BLOWUP FOR DEGENERATE AND SINGULAR PARABOLIC SYSTEM WITH NONLOCAL SOURCE

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We deal with the blowup properties of the solution to the degenerate and singular parabolic system with nonlocal source and homogeneous Dirichlet boundary conditions. The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution that exists globally or blows up in finite time are obtained. Furthermore, under certain conditions it is proved that the blowup set of the solution is the whole domain.

1. Introduction

In this paper, we consider the following degenerate and singular nonlinear reaction-diffusion equations with nonlocal source:

$$\begin{gathered}
x^{q_1}u_t - \left(x^{r_1}u_x\right)_x = \int_0^a v^{p_1}\,dx, \quad (x,t) \in (0,a)\times(0,T),\\
x^{q_2}v_t - \left(x^{r_2}v_x\right)_x = \int_0^a u^{p_2}\,dx, \quad (x,t) \in (0,a)\times(0,T),\\
u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, \quad t \in (0,T),\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in [0,a],
\end{gathered} \tag{1.1}$$

where $u_0(x), v_0(x) \in C^{2+\alpha}(\overline{D})$ for some $\alpha \in (0,1)$ are nonnegative nontrivial functions. $u_0(0) = u_0(a) = v_0(0) = v_0(a) = 0$, $u_0(x) \geq 0$, $v_0(x) \geq 0$, u_0, v_0 satisfy the compatibility condition, $T > 0$, $a > 0$, $r_1, r_2 \in [0,1)$, $|q_1| + r_1 \neq 0$, $|q_2| + r_2 \neq 0$, and $p_1 > 1$, $p_2 > 1$.

Let $D = (0,a)$ and $\Omega_t = D \times (0,t]$, $\overline{D}$ and $\overline{\Omega}_t$ are their closures, respectively. Since $|q_1| + r_1 \neq 0$, $|q_2| + r_2 \neq 0$, the coefficients of u_t, u_x, u_{xx} and v_t, v_x, v_{xx} may tend to 0 or ∞ as x tends to 0, we can regard the equations as degenerate and singular.

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Floater [9] and Chan and Liu [4] investigated the blowup properties of the following degenerate parabolic problem:

$$\begin{aligned} x^q u_t - u_{xx} &= u^p, \quad (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) &= 0, \quad t \in (0,T), \\ u(x,0) &= u_0(x), \quad x \in [0,a], \end{aligned} \tag{1.2}$$

where $q > 0$ and $p > 1$. Under certain conditions on the initial datum $u_0(x)$, Floater [9] proved that the solution $u(x,t)$ of (1.2) blows up at the boundary $x = 0$ for the case $1 < p \leq q+1$. This contrasts with one of the results in [10], which showed that for the case $q = 0$, the blowup set of solution $u(x,t)$ of (1.2) is a proper compact subset of D.

The motivation for studying problem (1.2) comes from Ockendon's model (see [14]) for the flow in a channel of a fluid whose viscosity depends on temperature

$$xu_t = u_{xx} + e^u, \tag{1.3}$$

where u represents the temperature of the fluid. In [9] Floater approximated e^u by u^p and considered (1.2). Budd et al. [2] generalized the results in [9] to the following degenerate quasilinear parabolic equation:

$$x^q u_t = (u^m)_{xx} + u^p, \tag{1.4}$$

with homogeneous Dirichlet conditions in the critical exponent $q = (p-1)/m$, where $q > 0$, $m \geq 1$, and $p > 1$. They pointed out that the general classification of blowup solution for the degenerate equation (1.4) stays the same for the quasilinear equation (see [2, 17])

$$u_t = (u^m)_{xx} + u^p. \tag{1.5}$$

For the case $p > q+1$, in [4] Chan and Liu continued to study problem (1.2). Under certain conditions, they proved that $x = 0$ is not a blowup point and the blowup set is a proper compact subset of D.

In [7], Chen and Xie discussed the following degenerate and singular semilinear parabolic equation:

$$\begin{aligned} u_t - (x^\alpha u_x)_x &= \int_0^a f(u(x,t))\,dx, \quad (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) &= 0, \quad t \in (0,T), \\ u(x,0) &= u_0(x), \quad x \in [0,a], \end{aligned} \tag{1.6}$$

they established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blowup of a positive solution.

In [6], Chen et al. consider the following degenerate nonlinear reaction-diffusion equation with nonlocal source:

$$
\begin{aligned}
x^q u_t - (x^\gamma u_x)_x &= \int_0^a u^p \, dx, \quad (x,t) \in (0,a) \times (0,T), \\
u(0,t) = u(a,t) &= 0, \quad t \in (0,T), \\
u(x,0) &= u_0(x), \quad x \in [0,a],
\end{aligned} \tag{1.7}
$$

they established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blowup of a positive solution. Furthermore, under certain conditions, it is proved that the blowup set of the solution is the whole domain.

In this paper, we generalize the results of [6] to parabolic system and investigate the effect of the singularity, degeneracy, and nonlocal reaction on the behavior of the solution of (1.1). The difficulties are the establishment of the corresponding comparison principle and the construction of a supersolution of (1.1). It is different from [4, 9] that under certain conditions the blowup set of the solution of (1.1) is the whole domain. But this is consistent with the conclusions in [1, 18, 19].

This paper is organized as follows: in the next section, we show the existence of a unique classical solution. In Section 3, we give some criteria for the solution $(u(x,t), v(x,t))$ to exist globally or blow up in finite time and in the last section, we discuss the blowup set.

2. Local existence

In order to prove the existence of a unique positive solution to (1.1), we start with the following comparison principle.

Lemma 2.1. *Let $b_1(x,t)$ and $b_2(x,t)$ be continuous nonnegative functions defined on $[0,a] \times [0,r]$ for any $r \in (0,T)$, and let $(u(x,t), v(x,t)) \in (C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r))^2$ satisfy*

$$
\begin{aligned}
x^{q_1} u_t - (x^{r_1} u_x)_x &\geq \int_0^a b_1(x,t) v(x,t) \, dx, \quad (x,t) \in (0,a) \times (0,r], \\
x^{q_2} v_t - (x^{r_2} v_x)_x &\geq \int_0^a b_2(x,t) u(x,t) \, dx, \quad (x,t) \in (0,a) \times (0,r], \\
u(0,t) \geq 0, \quad u(a,t) \geq 0, &\quad v(0,t) \geq 0, \quad v(a,t) \geq 0, \quad t \in (0,r], \\
u(x,0) \geq 0, &\quad v(x,0) \geq 0, \quad x \in [0,a].
\end{aligned} \tag{2.1}
$$

Then, $u(x,t) \geq 0$, $v(x,t) \geq 0$ on $[0,a] \times [0,T)$.

Proof. At first, similar to the proof of Lemma 2.1 in [20], by using [15, Lemma 2.2.1], we can easily obtain the following conclusion.

If $W(x,t)$ and $Z(x,t)\in C(\overline{\Omega}_r)\cap C^{2,1}(\Omega_r)$ satisfy

$$\begin{gathered} x^{q_1}W_t-(x^{r_1}W_x)_x\geq\int_0^a b_1(x,t)Z(x,t)dx,\quad (x,t)\in(0,a)\times(0,r],\\ x^{q_2}Z_t-(x^{r_2}Z_x)_x\geq\int_0^a b_2(x,t)W(x,t)dx,\quad (x,t)\in(0,a)\times(0,r],\\ W(0,t)>0,\quad W(a,t)\geq 0,\quad Z(0,t)>0,\quad Z(a,t)\geq 0,\quad t\in(0,r],\\ W(x,0)\geq 0,\quad Z(x,0)\geq 0,\quad x\in[0,a], \end{gathered}\tag{2.2}$$

then, $W(x,t)>0$, $Z(x,t)>0$, $(x,t)\in(0,a)\times(0,r]$.

