2$ for some $n\in\mathbb{N}$. Then, for every $\nu\in(0,1/(b_i-1))$ there exists $\overline{\varepsilon}_i>0$ such that for every $\varepsilon\in(0,\overline{\varepsilon}_i]$ and for every solution $v=(v_1,v_2)$ to system (2.11) satisfying (3.1) and (3.12),*

$$\frac{1}{b_i-1}-\nu<\left|v_i'(0)\right|<\frac{1}{b_i-1}\Longrightarrow v_i \text{ has at least } n \text{ zeros in } (0,\pi). \tag{3.54}$$

We point out that no information on the simplicity of the zeros is contained in (3.54).

Proof. Fix $i,j\in\{1,2\}$ with $i\neq j$. We take $\nu\in(0,1/(b_i-1))$ and $|k|\in(1/(b_i-1)-\nu, 1/(b_i-1))$. To simplify the notation, we set $s_h:=((2h+1)/2)(\pi/\sqrt{b_i})$ for every $h\in\{0,\dots,n-1\}$ and $s_n:=\pi$. By definition, $s_{h-1}<s_h$ for every $h\in\{1,\dots,n\}$. Lemma 3.8 ensures that the solution u to problem (3.28) where we choose $a=0$ satisfies $u(s_{h-1})u(s_h)<0$ for every $h\in\{1,\dots,n\}$. Furthermore,

$$u(s_h)=\frac{k}{\sqrt{b_i}}\sin\left(\frac{2h+1}{2}\pi\right)=\frac{k}{\sqrt{b_i}}(-1)^h \quad \forall h\in\{0,\dots,n-1\}, \tag{3.55}$$

whence it follows that

$$\left|u(s_h)\right| \geq \frac{|k|}{\sqrt{b_i}}\left|\sin\left(\sqrt{b_i}\pi\right)\right|>0 \quad \forall h\in\{0,\dots,n\}. \tag{3.56}$$

We now choose $\mu=\mu(\nu,b_i)\in(0,(|k|/\sqrt{b_i})|\sin(\sqrt{b_i}\pi)|)$ and we introduce $\overline{\varepsilon}_i=\overline{\varepsilon}_i(b_1, b_2,\sigma,\nu)$ by setting $\overline{\varepsilon_i}:=\underline{\varepsilon}_i$, with $\underline{\varepsilon}_i$ defined by (3.49). For every $\varepsilon\in(0,\overline{\varepsilon}_i]$ we consider a solution $v=(v_1,v_2)$ to system (2.11) with $v_i'(0)=k$ satisfying (3.1) and (3.12). From Lemma 3.9, we deduce

$$\left|v_i(t)-u(t)\right|\leq\mu \quad \forall t\in[0,\pi]. \tag{3.57}$$

Taking into account (3.56) and (3.57), we obtain that for every $h\in\{1,\dots,n\}$,

$$\begin{aligned} u(s_h)v_i(s_h) &= u(s_h)\left(v_i(s_h)-u(s_h)\right)+u(s_h)^2 \\ &\geq \left|u(s_h)\right|\left(-\left|v_i(s_h)-u(s_h)\right|+\left|u(s_h)\right|\right) \\ &\geq \left|u(s_h)\right|\left(-\mu+\frac{|k|}{\sqrt{b_i}}\left|\sin\left(\sqrt{b_i}\pi\right)\right|\right)>0. \end{aligned} \tag{3.58}$$

In particular, $v_i(s_{h-1})v_i(s_h) < 0$ for every $h \in \{1,\dots,n\}$. Being the sequence of s_h increasing, we can finally conclude that v_i has at least n zeros in (s_0,π), whence the thesis follows. □

In order to apply Theorem 2.2 and to improve the result of Proposition 3.10, we now prove that all the zeros of each component of the solution to problem (2.11) and (3.1) are simple provided that every initial slope satisfies (3.12) and the further condition

$$|v_i'(0)| > \frac{1}{b_i - 1} - \nu > 0 \tag{3.59}$$

for some $\nu \in (0, 1/(b_i - 1))$, and provided that ε is sufficiently small. The presence of the nonlinear terms in (2.11) containing the positive constant ε makes this assertion not true in general. Indeed, in general, a nontrivial component v_i of a solution v to system (2.11) is not strictly convex at a negative bump. The lack of convexity could lead to the existence of a nonsimple zero S of v_i, when v_i is negative in a left neighbourhood of S.

