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General decay for Kirchhoff plates with a boundary condition of memory type

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Abstract

In this paper we consider Kirchhoff plates with a memory condition at the boundary. For a wider class of relaxation functions, we establish a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.
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1 Introduction

We consider the following Kirchhoff plates with a memory condition at the boundary:

$$u_{tt} + \Delta^2 u + a(x)u_t = 0, \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial v} = 0, \quad \text{on } \Gamma_0 \times (0, \infty), \tag{1.2}$$

$$-u + \int_0^t g_1(t-s)\mathcal{B}_2 u(s)\,ds = 0, \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.3}$$

$$\frac{\partial u}{\partial v} + \int_0^t g_2(t-s)\mathcal{B}_1 u(s)\,ds = 0, \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.4}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad \text{in } \Omega, \tag{1.5}$$

where $a \in C^1(\bar{\Omega})$ and Ω is an open bounded set of $\mathbb{R}^2$ with a regular boundary Γ. We divide the boundary into two parts:

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with } \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset; \text{ and } \Gamma_0 \neq \emptyset.$$

Let us denote by $v = (v_1, v_2)$ the external unit normal to Γ, and let us denote by $\eta = (-v_2, v_1)$ the unit tangent positively oriented on Γ. We are denoting by $\mathcal{B}_1, \mathcal{B}_2$ the following differential operators:

$$\mathcal{B}_1 u = \Delta u + (1-\mu)B_1 u, \qquad \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial v} + (1-\mu)B_2 u,$$

where B_1 and B_2 are given by

$$B_1 u = 2v_1 v_2 u_{xy} - v_1^2 u_{yy} - v_2^2 u_{xx}, \qquad B_2 u = \frac{\partial}{\partial \eta}\big[(v_1^2 - v_2^2)u_{xy} + v_1 v_2(u_{yy} - u_{xx})\big],$$

and the constant μ, $0 < \mu < \frac{1}{2}$, represents Poisson's ratio.

In (1.1), u denotes the position of the plate. The integral equations (1.3) and (1.4) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. The relaxation functions $g_1, g_2 \in C^1(0, \infty)$ are positive and nondecreasing. This system models the small transversal vibrations of a thin plate whose Poisson coefficient is equal to μ. We assume that there exists $x_0 \in R^2$ such that

$$\Gamma_0 = \left\{ x \in \Gamma : \nu(x) \cdot (x - x_0) \le 0 \right\}, \tag{1.6}$$

$$\Gamma_1 = \left\{ x \in \Gamma : \nu(x) \cdot (x - x_0) > 0 \right\}. \tag{1.7}$$

If we denote the compactness of Γ_1 by $m(x) = x - x_0$, the condition (1.7) implies that there exists a small positive constant δ_0 such that $0 < \delta_0 \le m(x) \cdot \nu(x), \forall x \in \Gamma_1$.

The uniform stabilization of Kirchhoff plates with linear or nonlinear boundary feedback was investigated by several authors; see, for example, [1–3] among others. The uniform decay for plates with memory was studied in [4–6] and the references therein. There exists a large body of literature regarding viscoelastic problems with the memory term acting in the domain or at the boundary (see [7–12]). Rivera and Racke [13] investigated the decay results for magneto-thermo-elastic system. Santos *et al.* [14] studied the asymptotic behavior of the solutions of a nonlinear wave equation of Kirchhoff type with a boundary condition of memory type. Cavalcanti and Guesmia [15] proved the general decay rates of solutions to a nonlinear wave equation with a boundary condition of memory type. Park and Kang [16] studied the exponential decay for the multi-valued hyperbolic differential inclusion with a boundary condition of memory type. Kafini [17] showed the decay results for viscoelastic diffusion equations in the absence of instantaneous elasticity. They proved that the energy decays uniformly exponentially or algebraically at the same rate as the relaxation functions. In the present work, we generalize the earlier decay results of the solution of (1.1)-(1.5). More precisely, we show that the energy decays at the rate similar to the relaxation functions, which are not necessarily decaying like polynomial or exponential functions. In fact, our result allows a larger class of relaxation functions. Recently, Messaoudi and Soufyane [18], Santos and Soufyane [19], and Mustafa and Messaoudi [20] proved the general decay for the wave equation, von Karman plate system, and Timoshenko system with viscoelastic boundary conditions, respectively.