Next let $r_1'\in(r_1,1)$, $r_2'\in(r_2,1)$ be positive constants and

$$W(x,t)=u(x,t)+\eta\left(1+x^{r_1'-r_1}\right)e^{ct},\qquad Z(x,t)=v(x,t)+\eta\left(1+x^{r_2'-r_2}\right)e^{ct},\tag{2.3}$$

where $\eta>0$ is sufficiently small and c is a positive constant to be determined. Then $W(x,t)>0$, $Z(x,t)>0$ on the parabolic boundary of Ω_r, and in $(0,a)\times(0,r]$, we have

$$\begin{aligned} &x^{q_1}W_t-(x^{r_1}W_x)_x-\int_0^a b_1(x,t)Z(x,t)dx\\ &\quad\geq x^{q_1}\eta\left(1+x^{r_1'-r_1}\right)ce^{ct}+\frac{(r_1'-r_1)(1-r_1')\eta e^{ct}}{x^{2-r_1'}}-\int_0^a b_1(x,t)\eta\left(1+x^{r_2'-r_2}\right)e^{ct}dx\\ &\quad\geq\eta e^{ct}\left[cx^{q_1}+\frac{(r_1'-r_1)(1-r_1')}{x^{2-r_1'}}-a\left(1+a^{r_2'-r_2}\right)\max_{(x,t)\in[0,a]\times[0,r]}b_1(x,t)\right],\\ &x^{q_2}Z_t-(x^{r_2}Z_x)_x-\int_0^a b_2(x,t)W(x,t)dx\\ &\quad\geq\eta e^{ct}\left[cx^{q_2}+\frac{(r_2'-r_2)(1-r_2')}{x^{2-r_2'}}-a\left(1+a^{r_1'-r_1}\right)\max_{(x,t)\in[0,a]\times[0,r]}b_2(x,t)\right]. \end{aligned}\tag{2.4}$$

We will prove that the above inequalities are nonnegative in three cases.

Case 1. When

$$\begin{aligned} \max_{(x,t)\in[0,a]\times[0,r]}b_1(x,t)&\leq\frac{(r_1'-r_1)(1-r_1')}{a^{3-r_1'}\left(1+a^{r_2'-r_2}\right)},\\ \max_{(x,t)\in[0,a]\times[0,r]}b_2(x,t)&\leq\frac{(r_2'-r_2)(1-r_2')}{a^{3-r_2'}\left(1+a^{r_1'-r_1}\right)}. \end{aligned}\tag{2.5}$$

It is obvious that

$$\begin{aligned} x^{q_1}W_t-(x^{r_1}W_x)_x-\int_0^a b_1(x,t)Z(x,t)dx\geq 0,\\ x^{q_2}Z_t-(x^{r_2}Z_x)_x-\int_0^a b_2(x,t)W(x,t)dx\geq 0. \end{aligned}\tag{2.6}$$

Case 2. If

$$\max_{(x,t)\in[0,a]\times[0,r]} b_1(x,t) > \frac{(r_1'-r_1)(1-r_1')}{a^{3-r_1'}(1+a^{r_2'-r_2})},$$
$$\max_{(x,t)\in[0,a]\times[0,r]} b_2(x,t) > \frac{(r_2'-r_2)(1-r_2')}{a^{3-r_2'}(1+a^{r_1'-r_1})}. \tag{2.7}$$

Let x_0 and y_0 be the root of the algebraic equations

$$a(1+a^{r_2'-r_2})\max_{(x,t)\in[0,a]\times[0,r]} b_1(x,t) = \frac{(r_1'-r_1)(1-r_1')}{x^{2-r_1'}},$$
$$a(1+a^{r_1'-r_1})\max_{(x,t)\in[0,a]\times[0,r]} b_2(x,t) = \frac{(r_2'-r_2)(1-r_2')}{y^{2-r_2'}}, \tag{2.8}$$

and $C_1, C_2 > 0$ be sufficient large such that

$$C_1 > \begin{cases} \left(\max\limits_{(x,t)\in[0,a]\times[0,r]} b_1(x,t)\right)\dfrac{a(1+a^{r_2'-r_2})}{x_0^{q_1}} & \text{for } q_1 \ge 0, \\ \left(\max\limits_{(x,t)\in[0,a]\times[0,r]} b_1(x,t)\right)\dfrac{a(1+a^{r_2'-r_2})}{a^{q_1}} & \text{for } q_1 < 0, \end{cases}$$
$$C_2 > \begin{cases} \left(\max\limits_{(x,t)\in[0,a]\times[0,r]} b_2(x,t)\right)\dfrac{a(1+a^{r_1'-r_1})}{y_0^{q_2}} & \text{for } q_2 \ge 0, \\ \left(\max\limits_{(x,t)\in[0,a]\times[0,r]} b_2(x,t)\right)\dfrac{a(1+a^{r_1'-r_1})}{a^{q_2}} & \text{for } q_2 < 0. \end{cases} \tag{2.9}$$

Set $c = \max\{C_1, C_2\}$, then we have

$$\begin{aligned} &x^{q_1}W_t - (x^{r_1}W_x)_x - \int_0^a b_1(x,t)Z(x,t)\,dx \\ &\quad \ge \begin{cases} \eta e^{ct}\left[\dfrac{(r_1'-r_1)(1-r_1')}{x^{2-r_1'}} - a(1+a^{r_2'-r_2})\max\limits_{(x,t)\in[0,a]\times[0,r]} b_1(x,t)\right] & \text{for } x \le x_0, \\ \eta e^{ct}\left[cx^{q_1} - a(1+a^{r_2'-r_2})\max\limits_{(x,t)\in[0,a]\times[0,r]} b_1(x,t)\right] & \text{for } x > x_0, \end{cases} \\ &\quad \ge 0, \\ &x^{q_2}Z_t - (x^{r_2}Z_x)_x - \int_0^a b_2(x,t)W(x,t)\,dx \\ &\quad \ge \begin{cases} \eta e^{ct}\left[\dfrac{(r_2'-r_2)(1-r_2')}{x^{2-r_2'}} - a(1+a^{r_1'-r_1})\max\limits_{(x,t)\in[0,a]\times[0,r]} b_2(x,t)\right] & \text{for } x \le y_0, \\ \eta e^{ct}\left[cx^{q_2} - a(1+a^{r_1'-r_1})\max\limits_{(x,t)\in[0,a]\times[0,r]} b_2(x,t)\right] & \text{for } x > y_0, \end{cases} \\ &\quad \ge 0. \end{aligned} \tag{2.10}$$

Case 3. When

$$\max_{(x,t)\in[0,a]\times[0,r]} b_1(x,t) \le \frac{(r_1'-r_1)(1-r_1')}{a^{3-r_1'}(1+a^{r_2'-r_2})},$$
$$\max_{(x,t)\in[0,a]\times[0,r]} b_2(x,t) > \frac{(r_2'-r_2)(1-r_2')}{a^{3-r_2'}(1+a^{r_1'-r_1})}, \tag{2.11}$$

or

$$\max_{(x,t)\in[0,a]\times[0,r]} b_2(x,t) \le \frac{(r_2'-r_2)(1-r_2')}{a^{3-r_2'}(1+a^{r_1'-r_1})},$$
$$\max_{(x,t)\in[0,a]\times[0,r]} b_1(x,t) > \frac{(r_1'-r_1)(1-r_1')}{a^{3-r_1'}(1+a^{r_2'-r_2})}. \tag{2.12}$$

Combining Cases 1 with 2, it is easy to prove

$$x^{q_1}W_t - (x^{r_1}W_x)_x - \int_0^a b_1(x,t)Z(x,t)dx \ge 0,$$
$$x^{q_2}Z_t - (x^{r_2}Z_x)_x - \int_0^a b_2(x,t)W(x,t)dx \ge 0, \tag{2.13}$$

so we omit the proof here.

From the above three cases, we know that $W(x,t)>0$, $Z(x,t)>0$ on $[0,a]\times[0,r]$. Letting $\eta\to 0^+$, we have $u(x,t)\ge 0$, $v(x,t)\ge 0$ on $[0,a]\times[0,r]$. By the arbitrariness of $r\in(0,T)$, we complete the proof of Lemma 2.1. □

Obviously, $(\underline{u},\underline{v})=(0,0)$ is a subsolution of (1.1), we need to construct a supersolution.