The choice to restrict ourselves to the case where inequalities (3.12) and (3.59) hold is due to the fact that these inequalities guarantee the validity of conditions (2.19) in Lemma 2.5, whose application leads to the following proposition.

Proposition 3.11. *Assume that $b_i > 1$ for every $i \in \{1,2\}$. Fix $\nu \in (0, \min\{1/(b_1 - 1), 1/(b_2 - 1)\})$ and $\sigma > 0$. Then, there exist three positive constants $\delta_1 = \delta_1(b_1,\nu)$, $\delta_2 = \delta_2(b_2,\nu)$, and $\underline{\varepsilon}^* > 0$ such that for every $\varepsilon \in (0,\underline{\varepsilon}^*]$ and for every solution $v = (v_1,v_2)$ to problem (2.11) satisfying (3.1), (3.12), and (3.59) for every $i \in \{1,2\}$ it follows that*

$$v_i(t)^2 + v_i'(t)^2 > \delta_i \quad \forall t \in [0,\pi], \forall i \in \{1,2\}. \tag{3.60}$$

Proof. To prove the proposition we need to verify that the assumptions of Lemma 2.5 are satisfied. We fix $\nu \in (0, \min\{1/(b_1 - 1), 1/(b_2 - 1)\})$ and $\sigma > 0$. Every solution $v = (v_1,v_2)$ to problem (2.11) verifying (3.1) and (3.59) for each $i = 1,2$ satisfies

$$\max_{t\in[0,\pi]} |(v_i(t), v_i'(t))| \geq |v_i'(0)| > \frac{1}{b_i - 1} - \nu \quad \forall i \in \{1,2\}. \tag{3.61}$$

Moreover, Proposition 3.4 provides the existence of two positive constants $M = M(b_1,b_2,\sigma)$ and $\varepsilon^* = \varepsilon^*(b_1,b_2,\sigma)$ such that for every $\varepsilon \in (0,\varepsilon^*]$ and for every solution $v = (v_1,v_2)$ to problem (2.11) satisfying (3.12) and (3.1) it follows that

$$|v_j(t)| \leq |(v(t), v'(t))| \leq M \quad \forall t \in [0,\pi], \forall j \in \{1,2\}. \tag{3.62}$$

According to (3.61) and to (3.62), we observe that conditions (2.19) of Lemma 2.5 hold. Hence, by applying Lemma 2.5 the thesis follows. □

Remark 3.12. The statement of Proposition 3.10 can be improved. Indeed, suppose that all the assumptions of the proposition hold. Then, it is possible to prove that for every $\nu \in (0, \min\{1/(b_1 - 1), 1/(b_2 - 1)\})$ there exists $\overline{\varepsilon}_i^* > 0$ such that for every $\varepsilon \in (0,\overline{\varepsilon}_i^*]$ and for every solution $v = (v_1, v_2)$ to system (2.11) satisfying (3.1) and (3.12),

$$\frac{1}{b_i - 1} - \nu < |v_i'(0)| < \frac{1}{b_i - 1} \Longrightarrow v_i \text{ has exactly } n \text{ zeros in } (0,\pi], \qquad v_i(\pi) \neq 0. \quad (3.63)$$

To prove this result, we first denote by δ a positive constant satisfying the following inequality:

$$\operatorname{Rot}(\pi;(\xi;\xi')) + \delta < n+1 \quad (3.64)$$

for every solution ξ of problem (3.27) with any $k \in \mathbb{R} \setminus \{0\}$. To simplify the notation, in what follows we set $\operatorname{Rot}(\pi;(\xi,\xi')) := \operatorname{Rot}_{(3.27)}(\pi;k/|k|)$.