The paper is organized as follows. In Section 2 we present some notations and material needed for our work. In Section 3 we prove the general decay of the solutions to the Kirchhoff plates with a memory condition at the boundary.

2 Preliminaries

In this section, we present some material needed in the proof of our main result. We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Define the following space:

$$W = \left\{ w \in H^2(\Omega); w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}.$$

First, we shall use Eqs. (1.3) and (1.4) to estimate the values $\mathcal{B}_1$ and $\mathcal{B}_2$ on Γ_1. Denoting by

$$(g * v)(t) = \int_0^t g(t - s) v(s) \, ds,$$

the convolution product operator and differentiating Eqs. (1.3) and (1.4), we arrive at the following Volterra equations:

$$\mathcal{B}_2 u + \frac{1}{g_1(0)} g_1' * \mathcal{B}_2 u = \frac{1}{g_1(0)} u_t,$$

$$\mathcal{B}_1 u + \frac{1}{g_2(0)} g_2' * \mathcal{B}_1 u = -\frac{1}{g_2(0)} \frac{\partial u_t}{\partial \nu}.$$

Applying the Volterra inverse operator, we get

$$\mathcal{B}_2 u = \frac{1}{g_1(0)} \{ u_t + k_1 * u_t \},$$

$$\mathcal{B}_1 u = -\frac{1}{g_2(0)} \left\{ \frac{\partial u_t}{\partial \nu} + k_2 * \frac{\partial u_t}{\partial \nu} \right\},$$

where the resolvent kernels of $-\frac{g_i'}{g_i(0)}$ satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i', \quad \forall i = 1, 2.$$

Denoting by $\tau_i = \frac{1}{g_i(0)}$, for $i = 1, 2$, we obtain

$$\mathcal{B}_2 u = \tau_1 \{ u_t + k_1(0) u - k_1(t) u_0 + k_1' * u \}, \tag{2.1}$$

$$\mathcal{B}_1 u = -\tau_2 \left\{ \frac{\partial u_t}{\partial \nu} + k_2(0) \frac{\partial u}{\partial \nu} - k_2(t) \frac{\partial u_0}{\partial \nu} + k_2' * \frac{\partial u}{\partial \nu} \right\}. \tag{2.2}$$

Therefore, we use (2.1) and (2.2) instead of the boundary conditions (1.3) and (1.4).

Let us define the bilinear form $a(\cdot, \cdot)$ as follows:

$$a(u, v) = \int_\Omega \{ u_{xx} v_{xx} + u_{yy} v_{yy} + \mu(u_{xx} v_{yy} + u_{yy} v_{xx}) + 2(1 - \mu) u_{xy} v_{xy} \} \, dx \, dy. \tag{2.3}$$

We state the following lemma which will be useful in what follows.

Lemma 2.1 ([2]) *Let u and v be functions in $H^4(\Omega) \cap W$. Then we have*

$$\int_\Omega (\Delta^2 u) v \, dx = a(u, v) + \int_{\Gamma_1} \left\{ (\mathcal{B}_2 u) v - (\mathcal{B}_1 u) \frac{\partial v}{\partial \nu} \right\} d\Gamma. \tag{2.4}$$

Let us denote

$$(g \,\square\, v)(t) := \int_0^t g(t - s) |v(t) - v(s)|^2 \, ds.$$

The following lemma states an important property of the convolution operator.

Lemma 2.2 *For $g, v \in C^1([0, \infty) : \mathbb{R})$, we have*

$$(g * v) v_t = -\frac{1}{2} g(t) |v(t)|^2 + \frac{1}{2} g' \,\square\, v - \frac{1}{2} \frac{d}{dt} \left[g \,\square\, v - \left(\int_0^t g(s) \, ds \right) |v|^2 \right].$$

The proof of this lemma follows by differentiating the term $g \square v$.