Lemma 2.2. *There exists a positive constant t_0 ($t_0<T$) such that the problem (1.1) has a supersolution $(h_1(x,t),h_2(x,t))\in(C(\overline{\Omega}_{t_0})\cap C^{2,1}(\Omega_{t_0}))^2$.*

Proof. Let

$$\psi(x) = \left(\frac{x}{a}\right)^{1-r_1}\left(1-\frac{x}{a}\right)+\left(\frac{x}{a}\right)^{(1-r_1)/2}\left(1-\frac{x}{a}\right)^{1/2},$$
$$\varphi(x) = \left(\frac{x}{a}\right)^{1-r_2}\left(1-\frac{x}{a}\right)+\left(\frac{x}{a}\right)^{(1-r_2)/2}\left(1-\frac{x}{a}\right)^{1/2}, \tag{2.14}$$

and let K_0 be a positive constant such that $K_0\psi(x)\ge u_0(x)$, $K_0\varphi(x)\ge v_0(x)$.

Denote the positive constant $\int_0^1[s^{1-r_1}(1-s)+s^{(1-r_1)/2}(1-s)^{1/2}]^{p_2}ds$ by b_{20} and $\int_0^1[s^{1-r_2}(1-s)+s^{(1-r_2)/2}(1-s)^{1/2}]^{p_1}ds$ by b_{10}. Let $K_{10}\in(0,(1-r_1)/(2-r_1))$, $K_{20}\in(0,(1-r_2)/(2-r_2))$ be positive constants such that

$$K_{10}\le\left(2^{p_1+1}a^{3-r_1}b_{10}K_0^{p_1-1}\right)^{-2/(1-r_1)},$$
$$K_{20}\le\left(2^{p_2+1}a^{3-r_2}b_{20}K_0^{p_2-1}\right)^{-2/(1-r_2)}. \tag{2.15}$$

Let $(K_1(t), K_2(t))$ be the positive solution of the following initial value problem:

$$K_1'(t) = \begin{cases} \dfrac{b_{10}K_2^{p_1}(t)}{a^{q_1-1}K_{10}^{q_1}\left[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{1/2}(1-K_{10})^{(1-r_1)/2}\right]}, & q_1 \geq 0, \\ \dfrac{b_{10}K_2^{p_1}(t)}{a^{q_1-1}(1-K_{10})^{q_1}\left[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{1/2}(1-K_{10})^{(1-r_1)/2}\right]}, & q_1 < 0, \end{cases}$$

$$K_1(0) = K_0,$$

$$K_2'(t) = \begin{cases} \dfrac{b_{20}K_1^{p_2}(t)}{a^{q_2-1}K_{20}^{q_2}\left[K_{20}(1-K_{20})^{1-r_2} + K_{20}^{1/2}(1-K_{20})^{(1-r_2)/2}\right]}, & q_2 \geq 0, \\ \dfrac{b_{20}K_1^{p_2}(t)}{a^{q_2-1}(1-K_{20})^{q_2}\left[K_{20}(1-K_{20})^{1-r_2} + K_{20}^{1/2}(1-K_{20})^{(1-r_2)/2}\right]}, & q_2 < 0, \end{cases}$$

$$K_2(0) = K_0. \tag{2.16}$$

Since $K_1(t)$, $K_2(t)$ are increasing functions, we can choose $t_0 > 0$ such that $K_1(t) \leq 2K_0$, $K_2(t) \leq 2K_0$ for all $t \in [0, t_0]$. Set $h_1(x,t) = K_1(t)\psi(x)$, $h_2(x,t) = K_2(t)\varphi(x)$, then $h_1(x,t) \geq 0$, $h_2(x,t) \geq 0$ on $\overline{\Omega}_{t_0}$. We would like to show that $(h_1(x,t), h_2(x,t))$ is a supersolution of (1.1) in Ω_{t_0}. To do this, let us construct two functions J_1, J_2 by

$$\begin{aligned} J_1 &= x^{q_1}h_{1t} - (x^{r_1}h_{1x})_x - \int_0^a h_2^{p_1}\,dx, \quad (x,t) \in \Omega_{t_0}, \\ J_2 &= x^{q_2}h_{2t} - (x^{r_2}h_{2x})_x - \int_0^a h_1^{p_2}\,dx, \quad (x,t) \in \Omega_{t_0}. \end{aligned} \tag{2.17}$$

Then,

$$\begin{aligned} J_1 &= x^{q_1}h_{1t} - (x^{r_1}h_{1x})_x - \int_0^a h_2^{p_1}\,dx \\ &= x^{q_1}K_1'\psi(x) + \Bigg[\frac{2-r_1}{a^{2-r_1}} + \Bigg(\frac{(1-r_1)^2}{4}x^{(r_1-3)/2}(a-x)^{1/2} + \frac{1}{2}x^{(r_1-1)/2}(a-x)^{-1/2} \\ &\qquad + \frac{1}{4}x^{(1+r_1)/2}(a-x)^{-3/2}\Bigg) \times \frac{1}{a^{1-r_1/2}}\Bigg]K_1(t) - ab_{10}K_2^{p_1}(t) \\ &\geq x^{q_1}K_1'(t)\psi(x) + x^{(r_1-1)/2}(a-x)^{-1/2}\frac{K_1(t)}{2a^{1-r_1/2}} - ab_{10}K_2^{p_1}(t), \\ J_2 &\geq x^{q_2}K_2'(t)\varphi(x) + x^{(r_2-1)/2}(a-x)^{-1/2}\frac{K_2(t)}{2a^{1-r_2/2}} - ab_{20}K_1^{p_2}(t). \end{aligned} \tag{2.18}$$

For $(x,t) \in (0, aK_{10}) \times (0, t_0] \cup (a(1-K_{10}), a) \times (0, t_0]$, by (2.15), we have

$$\begin{aligned} J_1 &\geq x^{(r_1-1)/2}(a-x)^{-1/2}\frac{K_1(t)}{2a^{1-r_1/2}} - ab_{10}K_2^{p_1}(t) \\ &\geq \left[\frac{K_{10}^{(r_1-1)/2}}{2a^{2-r_1}}\right]K_1(t) - ab_{10}K_2^{p_1}(t_0) \\ &\geq \left[\frac{K_{10}^{(r_1-1)/2}}{2a^{2-r_1}}\right]K_0 - ab_{10}(2K_0)^{p_1} \\ &\geq 0. \end{aligned} \tag{2.19}$$

For $(x,t) \in (0, aK_{20}) \times (0, t_0] \cup (a(1-K_{20}), a) \times (0, t_0]$, by (2.15), we have

$$J_2 \geq \left[\frac{K_{20}^{(r_2-1)/2}}{2a^{2-r_2}}\right]K_0 - ab_{20}(2K_0)^{p_2} \geq 0. \tag{2.20}$$

For $(x,t) \in [aK_{10}, a(1-K_{10})] \times (0, t_0]$ by (2.16), we have

$$\begin{aligned} J_1 &\geq x^{q_1}K_1'(t)\psi(x) - ab_{10}K_2^{p_1}(t) \\ &\geq \begin{cases} a^{q_1}K_{10}^{q_1}K_1'(t)\left[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{1/2}(1-K_{10})^{(1-r_1)/2}\right] - ab_{10}K_2^{p_1}(t), & q_1 \geq 0, \\ a^{q_1}(1-K_{10})^{q_1}K_1'(t)\left[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{1/2}(1-K_{10})^{(1-r_1)/2}\right] - ab_{10}K_2^{p_1}(t), & q_1 < 0, \end{cases} \\ &\geq 0, \end{aligned} \tag{2.21}$$