We now claim that there exists a positive constant $\bar{\varepsilon}_i = \bar{\varepsilon}_i(\delta) > 0$ such that for every $\varepsilon \in (0,\bar{\varepsilon}_i]$ and for every solution $v = (v_1, v_2)$ to system (2.11) satisfying (3.1), (3.12), and (3.59),

$$\operatorname{Rot}(\pi;(v_i,v_i')) < \operatorname{Rot}_{(3.27)}\left(\pi;\frac{v_i'(0)}{|v_i'(0)|}\right) + \delta. \quad (3.65)$$

Note that $\operatorname{Rot}(\pi;(v_i,v_i'))$ is well-defined by Proposition 3.11 if we choose $\bar{\varepsilon}_i \in (0,\min\{\varepsilon^*, \underline{\varepsilon}^*\})$, where ε^* and $\underline{\varepsilon}^*$ are given in Propositions 3.4 and 3.11, respectively. We only sketch the proof of the claim, since its procedure is standard (we refer to the proof of [10, Lemma 3.4] for more details). Arguing by contradiction, we suppose that for every $m \in \mathbb{N}$ there are $\varepsilon_m \in (0,[\delta_i(b_j-1)]/\{mM[1+M(b_j-1)]\}]$ and a solution v^m of (2.11) satisfying (3.1), (3.12), and (3.59) such that

$$\operatorname{Rot}(\pi;(v_i^m,(v_i^m)')) \geq \operatorname{Rot}_{(3.27)}\left(\pi;\frac{(v_i^m)'(0)}{|(v_i^m)'(0)|}\right) + \delta. \quad (3.66)$$

For each $m \in \mathbb{N}$, we take $\alpha_m \in \{0,\pi\}$ satisfying $\cos\alpha_m = (v_i^m)'(0)/|(v_i^m)'(0)|$. It is not restrictive to assume that $\lim_{m\to+\infty}\alpha_m = \alpha$ for some $\alpha \in \{0,\pi\}$. Hence, from the continuity of $\operatorname{Rot}_{(3.27)}(\pi;\cdot)$, we can conclude that

$$\liminf_{m\to+\infty} \operatorname{Rot}(\pi;(v_i^m,(v_i^m)')) \geq \operatorname{Rot}_{(3.27)}(\pi;\cos\alpha) + \delta. \quad (3.67)$$

On the other hand, we consider the polar coordinates $(\vartheta_m(t),\rho_m(t))$ to represent $(v_i^m, (v_i^m)')$. The inequality

$$b_i\left[\left(\frac{\sin t}{b_i-1} + v_i^m(t)\right)^+ - \frac{\sin t}{b_i-1}\right]v_i^m \leq b_i v_i^m(t)^2 \quad \forall t \in [0,\pi] \quad (3.68)$$

and the estimates provided by Propositions 3.4 and 3.11 ensure the validity of

$$\vartheta_m'(t) \leq \frac{(v_i^m)'(t)^2 + b_i v_i^m(t)^2}{(v_i^m)'(t)^2 + v_i^m(t)^2} + \varepsilon_m \frac{M}{\delta_i}\left(\frac{1}{b_j-1} + M\right) \leq \cos^2\vartheta_m(t) + b_i\sin^2\vartheta_m(t) + \frac{1}{m}. \quad (3.69)$$

A result on differential inequalities [23] and a well-known theorem on continuous dependence of the solutions lead to

$$\limsup_{m\to+\infty} \operatorname{Rot}\left(\pi;\left(v_i^m,\left(v_i^m\right)'\right)\right) = \limsup_{m\to+\infty} \frac{\vartheta_m(\pi)-\alpha_m}{\pi} \leq \operatorname{Rot}_{(3.27)}(\pi;\cos\alpha), \tag{3.70}$$

which contradicts (3.67). Thus, the claim is proved.