We formulate the following assumption:

(A1) Let $a \in C^1(\bar{\Omega})$ satisfy $a(x) \geq a_0 > 0$ in Ω for some a_0.

Let us introduce the energy function

$$E(t) = \frac{1}{2}\left[\int_{\Omega} |u_t|^2 \, dx + a(u, u) + \tau_1 \int_{\Gamma_1} \left(k_1(t)|u|^2 - k_1' \square u \right) d\Gamma \right.$$

$$\left. + \tau_2 \int_{\Gamma_1} \left(k_2(t)\left|\frac{\partial u}{\partial \nu}\right|^2 - k_2' \square \frac{\partial u}{\partial \nu} \right) d\Gamma \right].$$

In these conditions, we are able to prove the existence of a strong solution.

Theorem 2.1 *Let $k_i \in C^2(\mathbb{R}^+)$ be such that*

$$k_i, -k_i', k_i'' \geq 0 \quad (i = 1, 2).$$

If $(u_0, u_1) \in (H^4(\Omega) \cap W) \times W$ satisfy the compatibility condition

$$\mathcal{B}_2 u_0 - \tau_1 u_1 = 0, \qquad \mathcal{B}_1 u_0 + \tau_2 \frac{\partial u_1}{\partial \nu} \quad \text{on } \Gamma_1, \tag{2.5}$$

then there is only one solution u of the system (1.1)-(1.5) *satisfying*

$$u \in L^\infty\big(0, T : H^4(\Omega) \cap W\big), \qquad u_t \in L^\infty(0, T : W), \qquad u_{tt} \in L^\infty\big(0, T : L^2(\Omega)\big).$$

Proof See Park and Kang [16]. □

3 General decay

In this section, we show that the solution of the system (1.1)-(1.5) may have a general decay not necessarily of exponential or polynomial type. For this we consider that the resolvent kernels satisfy the following hypothesis:

(H) $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice differentiable function such that

$$k_i(0) > 0, \qquad \lim_{t \to \infty} k_i(t) = 0, \qquad k_i'(t) \leq 0,$$

and there exists a nonincreasing continuous function $\xi_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$k_i''(t) \geq -\xi_i(t)k_i'(t), \quad i = 1, 2, \forall t \geq 0. \tag{3.1}$$

The following identity will be used later.

Lemma 3.1 ([2]) *For every $v \in H^4(\Omega)$ and for every $\mu \in \mathbb{R}$, we have*

$$\int_{\Omega} (m \cdot \nabla v)\Delta^2 v \, dx = a(v, v) + \frac{1}{2}\int_{\Gamma} m \cdot \nu \left[v_{xx}^2 + v_{yy}^2 + 2\mu v_{xx}v_{yy} + 2(1 - \mu)v_{xy}^2 \right] d\Gamma$$

$$+ \int_{\Gamma} \left[(\mathcal{B}_2 v)m \cdot \nabla v - (\mathcal{B}_1 v)\frac{\partial}{\partial \nu}(m \cdot \nabla v) \right] d\Gamma.$$

Our point of departure will be to establish some inequalities for the strong solution of the system (1.1)-(1.5).

Lemma 3.2 *The energy functional E satisfies, along the solution of* (1.1)-(1.5), *the estimate*

$$E'(t) \leq -\int_\Omega a(x)|u_t|^2 \, dx - \frac{\tau_1}{2} \int_{\Gamma_1} \left(|u_t|^2 - k_1^2(t)|u_0|^2 - k_1'(t)|u|^2 + k_1'' \square u \right) d\Gamma$$

$$- \frac{\tau_2}{2} \int_{\Gamma_1} \left(\left| \frac{\partial u_t}{\partial v} \right|^2 - k_2^2(t) \left| \frac{\partial u_0}{\partial v} \right|^2 - k_2'(t) \left| \frac{\partial u}{\partial v} \right|^2 + k_2'' \square \frac{\partial u}{\partial v} \right) d\Gamma. \tag{3.2}$$