For $(x,t) \in [aK_{20}, a(1-K_{20})] \times (0, t_0]$ by (2.16), we have

$$\begin{aligned} J_2 &\geq x^{q_2}K_2'(t)\varphi(x) - ab_{20}K_1^{p_2}(t) \\ &\geq \begin{cases} a^{q_2}K_{20}^{q_2}K_2'(t)\left[K_{20}(1-K_{20})^{1-r_2} + K_{20}^{1/2}(1-K_{20})^{(1-r_2)/2}\right] - ab_{20}K_1^{p_2}(t), & q_2 \geq 0, \\ a^{q_2}(1-K_{20})^{q_2}K_2'(t)\left[K_{20}(1-K_{20})^{1-r_2} + K_{20}^{1/2}(1-K_{20})^{(1-r_1)/2}\right] - ab_{20}K_1^{p_2}(t), & q_2 < 0, \end{cases} \\ &\geq 0. \end{aligned} \tag{2.22}$$

Thus, $J_1(x,t) \geq 0, J_2(x,t) \geq 0$ in Ω_{t_0}. It follows from $h_1(0,t) = h_1(a,t) = h_2(0,t) = h_2(a,t) = 0$ and $h_1(x,0) = K_0\psi(x) \geq u_0(x)$, $h_2(x,0) = K_0\varphi(x) \geq v_0(x)$ that $(h_1(x,t), h_2(x,t))$ is a supersolution of (1.1) in Ω_{t_0}. The proof of Lemma 2.2 is complete. □

To show the existence of the classical solution $(u(x,t), v(x,t))$ of (1.1), let us introduce a cutoff function $\rho(x)$. By Dunford and Schwartz [8, page 1640], there exists a nondecreasing $\rho(x) \in C^3(R)$ such that $\rho(x) = 0$ if $x \le 0$ and $\rho(x) = 1$ if $x \ge 1$. Let $0 < \delta < \min\{(1-r_1)/(2-r_1)a, (1-r_2)/(2-r_2)a\}$,

$$\rho_\delta(x) = \begin{cases} 0, & x \le \delta, \\ \rho\left(\dfrac{x}{\delta} - 1\right), & \delta < x < 2\delta, \\ 1, & x \ge 2\delta, \end{cases} \tag{2.23}$$

and $u_{0\delta}(x) = \rho_\delta(x)u_0(x)$, $v_{0\delta}(x) = \rho_\delta(x)v_0(x)$. We note that

$$\frac{\partial u_{0\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \le \delta, \\ -\dfrac{x}{\delta^2}\rho'\left(\dfrac{x}{\delta} - 1\right)u_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta, \end{cases}$$
$$\frac{\partial v_{0\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \le \delta, \\ -\dfrac{x}{\delta^2}\rho'\left(\dfrac{x}{\delta} - 1\right)v_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta. \end{cases} \tag{2.24}$$

Since ρ is nondecreasing, we have $\partial u_{0\delta}(x)/\partial\delta \le 0$, $\partial v_{0\delta}(x)/\partial\delta \le 0$. From $0 \le \rho(x) \le 1$, we have $u_0(x) \ge u_{0\delta}(x)$, $v_0(x) \ge v_{0\delta}(x)$ and $\lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$, $\lim_{\delta\to 0} v_{0\delta}(x) = v_0(x)$.

Let $D_\delta = (\delta, a)$, let $w_\delta = D_\delta \times (0, t_0]$, let $\overline{D}_\delta$ and $\overline{w}_\delta$ be their respective closures, and let $S_\delta = \{\delta, a\} \times (0, t_0]$. We consider the following regularized problem:

$$\begin{gathered} x^{q_1}u_{\delta t} - \left(x^{r_1}u_{\delta x}\right)_x = \int_\delta^a v_\delta^{p_1}\,dx, \quad (x,t) \in w_\delta, \\ x^{q_2}v_{\delta t} - \left(x^{r_2}v_{\delta x}\right)_x = \int_\delta^a u_\delta^{p_2}\,dx, \quad (x,t) \in w_\delta, \\ u_\delta(\delta,t) = u_\delta(a,t) = v_\delta(\delta,t) = v_\delta(a,t) = 0, \quad t \in (0,t_0], \\ u_\delta(x,0) = u_{0\delta}(x), \quad v_\delta(x,0) = v_{0\delta}(x), \quad x \in \overline{D}_\delta. \end{gathered} \tag{2.25}$$

By using Schauder's fixed point theorem, we have the following.

Theorem 2.3. *The problem (2.25) admits a unique nonnegative solution $(u_\delta, v_\delta) \in (C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$. Moreover, $0 \le u_\delta \le h_1(x,t)$, $0 \le v_\delta \le h_2(x,t)$, $(x,t) \in \overline{w}_\delta$, where $h_1(x,t)$, $h_2(x,t)$ are given by Lemma 2.2.*

Proof. By the proof of Lemma 2.1, we know that there exists at most one nonnegative solution (u_δ, v_δ). To prove existence, we use Schauder's fixed point theorem.

Let

$$\begin{gathered} X_1 = \{v_1 \in C^{\alpha,\alpha/2}(\overline{w}_\delta) : 0 \le v_1(x,t) \le h_2(x,t),\ (x,t) \in \overline{w}_\delta\}, \\ X_2 = \{u_1 \in C^{\alpha,\alpha/2}(\overline{w}_\delta) : 0 \le u_1(x,t) \le h_1(x,t),\ (x,t) \in \overline{w}_\delta\}. \end{gathered} \tag{2.26}$$

Obviously, X_1, X_2 are closed convex subsets of Banach space $C^{\alpha,\alpha/2}(\overline{w}_\delta)$. In order to get the conclusion, we have to define another set: $X = X_1 \times X_2$. Obviously $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$ is a Banach space with the norm

$$||(v_1,u_1)||_{\alpha,\alpha/2} = ||v_1||_{\alpha,\alpha/2} + ||u_1||_{\alpha,\alpha/2}, \quad \text{for any } (v_1,u_1) \in \left(C^{\alpha,\alpha/2}(\overline{w}_\delta)\right)^2, \tag{2.27}$$

and X is a closed convex subset of Banach space $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$. For any $v_1 \in X_1$, $u_1 \in X_2$, let us consider the following linearized uniformly parabolic problem:

$$\begin{gathered}
x^{q_1} W_{\delta t} - \left(x^{r_1} W_{\delta x}\right)_x = \int_\delta^a v_1^{p_1}\,dx, \quad (x,t) \in w_\delta, \\
x^{q_2} Z_{\delta t} - \left(x^{r_2} Z_{\delta x}\right)_x = \int_\delta^a u_1^{p_2}\,dx, \quad (x,t) \in w_\delta, \\
W_\delta(\delta,t) = W_\delta(a,t) = Z_\delta(\delta,t) = Z_\delta(a,t) = 0, \quad t \in (0,t_0], \\
W_\delta(x,0) = u_{0\delta}(x), \quad Z_\delta(x,0) = v_{0\delta}(x), \quad x \in [\delta,a].
\end{gathered} \tag{2.28}$$

It is easy to see that $(\underline{W}(x,t),\underline{Z}(x,t)) = (0,0)$ and $(\overline{W}(x,t),\overline{Z}(x,t)) = (h_1(x,t),h_2(x,t))$ are subsolution and supersolution of problem (2.28). We also note that $x^{-q_1+r_1}$, $x^{-q_1-1+r_1}$, x^{-q_1}, $x^{-q_2+r_2}$, $x^{-q_2-1+r_2}$, $x^{-q_2} \in C^{\alpha,\alpha/2}(\overline{w}_\delta)$, and $x^{-q_1}\int_\delta^a v_1^{p_1}\,dx$, $x^{-q_2}\int_\delta^a u_1^{p_2}\,dx \in C^{\alpha,\alpha/2}(\overline{w}_\delta)$, $u_{0\delta}(x), v_{0\delta}(x) \in C^{2+\alpha}(\overline{D}_\delta)$. It follows from Theorem 4.2.2 of Laddle et al. [11, page 143] that the problem (2.28) has a unique solution $(W_\delta(x,t;v_1,u_1),Z_\delta(x,t;v_1,u_1)) \in (C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$, which satisfies $0 \le W_\delta(x,t;v_1,u_1) \le h_1(x,t)$, $0 \le Z_\delta(x,t;v_1,u_1) \le h_2(x,t)$. Thus, we can define a mapping Y from X into $(C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta)^2$, such that

$$Y(v_1(x,t),u_1(x,t)) = (W_\delta(x,t;v_1,u_1),Z_\delta(x,t;v_1,u_1)), \tag{2.29}$$

where $(W_\delta(x,t;v_1,u_1),Z_\delta(x,t;v_1,u_1))$ denotes the unique solution of (2.28) corresponding to $(v_1(x,t),u_1(x,t)) \in X$. To use Schauder's fixed point theorem, we need to verify the fact that Y maps X into itself is continuous and compact.