By recalling inequalities (3.64) and (3.65), we can finally deduce the existence of a positive constant $\bar{\varepsilon}_i$ such that for every $\varepsilon \in (0,\bar{\varepsilon}_i]$ and for every solution $v = (v_1, v_2)$ to system (2.11) satisfying (3.1), (3.12), and (3.59),

$$\operatorname{Rot}\left(\pi;\left(v_i,v_i'\right)\right) < n+1. \tag{3.71}$$

Hence, the assertion (3.63) comes from an application of Proposition 3.10.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We take $(n_1,n_2) \in \mathbb{N}^2 \setminus \{(1,1)\}$ with $n_1 \leq h$, $n_2 \leq k$ and consider $(s_1,s_2) \in \tau$ with $s_i = -1$ whenever $n_i = 1$. Fix $\sigma > 0$ and $\nu \in (0,\min\{1/(b_1-1),1/(b_2-1)\})$. Furthermore, we define $\varepsilon_0 := \min\{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2,\underline{\varepsilon}^*\}$ where $\hat{\varepsilon}_i$, $\bar{\varepsilon}_i$, and $\underline{\varepsilon}^*$ follow from the application of Propositions 3.5, 3.10, and 3.11, respectively. For every $\varepsilon \in (0,\varepsilon_0]$, we consider $v = (v_1,v_2)$ such that

$$\begin{gathered} v \text{ solves (2.11) and (3.1)}, \\ \operatorname{sgn}\left(v_i'(0)\right) = s_i, \quad \frac{1}{b_i-1} - \frac{\nu}{2} \leq \left|v_i'(0)\right| \leq 2\pi^2\frac{b_i}{b_i-1} + \sigma \quad \forall i \in \{1,2\}. \end{gathered} \tag{3.72}$$

We first note that for each $i \in \{1,2\}$, Proposition 3.10 implies that $\operatorname{Rot}(\pi;(v_i,v_i')) > n_i$ for every function v satisfying (3.72) and the additional condition $|v_i'(0)| = 1/(b_i-1) - \nu/2$.

Moreover, if $n_i \neq 1$, then Proposition 3.5 guarantees that $\operatorname{Rot}(\pi;(v_i,v_i')) < 2 \leq n_i$ for every function v satisfying (3.72) and the additional condition $|v_i'(0)| = 2\pi^2(b_i/(b_i-1)) + \sigma$.

On the other hand, if $n_i = 1$ and $s_i = -1$, then Proposition 3.1 guarantees that $\operatorname{Rot}(\pi;(v_i,v_i')) < 1 = n_i$ for every function v satisfying (3.72) and the additional condition $|v_i'(0)| = 2\pi^2(b_i/(b_i-1)) + \sigma$.

In particular, all the assumptions of Theorem 2.2 are satisfied. Thus, there is at least one solution v of problem (1.5) with $\operatorname{sgn}(v_i'(0)) = s_i$ such that v_i has exactly $n_i - 1$ zeros in $(0,\pi)$ for each $i \in \{1,2\}$. This completes the proof. □

4. Some remaining open questions

As we suggested in the introduction, this paper represents a beginning in this area. There are many open questions.

(1) In the uncoupled two-by-two system, one has a total of $4hk$ solution pairs. For technical reasons, our proof only gives a smaller number. Is this result exact or can one find the missing solutions?

(2) Can one obtain the corresponding results for $n \times n$ systems?

(3) Can one obtain corresponding results if the second-order differential operator is replaced with a fourth-order differential operator with corresponding boundary conditions?

(4) Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of the matrix?

(5) Can one find similar results for the corresponding partial differential equation setting, or at least the case of the radially symmetric Laplacian on the ball?

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