Proof Multiplying Eq. (1.1) by u_t, integrating over Ω, and using Lemma 2.1, we get

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_\Omega |u_t|^2 \, dx + a(u, u) \right\} = -\int_{\Gamma_1} (\mathcal{B}_2 u) u_t \, d\Gamma + \int_{\Gamma_1} (\mathcal{B}_1 u) \frac{\partial u_t}{\partial v} \, d\Gamma - \int_\Omega a(x)|u_t|^2 \, dx.$$

Substituting the boundary terms by (2.1) and (2.2) and using Lemma 2.2 and the Young inequality, our conclusion follows. □

Let us consider the following binary operator:

$$(k \circ \varphi)(t) := \int_0^t k(t - s)\big(\varphi(t) - \varphi(s)\big) \, ds.$$

Then applying the Holder inequality for $0 \leq \alpha \leq 1$, we have

$$\big|(k \circ \varphi)(t)\big|^2 \leq \left[\int_0^t |k(s)|^{2(1-\alpha)} \, ds \right] \big(|k|^{2\alpha} \square \varphi\big)(t). \tag{3.3}$$

Let us define the functional

$$\psi(t) = \int_\Omega (m \cdot \nabla u) u_t \, dx.$$

The following lemma plays an important role in the construction of the desired functional.

Lemma 3.3 *Suppose that the initial data* $(u_0, u_1) \in (H^4(\Omega) \cap W) \times W$ *and satisfies the compatibility condition* (2.5). *Then the solution of the system* (1.1)-(1.5) *satisfies*

$$\frac{d}{dt} \psi(t) \leq \frac{1}{2} \int_{\Gamma_1} m \cdot v |u_t|^2 \, d\Gamma - \frac{1}{2} \int_\Omega |u_t|^2 \, dx$$

$$- \left(1 - \frac{C_0}{2} - \frac{\epsilon \lambda_0}{2} \right) a(u, u) + \frac{1}{2} \int_\Omega a(x)|u_t|^2 \, dx$$

$$+ \frac{2\tau_1^2}{\epsilon} \int_{\Gamma_1} \left\{ |u_t|^2 + k_1^2(t)|u|^2 + k_1^2(t)|u_0|^2 + |k_1' \circ u|^2 \right\} d\Gamma$$

$$+ \frac{2\tau_2^2}{\epsilon} \int_{\Gamma_1} \left\{ \left| \frac{\partial u_t}{\partial v} \right|^2 + k_2^2(t) \left| \frac{\partial u}{\partial v} \right|^2 + k_2^2(t) \left| \frac{\partial u_0}{\partial v} \right|^2 + \left| k_2' \circ \frac{\partial u}{\partial v} \right|^2 \right\} d\Gamma$$

$$- \left(\frac{1}{2} - \frac{\epsilon \lambda_0}{2\delta_0} \right) \int_{\Gamma_1} m \cdot v \big[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx} u_{yy} + 2(1-\mu) u_{xy}^2 \big] \, d\Gamma.$$

Proof Differentiating ψ, using Eq. (1.1), and taking $v = u$ in Lemma 3.1, we get

$$
\begin{aligned}
\frac{d}{dt}\psi(t) &= \int_\Omega (m \cdot \nabla u_t)u_t\,dx + \int_\Omega (m \cdot \nabla u)u_{tt}\,dx \\
&= \frac{1}{2}\int_{\Gamma_1} m \cdot v |u_t|^2\,d\Gamma - \int_\Omega |u_t|^2\,dx - a(u,u) \\
&\quad - \int_\Gamma \left[(\mathcal{B}_2 u)(m \cdot \nabla u) - (\mathcal{B}_1 u)\frac{\partial}{\partial v}(m \cdot \nabla u) \right]d\Gamma \\
&\quad - \frac{1}{2}\int_\Gamma m \cdot v\left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2 \right]d\Gamma \\
&\quad - \int_\Omega a(x)u_t(m \cdot \nabla u)\,dx.
\end{aligned}
\tag{3.4}
$$