In fact, $YX \subset X$ and the embedding operator form Banach space $(C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$ to the Banach space $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$ is compact. Therefore Y is compact. To show Y is continuous in X_1 let us consider a sequence $\{v_{1n}(x,t)\}$ which converges to $v_1(x,t)$ uniformly in the norm $\|\cdot\|_{\alpha,\alpha/2}$. We know that $v_1(x,t) \in X_1$. Analogously, in X_2 we consider a sequence $\{u_{1n}(x,t)\}$ which converges to $u_1(x,t)$ uniformly in the norm $\|\cdot\|_{\alpha,\alpha/2}$ and $u_1(x,t) \in X_2$. So we get a sequence $\{(v_{1n}(x,t),u_{1n}(x,t))\} \subset X$, which converges to $(v_1(x,t),u_1(x,t))$ uniformly in the norm $\|(\cdot,\cdot)\|_{\alpha,\alpha/2}$ and $(v_1(x,t),u_1(x,t)) \in X$. Let $(W_\delta n(x,t),Z_\delta n(x,t))$ and $(W_\delta(x,t),Z_\delta(x,t))$ be the solution of problem (2.28) corresponding to $(v_{1n}(x,t),u_{1n}(x,t))$ and $(v_1(x,t),u_1(x,t))$, respectively. Without loss of generality, let us assume that

$$\begin{gathered}
||v_{1n}(x,t)||_{\alpha,\alpha/2} \le ||v_1(x,t)||_{\alpha,\alpha/2} + 1, \quad \text{for any } n \ge 1, \\
||u_{1n}(x,t)||_{\alpha,\alpha/2} \le ||u_1(x,t)||_{\alpha,\alpha/2} + 1, \quad \text{for any } n \ge 1.
\end{gathered} \tag{2.30}$$

Let $W(x,t) = W_{\delta n}(x,t) - W_\delta(x,t)$, $Z(x,t) = Z_{\delta n}(x,t) - Z_\delta(x,t)$. Then we have

$$\begin{gathered}
x^{q_1} W_t - (x^{r_1} W_x)_x = \int_\delta^a (v_{1n}^{p_1} - v_1^{p_1})\,dx, \quad (x,t) \in w_\delta, \\
x^{q_2} Z_t - (x^{r_2} Z_x)_x = \int_\delta^a (u_{1n}^{p_2} - u_1^{p_2})\,dx, \quad (x,t) \in w_\delta, \\
W(\delta,t) = W(a,t) = Z(\delta,t) = Z(a,t) = 0, \quad t \in (0,t_0], \\
W(x,0) = 0, \quad Z(x,0) = 0, \quad x \in \overline{D}_\delta.
\end{gathered} \tag{2.31}$$

From Theorem 4.5.2 of Ladyženskaja et al. [12, page 320], there exist positive constants C_1 (independent of v_{1n} and v_1), C_2 (independent of u_{1n} and u_1) such that

$$\begin{aligned}
\|W\|_{2+\alpha,1+\alpha/2} &\le C_1 \left\| \int_\delta^a (v_{1n}^{p_1} - v_1^{p_1})\,dx \right\|_{\alpha,\alpha/2} \\
&\le C_1 a p_1 \left\| (v_1 + \tau(v_{1n} - v_1))^{p_1 - 1} \right\|_{\alpha,\alpha/2} \left\| v_{1n} - v_1 \right\|_{\alpha,\alpha/2} \\
&\le C_1 a p_1 \left[3\left(\|v_1\|_{\alpha,\alpha/2} + 1 \right) \right]^{p_1 - 1} \|v_{1n} - v_1\|_{\alpha,\alpha/2}, \\
\|Z\|_{2+\alpha,1+\alpha/2} &\le C_2 a p_2 \left[3\left(\|u_1\|_{\alpha,\alpha/2} + 1 \right) \right]^{p_2 - 1} \|u_{1n} - u_1\|_{\alpha,\alpha/2},
\end{aligned} \tag{2.32}$$

where $\tau \in (0,1)$. So,

$$\begin{aligned}
\|(W,Z)\|_{2+\alpha,1+\alpha/2} &= \|W\|_{2+\alpha,1+\alpha/2} + \|Z\|_{2+\alpha,1+\alpha/2} \\
&\le C_1 a p_1 \left[3\left(\|v_1\|_{\alpha,\alpha/2} + 1 \right) \right]^{p_1 - 1} \|v_{1n} - v_1\|_{\alpha,\alpha/2} \\
&\quad + C_2 a p_2 \left[3\left(\|u_1\|_{\alpha,\alpha/2} + 1 \right) \right]^{p_2 - 1} \|u_{1n} - u_1\|_{\alpha,\alpha/2} \\
&\le C \|(v_{1n} - v_1, u_{1n} - u_1)\|_{\alpha,\alpha/2}.
\end{aligned} \tag{2.33}$$

This shows that the mapping Y is continuous. By Schauder's fixed point theorem, we complete the proof of Theorem 2.3. □

Now we can prove the following local existence result.

Theorem 2.4. *There exists some t_0 ($< T$) such that problem (1.1) has a unique nonnegative solution $(u(x,t), v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$.*

Proof. By Theorem 2.3, the problem (2.25) has a unique nonnegative solution $(u_\delta, v_\delta) \in (C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$. It follows from Lemma 2.1 that $(u_{\delta 1}, v_{\delta 1}) \le (u_{\delta 2}, v_{\delta 2})$ if $\delta 1 > \delta 2$. Therefore, $\lim_{\delta\to 0}(u_\delta(x,t), v_\delta(x,t))$ exists for all $(x,t) \in (0,a] \times [0,t_0]$. Let $(u(x,t), v(x,t)) = \lim_{\delta\to 0}(u_\delta(x,t), v_\delta(x,t))$, $(x,t) \in (0,a] \times [0,t_0]$ and define $(u(0,t), v(0,t)) = (0,0)$, $t \in [0,t_0]$. We would like to show that $(u(x,t), v(x,t))$ is a classical solution of (1.1) in Ω_{t_0}. For any $(x_1,t_1) \in \Omega_{t_0}$, there exist three domains $Q' = (a_1', a_2') \times (t_2', t_3']$, $Q'' = (a_1'', a_2'') \times (t_2'', t_3'']$, and $Q''' = (a_1''', a_2''') \times (t_2''', t_3''']$ such that $(x_1,t_1) \in Q' \subset Q'' \subset Q''' \subset (0,a) \times (0,t_0]$ with $0 < a_1''' < a_1'' < a_1' < x_1 < a_2' < a_2'' < a_2''' < a$, $0 \le t_2''' \le t_2'' \le t_2' < t_1 < t_3' \le t_3'' \le t_3''' \le t_0$. Since

$(u_\delta(x,t),v_\delta(x,t)) \le (h_1(x,t),h_2(x,t))$ in Q''' and $h_1(x,t)$, $h_2(x,t)$ are finite on $\overline{Q}'''$, for any constant $\tilde{q} > 1$ and some positive constants K_3, K_4, we have

$$\begin{aligned}
&\text{(i)}\ \ ||u_\delta||_{L^{\tilde{q}}(Q''')} \le ||h_1||_{L^{\tilde{q}}(Q''')} \le K_3, \qquad ||v_\delta||_{L^{\tilde{q}}(Q''')} \le ||h_2||_{L^{\tilde{q}}(Q''')} \le K_3,\\
&\text{(ii)}\ \ ||x^{-q_1}\int_\delta^a v_\delta^{p_1}\,dx||_{L^{\tilde{q}}(Q''')} \le (a_1^*)^{-q_1}\left|\left|\int_0^a h_2^{p_1}\,dx\right|\right|_{L^{\tilde{q}}(Q''')} \le K_4,\\
&\qquad \left|\left|x^{-q_2}\int_\delta^a u_\delta^{p_2}\,dx\right|\right|_{L^{\tilde{q}}(Q''')} \le (a_2^*)^{-q_2}\left|\left|\int_0^a h_1^{p_2}\,dx\right|\right|_{L^{\tilde{q}}(Q''')} \le K_4,
\end{aligned}\tag{2.34}$$

where $a_1^* = a_1'''$ if $q_1 \ge 0$, $a_1^* = a_2'''$ if $q_1 < 0$, and $a_2^* = a_1'''$ if $q_2 \ge 0$, $a_2^* = a_2'''$ if $q_2 < 0$.

By the local L^p estimate of Ladyženskaja et al. [12, pages 341-342, 352], $(u_\delta,v_\delta) \in (W_{\tilde{q}}^{2,1}(Q''))^2$. By the embedding theorem in [12, pages 61 and 80], $W_{\tilde{q}}^{2,1}(Q'')\hookrightarrow H^{\alpha,\alpha/2}(Q'')$ if we choose $\tilde{q} > 2/(1-\alpha)$. Then, $\|u_\delta\|_{H^{\alpha,\alpha/2}(Q'')} \le K_5$ and $\|v_\delta\|_{H^{\alpha,\alpha/2}(Q'')} \le K_5$ for some positive constant K_5, and we have

$$\begin{aligned}
&\left|\left|x^{-q_1}\int_\delta^a v_\delta^{p_1}\,dx\right|\right|_{H^{\alpha,\alpha/2}(Q'')}\\
&\quad \le (a_1^*)^{-q_1}\left|\left|\int_\delta^a h_2^{p_1}\,dx\right|\right|_\infty + \sup_{(x,t)\in Q''\ (\tilde{x},t)\in Q''} \frac{|\int_\delta^a v_\delta^{p_1}\,dx|\cdot|x^{-q_1}-\tilde{x}^{-q_1}|}{|x-\tilde{x}|^\alpha}\\
&\qquad + \sup_{(\tilde{x},t)\in Q''\ (\tilde{x},\tilde{t})\in Q''} \frac{|\tilde{x}^{-q_1}|\cdot|\int_\delta^a p_1(v_\delta(x,\tilde{t})+\tau(v_\delta(x,t)-v_\delta(x,\tilde{t})))^{p_1-1}(v_\delta(x,t)-v_\delta(x,\tilde{t}))\,dx|}{|t-\tilde{t}|^{\alpha/2}}\\
&\quad \le (a_1^*)^{-q_1}\left|\left|\int_0^a h_2^{p_1}\,dx\right|\right|_\infty + \left|\left|\int_0^a h_2^{p_1}\,dx\right|\right|_\infty\cdot||x^{-q_1}||_{H^\alpha(a_1'',a_2'')}\\
&\qquad + (a_1^*)^{-q_1}\left|\left|\int_0^a p_1h_2^{p_1-1}\,dx\right|\right|_\infty\cdot||v_\delta||_{H^{\alpha,\alpha/2}(Q'')} \le K_6,\\
&\left|\left|x^{-q_2}\int_\delta^a u_\delta^{p_2}\,dx\right|\right|_{H^{\alpha,\alpha/2}(Q'')} \le K_6,
\end{aligned}\tag{2.35}$$

for some positive constant K_6, which is independent of δ, where $\tau \in (0,1)$. By Ladyženskaja et al. [12, Theorem 4.10.1, pages 351-352], we have

$$||u_\delta||_{H^{2+\alpha,1+\alpha/2}(Q')} \le K_7, \qquad ||v_\delta||_{H^{2+\alpha,1+\alpha/2}(Q')} \le K_7,\tag{2.36}$$

for some positive constant K_7 independent of δ. This implies that u_δ, $u_{\delta t}$, $u_{\delta x}$, $u_{\delta xx}$ and v_δ, $v_{\delta t}$, $v_{\delta x}$, $v_{\delta xx}$ are equicontinuous in Q'. By the Ascoli-Arzela theorem, we know that

$$\|u\|_{H^{2+\alpha',1+\alpha'/2}(Q')} \le K_8, \qquad \|v\|_{H^{2+\alpha',1+\alpha'/2}(Q')} \le K_8,\tag{2.37}$$

for some $\alpha' \in (0,\alpha)$ and some positive constant K_8 independent of δ, and that the derivatives of u and v are uniform limits of the corresponding partial derivatives of u_δ

and v_δ, respectively. Hence $(u(x,t),v(x,t))$ satisfies (1.1), and $\lim_{t\to 0}(u(x,t),v(x,t)) = \lim_{t\to 0}\lim_{\delta\to 0}(u_\delta(x,t),v_\delta(x,t)) = \lim_{\delta\to 0}(u_{0\delta}(x,t),v_{0\delta}(x,t)) = (u_0(x),v_0(x))$. It follows from $0 \le u(x,t) \le h_1(x,t)$, $0 \le v(x,t) \le h_2(x,t)$ and $h_1(x,t) \to 0$, $h_2(x,t) \to 0$ as $x \to 0$ or $x \to a$ that $\lim_{x\to 0}(u(x,t),v(x,t)) = \lim_{x\to a}(u(x,t),v(x,t)) = (0,0)$, thus $(u,v) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ is the solution of (1.1) in Ω_{t_0}. We complete the proof of Theorem 2.4. □

By using Lemma 2.1, there exists at most one nonnegative solution of (1.1). Similar to the proof of [9, Theorem 2.5], we obtain the following constitutional result.

Theorem 2.5. *Let T be the supremum over t_0 for which there is a unique nonnegative solution $(u(x,t),v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$ of (1.1). Then (1.1) has a unique nonnegative solution $(u(x,t),v(x,t)) \in (C([0,a] \times [0,T)) \cap C^{2,1}((0,a) \times (0,T)))^2$. If $T < +\infty$, then $\limsup_{t\to T} \max_{x\in[0,a]}(|u(x,t)| + |v(x,t)|) = +\infty$.*

3. Blowup of solution

In this section, we give some global existence and blowup result of the solution of (1.1).

3.1. Existence and nonexistence of the global solution. In this subsection, we would assume $q_1 > r_1 - 1$, $q_2 > r_2 - 1$.

First, the solution of the following elliptic boundary value problem:

$$-\left(x^{r_1}\psi'(x)\right)' = 1, \qquad x \in (0,a);\ \psi(0) = \psi(a) = 0, \tag{3.1}$$

is given by $\psi(x) = (a^{2-r_1}/(2-r_1))(x/a)^{1-r_1}(1-x/a)$.

Analogously, the solution of the following elliptic boundary value problem:

$$-\left(x^{r_2}\varphi'(x)\right)' = 1, \quad x \in (0,a);\ \varphi(0) = \varphi(a) = 0, \tag{3.2}$$

is given by $\varphi(x) = (a^{2-r_2}/(2-r_2))(x/a)^{1-r_2}(1-x/a)$.

By direction computation, we have

$$\begin{aligned} \int_0^a \psi^{p_2}\,dx &= \frac{a^{(2-r_1)p_2+1}B\left(p_2(1-r_1)+1,p_2+1\right)}{(2-r_1)^{p_2}}, \\ \int_0^a \varphi^{p_1}\,dx &= \frac{a^{(2-r_2)p_1+1}B\left(p_1(1-r_2)+1,p_1+1\right)}{(2-r_2)^{p_1}}, \end{aligned} \tag{3.3}$$

where $B(l,m)$ is a Beta function defined by $B(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$.

Let

$$\begin{aligned} a_1 &= \frac{a_2^{p_1}\left[a^{(2-r_2)p_1+1}B\left(p_1(1-r_2)+1,p_1+1\right)\right]}{(2-r_2)^{p_1}}, \\ a_2 &= \frac{a_1^{p_2}\left[a^{(2-r_1)p_2+1}B\left(p_2(1-r_1)+1,p_2+1\right)\right]}{(2-r_1)^{p_2}}, \end{aligned} \tag{3.4}$$

then we have the following global existence result.