Let us next examine the integrals over Γ_0 in (3.4). Since $u = \frac{\partial u}{\partial v} = 0$ on Γ_0, we have $B_1 u = B_2 u = 0$ on Γ_0 and

$$
\begin{aligned}
&\frac{\partial}{\partial v}(m \cdot \nabla u) = (m \cdot v)\Delta u, \\
&u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2 = (\Delta u)^2 \quad \text{on } \Gamma_0
\end{aligned}
\tag{3.5}
$$

since

$$
u_{xx}u_{yy} - (u_{xy})^2 = 0 \quad \text{on } \Gamma_0.
$$

Therefore, from (3.4) and (3.5), we have

$$
\begin{aligned}
\frac{d}{dt}\psi(t) &\le \frac{1}{2}\int_{\Gamma_1} m \cdot v |u_t|^2\,d\Gamma - \int_\Omega |u_t|^2\,dx - a(u,u) + \frac{1}{2}\int_{\Gamma_0} m \cdot v(\Delta u)^2\,d\Gamma \\
&\quad - \frac{1}{2}\int_{\Gamma_1} m \cdot v\left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2 \right]d\Gamma \\
&\quad - \int_{\Gamma_1} (\mathcal{B}_2 u)(m \cdot \nabla u)\,d\Gamma + \int_{\Gamma_1} (\mathcal{B}_1 u)\frac{\partial}{\partial v}(m \cdot \nabla u)\,d\Gamma \\
&\quad - \int_\Omega a(x)u_t(m \cdot \nabla u)\,dx.
\end{aligned}
\tag{3.6}
$$

Using the Young inequality, we have

$$
\left| \int_{\Gamma_1} (\mathcal{B}_2 u)(m \cdot \nabla u)\,d\Gamma \right| \le \frac{1}{2\epsilon}\int_{\Gamma_1} |\mathcal{B}_2 u|^2\,d\Gamma + \frac{\epsilon}{2}\int_{\Gamma_1} |m \cdot \nabla u|^2\,d\Gamma,
\tag{3.7}
$$

$$
\left| \int_{\Gamma_1} (\mathcal{B}_1 u)\frac{\partial}{\partial v}(m \cdot \nabla u)\,d\Gamma \right| \le \frac{1}{2\epsilon}\int_{\Gamma_1} |\mathcal{B}_1 u|^2\,d\Gamma + \frac{\epsilon}{2}\int_{\Gamma_1} \left| \frac{\partial}{\partial v}(m \cdot \nabla u) \right|^2 d\Gamma,
\tag{3.8}
$$

where ϵ is a positive constant. Since the bilinear form $a(u,u)$ is strictly coercive on W, using the trace theory, we obtain

$$
\begin{aligned}
&\int_{\Gamma_1} |m \cdot \nabla u|^2\,d\Gamma + \int_{\Gamma_1} \left| \frac{\partial}{\partial v}(m \cdot \nabla u) \right|^2 d\Gamma \\
&\le \lambda_0 a(u,u) + \frac{\lambda_0}{\delta_0}\int_{\Gamma_1} m \cdot v\left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2 \right]d\Gamma,
\end{aligned}
\tag{3.9}
$$

where λ_0 is a constant depending on Ω and μ. Further, one has

$$\left| \int_\Omega a(x) u_t (m \cdot \nabla u) \, dx \right| \le \frac{1}{2} \int_\Omega a(x) |u_t|^2 \, dx + \frac{C_0}{2} a(u, u), \tag{3.10}$$

where $\int_\Omega |\nabla u|^2 \, dx \le C_0 a(u, u)$ with some constant $C_0 > 0$. Substituting the inequalities (3.7)-(3.10) into (3.6) and taking into account the fact that $m \cdot \nu \le 0$ on Γ_0, we have

$$\begin{aligned}
\frac{d}{dt} \psi(t) \le {} & \frac{1}{2} \int_{\Gamma_1} m \cdot \nu |u_t|^2 \, d\Gamma - \frac{1}{2} \int_\Omega |u_t|^2 \, dx \\
& - \left(1 - \frac{C_0}{2} - \frac{\epsilon \lambda_0}{2} \right) a(u, u) + \frac{1}{2\epsilon} \int_{\Gamma_1} |\mathcal{B}_1 u|^2 + |\mathcal{B}_2 u|^2 \, d\Gamma \\
& - \left(\frac{1}{2} - \frac{\epsilon \lambda_0}{2\delta_0} \right) \int_{\Gamma_1} m \cdot \nu \left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx} u_{yy} + 2(1 - \mu) u_{xy}^2 \right] d\Gamma \\
& + \frac{1}{2} \int_\Omega a(x) |u_t|^2 \, dx.
\end{aligned}$$