THEOREM 3.1. *Let $(u(x,t),v(x,t))$ be the solution of (1.1). If $u_0(x)\le a_1\psi(x)$, $v_0(x)\le a_2\varphi(x)$, then $(u(x,t),v(x,t))$ exists globally.*

Proof. Let $\overline{u}=a_1\psi(x)$, $\overline{v}=a_2\varphi(x)$, then we have

$$\begin{aligned}
&x^{q_1}\overline{u}_t(x,t)-\left(x^{r_1}\overline{u}_x(x,t)\right)_x\\
&\quad=-\left(x^{r_1}a_1\psi'(x)\right)'=a_1\\
&\quad=a_2^{p_1}\left[a^{(2-r_2)p_1+1}B\frac{(p_1(1-r_2)+1,p_1+1)}{(2-r_2)^{p_1}}\right]\\
&\quad=\int_0^a(a_2\varphi)^{p_1}dx=\int_0^a\overline{v}^{p_1}(x,t)dx,\quad (x,t)\in(0,a)\times(0,T),
\end{aligned}\tag{3.5}$$

$$\begin{gathered}
x^{q_2}\overline{v}_t(x,t)-\left(x^{r_2}\overline{v}_x(x,t)\right)_x=\int_0^a\overline{u}^{p_2}(x,t)dx,\quad (x,t)\in(0,a)\times(0,T),\\
\overline{u}(0,t)=\overline{u}(a,t)=\overline{v}(0,t)=\overline{v}(a,t)=0,\quad t\in(0,T),\\
\overline{u}(x,0)=a_1\psi(x)\ge u_0(x),\quad \overline{v}(x,0)=a_2\varphi(x)\ge v_0(x),\quad x\in[0,a],
\end{gathered}$$

that is to say $(\overline{u}(x,t),\overline{v}(x,t))=(a_1\psi(x),a_2\varphi(x))$ is a supersolution of (1.1). By Theorem 2.5, $T=+\infty$, that is, $(u(x,t),v(x,t))$ exists globally. The proof of Theorem 3.1 is complete. $\square$

Next we consider the following eigenvalue problem:

$$\begin{gathered}
-\left(x^{r_1}\varphi_1'(x)\right)'=\lambda_1x^{q_1}\varphi_1(x),\quad x\in(0,a),\\
\varphi_1(0)=\varphi_1(a)=0.
\end{gathered}\tag{3.6}$$

By transformation $\varphi_1(x)=x^{(1-r_1)/2}y_1(x)$, the above differential equation becomes

$$x^2y_1''(x)+xy_1'(x)-\frac{(1-r_1)^2}{4}y_1(x)+\lambda_1x^{q_1+2-r_1}y_1(x)=0,\quad x\in(0,a).\tag{3.7}$$

Again, by transformation $x=z^{2/(q_1+2-r_1)}$, the problem (3.6) becomes

$$\begin{gathered}
z^2y_1''(z)+zy_1'(z)+\left[\frac{4\lambda_1^2z^2}{(q_1+2-r_1)^2}-\frac{(1-r_1)^2}{(q_2+2-r_1)^2}\right]y_1(z)=0,\quad z\in(0,b_1),\\
y_1(0)=y_1(b_1)=0,
\end{gathered}\tag{3.8}$$

where $b_1=a^{(q_1+2-r_1)/2}$. Equation (3.8) is a Bessel equation. Its general solution is given by

$$y_1(z)=AJ_{(1-r_1)/(q_1+2-r_1)}\left(\frac{2\sqrt{\lambda_1}}{q_1+2-r_1}z\right)+BJ_{-(1-r_1)/(q_1+2-r_1)}\left(\frac{2\sqrt{\lambda_1}}{q_1+2-r_1}z\right),\tag{3.9}$$

where A and B are arbitrary constants, $J_{(1-r_1)/(q_1+2-r_1)}$ and $J_{-(1-r_1)/(q_1+2-r_1)}$ denote Bessel functions of the first kind of orders $(1-r_1)/(q_1+2-r_1)$ and $-(1-r_1)/(q_1+2-r_1)$, respectively. Let μ_1 be the first root of $J_{(1-r_1)/(q_1+2-r_1)}(2\sqrt{\lambda_1}b_1/(q_1+2-r_1))$. By Mclachlan

[13, pages 29 and 75], it is positive. It is obvious that μ_1 is the first eigenvalue of problem (3.6); also we can easily obtain the corresponding eigenfunction

$$\varphi_1(x) = k_1 x^{(1-r_1)/2} J_{(1-r_1)/(q_1+2-r_1)}\left(\frac{2\sqrt{\mu_1}}{q_1+2-r_1} x^{(q_1+2-r_1)/2}\right), \tag{3.10}$$

which is positive for $x \in (0,a)$. Since $q_1 > r_1 - 1$, we can choose $k_1 > 0$ such that

$$\max_{x\in[0,a]} x^{q_1}\varphi_1(x) = 1. \tag{3.11}$$

Analogously, we consider the following eigenvalue problem:

$$\begin{gathered} -\left(x^{r_2}\varphi_2'(x)\right)' = \lambda_2 x^{q_2}\varphi_2(x), \quad x \in (0,a), \\ \varphi_2(0) = \varphi_2(a) = 0. \end{gathered} \tag{3.12}$$

By using the same method as above, let μ_2 be the first root of $J_{(1-r_2)/(q_2+2-r_2)}(2\sqrt{\lambda_2}b_2/(q_2 + 2 - r_2))$, where $b_2 = a^{(q_2+2-r_2)/2}$. By Mclachlan [13, pages 29 and 75], it is positive. It is obvious that μ_2 is the first eigenvalue of problem (3.12); also we can easily obtain the corresponding eigenfunction

$$\varphi_2(x) = k_2 x^{(1-r_2)/2} J_{(1-r_2)/(q_2+2-r_2)}\left(\frac{2\sqrt{\mu_2}}{q_2+2-r_2} x^{(q_2+2-r_2)/2}\right), \tag{3.13}$$

which is positive for $x \in (0,a)$. Since $q_2 > r_2 - 1$, we can choose $k_2 > 0$ such that

$$\max_{x\in[0,a]} x^{q_2}\varphi_2(x) = 1. \tag{3.14}$$

Since $u_0(x), v_0(x)$ are both nonnegative nontrivial functions, there exists a constant $\delta > 0$, such that $\int_0^a x^{q_1}\varphi_1(x)u_0(x)dx \geq \delta$, $\int_0^a x^{q_2}\varphi_2(x)v_0(x)dx \geq \delta$. Then, we have the following theorem.

Theorem 3.2. *Let $(u(x,t), v(x,t))$ be the solution of the problem (1.1), then the solution of (1.1) blows up in finite time if*

$$\begin{gathered} \int_0^a \varphi_1(x)dx\left(\int_0^a x^{q_2}\varphi_2(x)dx\right)^{1-p_1}\left(\int_0^a x^{q_2}\varphi_2(x)v_0(x)dx\right)^{p_1} > \max\{\mu_1,\mu_2\}\int_0^a x^{q_1}\varphi_1(x)u_0(x)dx, \\ \int_0^a \varphi_2(x)dx\left(\int_0^a x^{q_1}\varphi_1(x)dx\right)^{1-p_2}\left(\int_0^a x^{q_1}\varphi_1(x)u_0(x)dx\right)^{p_2} > \max\{\mu_1,\mu_2\}\int_0^a x^{q_2}\varphi_2(x)v_0(x)dx. \end{gathered} \tag{3.15}$$

Proof. We set

$$U(t) = \int_0^a x^{q_1}\varphi_1(x)u(x,t)dx, \qquad V(t) = \int_0^a x^{q_2}\varphi_2(x)v(x,t)dx. \tag{3.16}$$

Multiplying (1.1) by $\varphi_1(x)$ and integrating it over x from 0 to a, we have

$$\int_0^a x^{q_1} u_t \varphi_1 dx = \int_0^a (x^{r_1} u_x)_x \varphi_1 dx + \int_0^a \varphi_1 dx \int_0^a v^{p_1} dx. \tag{3.17}$$

Integrating by part, using Jensen's inequality, we have

$$\begin{aligned} U'(t) &= \int_0^a x^{q_1} u_t \varphi_1 dx \\ &\geq -\mu_1 \int_0^a x^{q_1} \varphi_1(x) u(x,t) dx + \int_0^a \varphi_1(x) dx \int_0^a x^{q_2} \varphi_2(x) v^{p_1} dx \\ &\geq -\mu_1 U(t) + \int_0^a \varphi_1(x) dx \left(\int_0^a x^{q_2} \varphi_2(x) dx \right)^{1-p_1} \left(\int_0^a x^{q_2} \varphi_2(x) v \, dx \right)^{p_1} \\ &= -\mu_1 U(t) + \int_0^a \varphi_1(x) dx \left(\int_0^a x^{q_2} \varphi_2(x) dx \right)^{1-p_1} V^{p_1}(t), \\ V'(t) &\geq -\mu_2 V(t) + \int_0^a \varphi_2(x) dx \left(\int_0^a x^{q_1} \varphi_1(x) dx \right)^{1-p_2} U^{p_2}(t). \end{aligned} \tag{3.18}$$

If we set

$$C_1 = \int_0^a \varphi_1(x) dx \left(\int_0^a x^{q_2} \varphi_2(x) dx \right)^{1-p_1}, \qquad C_2 = \int_0^a \varphi_2(x) dx \left(\int_0^a x^{q_1} \varphi_1(x) dx \right)^{1-p_2}, \tag{3.19}$$

then we have

$$\begin{aligned} U'(t) &\geq -\mu_1 U(t) + C_1 V^{p_1}(t), \\ V'(t) &\geq -\mu_2 V(t) + C_2 U^{p_2}(t). \end{aligned} \tag{3.20}$$

If we set $\tilde{U} = (C_1 C_2^{p_1})^{1/(p_1 p_2 - 1)} U$, $\tilde{V} = (C_2 C_1^{p_2})^{1/(p_1 p_2 - 1)} V$, $\mu = \max\{\mu_1, \mu_2\}$, then we have

$$\begin{aligned} \tilde{U}'(t) &\geq -\mu \tilde{U}(t) + \tilde{V}^{p_1}(t), \\ \tilde{V}'(t) &\geq -\mu \tilde{V}(t) + \tilde{U}^{p_2}(t). \end{aligned} \tag{3.21}$$

Since $\tilde{U}(0) > 0$, $\tilde{V}(0) > 0$ and $\tilde{U}^{p_2}(0)/\mu > \tilde{V}(0) > \mu \tilde{U}^{1/p_1}(0)$, we get from [16, Corollary 1] that $(\tilde{U}, \tilde{U})$ blows up in finite time. Therefore, the solution of (1.1) blows up in finite time. The proof of Theorem 3.2 is complete. □

Remark 3.3. Since the system (1.1) is completely coupled, we know that if the solution (u, v) blows up in finite time, then u and v blow up simultaneously.

3.2. Global blowup. In this subsection, we discuss the global blowup in two special cases.

Case 1. $q_1 > 0$, $r_1 = 0$ or $q_2 > 0$, $r_2 = 0$.

Chan et al. [3, 5] proved that there exists Green's function $G(x,\xi,t-\tau)$ associated with the operator $L = x^{q_1}(\partial/\partial t) - \partial^2/\partial x^2$ with the first boundary condition, and obtained the following lemmas.

Lemma 3.4. (a) *For $t > \tau$, $G(x,\xi,t-\tau)$ is continuous for $(x,t,\xi,\tau) \in ([0,a] \times (0,T]) \times ((0,a] \times [0,T))$.*

(b) *For each fixed $(\xi,\tau) \in (0,a] \times [0,T)$, $G_t(x,\xi,t-\tau) \in C([0,a] \times (\tau,T])$.*

(c) *In $\{(x,t,\xi,\tau) : x$ and ξ are in $(0,a)$, $T \geq t > \tau \geq 0\}$, $G(x,\xi,t-\tau)$ is positive.*

Lemma 3.5. *For fixed $x_0 \in (0,a]$, given any $x \in (0,a)$ and any finite time T, there exist positive constants C_1 (depending on x and T) and C_2 (depending on T) such that*

$$\int_0^a G(x,\xi,t)d\xi > C_1, \quad \int_0^a G(x_0,\xi,t)\,d\xi < C_2, \quad \text{for } 0 \leq t \leq T. \tag{3.22}$$

Now we give the global blowup result

Theorem 3.6. *Under the assumption of Case 1, if the solution of (1.1) blows up at the point $x_0 \in (0,a)$, then the blowup set of the solution of (1.1) is $[0,a]$.*

Proof. From the remark, we know that u and v blow up simultaneously if the solution (u,v) blows up in finite time. Without loss of generality, we assume $q_1 > 0$, $r_1 = 0$, and $u(x,t)$ blows up in finite time T. By Green's second identity we have

$$u(x,t) = \int_0^a \xi^{q_1} G(x,\xi,t)u_0(\xi)d\xi + \int_0^t \int_0^a G(x,\xi,t-\tau) \int_0^a v^{p_1}(y,\tau)dy\,d\xi\,d\tau \tag{3.23}$$

for any $(x,t) \in (0,a) \times (0,T)$. According to the conditions, $u(x,t)$ blows up at $x = x_0$, then $\limsup_{t\to T} u(x_0,t) = +\infty$. By (3.23) and Lemma 3.5, we have

$$\begin{aligned} u(x_0,t) &= \int_0^a \xi^{q_1} G(x_0,\xi,t)u_0(\xi)d\xi + \int_0^t \int_0^a G(x_0,\xi,\tau) \int_0^a v^{p_1}(y,t-\tau)dy\,d\xi\,d\tau \\ &\leq C_2 a^{q_1} \max_{x\in[0,a]} u_0(x) + C_2 \int_0^t \int_0^a v^{p_1}(y,t-\tau)dy\,d\tau. \end{aligned} \tag{3.24}$$

Since $\limsup_{t\to T} u(x_0,t) = +\infty$, we have

$$\lim_{t\to T} \int_0^t \int_0^a v^{p_1}(y,t-\tau)dy\,d\tau = +\infty. \tag{3.25}$$

On the other hand, for any $x \in (0,a)$, we have

$$\begin{aligned} u(x,t) &\geq \int_0^a \xi^{q_1} G(x,\xi,t)u_0(\xi)d\xi + C_1 \int_0^t \int_0^a v^{p_1}(y,t-\tau)dy\,d\tau \\ &\geq C_1 \int_0^t \int_0^a v^{p_1}(y,t-\tau)dy\,d\tau, \quad t \in (0,T). \end{aligned} \tag{3.26}$$

It follows from the above inequality and (3.25) that $\limsup_{t\to T} u(x,t) = +\infty$.

For any $\widetilde{x} \in \{0,a\}$, we can choose a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (\widetilde{x},T)$ $(n \to +\infty)$ and $\lim_{n\to\infty} u(x_n,t_n) = +\infty$. Thus the blowup set is the whole domain $[0,a]$, and we complete the proof of Theorem 3.6. □

Case 2. $q_1 = 0$, $0 \le r_1 < 1$ or $q_2 = 0$, $0 \le r_2 < 1$.

We will prove that the blowup set is the whole domain under the following assumption:

(H) there exists M $(0 < M < +\infty)$ such that $(x^{r_1} u_{0x}(x))_x \le M$ or $(x^{r_2} v_{0x}(x))_x \le M$ in $(0,a)$.

Theorem 3.7. *Under the assumptions of* (H) *and Case 2, if the solution of (1.1) blows up at the point* $x_0 \in (0,a)$, *then the blowup set of the solution of (1.1) is* $[0,a]$.

Proof. The proof is similar to the proof of [7, Theorem 4.3], so we omit it. The proof of Theorem 3.7 is complete. □

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