Since the boundary conditions (2.1) and (2.2) can be written as

$$\mathcal{B}_2 u = \tau_1 \big\{ u_t + k_1(t) u - k_1(t) u_0 - k_1' \circ u \big\},$$

$$\mathcal{B}_1 u = -\tau_2 \left\{ \frac{\partial u_t}{\partial \nu} + k_2(t) \frac{\partial u}{\partial \nu} - k_2(t) \frac{\partial u_0}{\partial \nu} - k_2' \circ \frac{\partial u}{\partial \nu} \right\},$$

our conclusion follows. □

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + \psi(t),$$

with $N > 0$. Now we are in a position to show the main result of this paper.

Theorem 3.1 *Let* $(u_0, u_1) \in W \times L^2(\Omega)$. *Suppose the resolvent kernels* k_1, k_2 *satisfy the conditions* (H) *and* (3.1). *Then there exist constants* $\omega, C > 0$ *such that for some* t_0 *large enough, the solution of* (1.1)-(1.5) *satisfies*

$$E(t) \le C E(0) e^{-\omega \int_0^t \xi(s) \, ds}, \quad \forall t \ge t_0, \, \text{if} \, u_0 = \frac{\partial u_0}{\partial \nu} = 0 \, \text{on} \, \Gamma_1. \tag{3.11}$$

Otherwise,

$$E(t) \le C \left(E(0) + \int_0^t k_0(s) e^{\omega \int_{t_0}^s \xi(\tau) \, d\tau} \, ds \right) e^{-\omega \int_0^t \xi(s) \, ds} \tag{3.12}$$

for all $t \ge t_0$, *where*

$$\xi(t) = \min\big\{ \xi_1(t), \xi_2(t) \big\}$$

and

$$k_0(t) = \int_{\Gamma_1} k_1^2(t) |u_0|^2 \, d\Gamma + \int_{\Gamma_1} k_2^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \, d\Gamma.$$

Proof Applying the inequality (3.3) with $\alpha = \frac{1}{2}$ in Lemma 3.3 and from Lemma 3.2, we obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq &-\left(N - \frac{1}{2}\right) \int_\Omega a(x)|u_t|^2 \, dx \\
&- \frac{\tau_1 N}{2} \int_{\Gamma_1} \left\{|u_t|^2 - k_1^2(t)|u_0|^2 - k_1'(t)|u|^2 + k_1'' \square u\right\} d\Gamma \\
&- \frac{\tau_2 N}{2} \int_{\Gamma_1} \left\{\left|\frac{\partial u_t}{\partial \nu}\right|^2 - k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 - k_2'(t)\left|\frac{\partial u}{\partial \nu}\right|^2 + k_2'' \square \frac{\partial u}{\partial \nu}\right\} d\Gamma \\
&- \left(1 - \frac{C_0}{2} - \frac{\epsilon \lambda_0}{2}\right) a(u,u) + \frac{1}{2} \int_{\Gamma_1} m \cdot \nu |u_t|^2 \, d\Gamma \\
&+ \frac{2\tau_1^2}{\epsilon} \int_{\Gamma_1} \left\{|u_t|^2 + k_1^2(t)|u|^2 + k_1^2(t)|u_0|^2 - k_1(0)k_1' \square u\right\} d\Gamma \\
&+ \frac{2\tau_2^2}{\epsilon} \int_{\Gamma_1} \left\{\left|\frac{\partial u_t}{\partial \nu}\right|^2 + k_2^2(t)\left|\frac{\partial u}{\partial \nu}\right|^2 + k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 - k_2(0)k_2' \square \frac{\partial u}{\partial \nu}\right\} d\Gamma \\
&- \frac{1}{2} \int_\Omega |u_t|^2 \, dx - \left(\frac{1}{2} - \frac{\epsilon \lambda_0}{2\delta_0}\right) \\
&\times \int_{\Gamma_1} m \cdot \nu \left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx} u_{yy} + 2(1-\mu)u_{xy}^2\right] d\Gamma.
\end{aligned}
\tag{3.13}
$$

Then, choosing N large enough and $0 < \epsilon < \min\{\frac{2-C_0}{\lambda_0}, \frac{\delta_0}{\lambda_0}\}$, we obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq &-c_0 E(t) + c \int_{\Gamma_1} k_1^2(t)|u_0|^2 \, d\Gamma + c \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma \\
&- c \int_{\Gamma_1} k_1' \square u \, d\Gamma - c \int_{\Gamma_1} k_2' \square \frac{\partial u}{\partial \nu} \, d\Gamma, \quad \forall t \geq t_0.
\end{aligned}
$$

On the other hand, we can choose N even larger so that

$$
\mathcal{L}(t) \sim E(t).
\tag{3.14}
$$

If $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$, $t \geq 0$, then using (3.1) and (3.2), we have

$$
\begin{aligned}
\xi(t)\mathcal{L}'(t) \leq &-c_0 \xi(t) E(t) + c\xi(t) \int_{\Gamma_1} k_1^2(t)|u_0|^2 \, d\Gamma + c\xi(t) \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma \\
&- c\xi_1(t) \int_{\Gamma_1} k_1' \square u \, d\Gamma - c\xi_2(t) \int_{\Gamma_1} k_2' \square \frac{\partial u}{\partial \nu} \, d\Gamma \\
\leq &-c_0 \xi(t) E(t) + c\xi(t) \int_{\Gamma_1} k_1^2(t)|u_0|^2 \, d\Gamma + c\xi(t) \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma \\
&+ c \int_{\Gamma_1} k_1'' \square u \, d\Gamma + c \int_{\Gamma_1} k_2'' \square \frac{\partial u}{\partial \nu} \, d\Gamma \\
\leq &-c_0 \xi(t) E(t) + c\xi(t) \int_{\Gamma_1} k_1^2(t)|u_0|^2 \, d\Gamma \\
&+ c\xi(t) \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma - cE'(t), \quad \forall t \geq t_0,
\end{aligned}
$$

which gives

$$\xi(t)\mathcal{L}'(t) + cE'(t)$$
$$\leq -c_0\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} k_1^2(t)|u_0|^2\,d\Gamma + c\xi(t)\int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma, \quad \forall t \geq t_0.$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t) \leq 0$, for a.e. t, then we infer that

$$(\xi\mathcal{L} + cE)'(t)$$
$$\leq \xi(t)\mathcal{L}'(t) + cE'(t)$$
$$\leq -c_0\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} k_1^2(t)|u_0|^2\,d\Gamma + c\xi(t)\int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma, \quad \forall t \geq t_0.$$

Since using (3.14),

$$F = \xi\mathcal{L} + cE \sim E, \tag{3.15}$$

we obtain for some positive constant ω,

$$F'(t) \leq -\omega\xi(t)F(t) + c\int_{\Gamma_1} k_1^2(t)|u_0|^2\,d\Gamma + c\int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma, \quad \forall t \geq t_0. \tag{3.16}$$

Case 1: If $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_1, then (3.16) reduces to

$$F'(t) \leq -\omega\xi(t)F(t), \quad \forall t \geq t_0.$$

A simple integration over (t_0, t) yields

$$F(t) \leq F(t_0)e^{-\omega\int_{t_0}^{t}\xi(s)\,ds}, \quad \forall t \geq t_0.$$

By using (3.2) and (3.15), we then obtain for some positive constant C

$$E(t) \leq CE(0)e^{-\omega\int_{0}^{t}\xi(s)\,ds}, \quad \forall t \geq t_0.$$

Thus, the estimate (3.11) is proved.

Case 2: If $(u_0, \frac{\partial u_0}{\partial \nu}) \neq (0,0)$ on Γ_1, then (3.16) gives

$$F'(t) \leq -\omega\xi(t)F(t) + ck_0(t), \quad \forall t \geq t_0, \tag{3.17}$$

where

$$k_0(t) = \int_{\Gamma_1} k_1^2(t)|u_0|^2\,d\Gamma + \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma.$$

In this case, we introduce

$$G(t) := F(t) - ce^{-\omega\int_{t_0}^{t}\xi(s)\,ds}\int_{t_0}^{t} k_0(s)e^{\omega\int_{t_0}^{s}\xi(\tau)\,d\tau}\,ds. \tag{3.18}$$

A simple differentiation of G, using (3.17), leads to

$$G'(t) = F'(t) + \omega\xi(t)ce^{-\omega\int_{t_0}^t \xi(s)\,ds} \int_{t_0}^t k_0(s)e^{\omega\int_{t_0}^s \xi(\tau)\,d\tau}\,ds - ck_0(t) \leq -\omega\xi(t)G(t), \quad \forall t \geq t_0.$$

Again, a simple integration over (t_0, t) yields

$$G(t) \leq G(t_0)e^{-\omega\int_{t_0}^t \xi(s)\,ds}, \quad \forall t \geq t_0,$$

which implies, for all $t \geq t_0$,

$$F(t) \leq \left(F(t_0) + c\int_{t_0}^t k_0(s)e^{\omega\int_{t_0}^s \xi(\tau)\,d\tau}\,ds\right)e^{-\omega\int_{t_0}^t \xi(s)\,ds}.$$

By using (3.15), we deduce

$$E(t) \leq C\left(E(0) + \int_0^t k_0(s)e^{\omega\int_{t_0}^s \xi(\tau)\,d\tau}\,ds\right)e^{-\omega\int_{t_0}^t \xi(s)\,ds}, \quad \forall t \geq t_0.$$

Consequently, by the boundedness of ξ, (3.12) is established. $\qquad\square$

Remark 3.1 Estimates (3.11) and (3.12) are also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $E(t)$ and ξ.

Remark 3.2 Note that the exponential and polynomial decay estimates are only particular cases of (3.11) and (3.12). More precisely, we have exponential decay for $\xi_1(t) \equiv c_1$ and $\xi_2(t) \equiv c_2$ and polynomial decay for $\xi_1(t) = c_1(1 + t)^{-1}$ and $\xi_2(t) \equiv c_2$, where c_1 and c_2 are positive constants.

Example 3.1 As in [20], we give some examples to illustrate the energy decay rates given by (3.11).

(1) If $k_1(t) = k_2(t) = ae^{-b(1+t)^p}$, $0 < p \leq 1$, then for $i = 1, 2$, $k_i''(t) \geq -\xi(t)k_i'(t)$, where $\xi(t) = bp(1 + t)^{p-1}$. For suitably chosen positive constants a and b, k_i satisfies (H) and (3.11) gives

$$E(t) \leq ce^{-\omega b(1+t)^p}.$$

(2) If $k_1(t) = \frac{a_1}{(1+t)^q}$, $q > 0$, and $k_2(t) = a_2e^{-b(1+t)^p}$, $0 < p \leq 1$, then for $i = 1, 2$, $k_i''(t) \geq -\xi(t)k_i'(t)$, where $\xi(t) = q(1 + t)^{-1}$. Then

$$E(t) \leq \frac{c}{(1 + t)^{\omega q}}.$$

The above two examples are included in the following more general one.

(3) For any nonincreasing functions $k_i(t)$, $i = 1, 2$, which satisfy (H), $\xi_i = -\frac{k_i'}{k_i}$ are also nonincreasing differentiable functions, and $c\xi_1(t) \leq \xi_2(t)$, for some $0 < c \leq 1$, (3.11) gives

$$E(t) \leq c\big[k_1(t)\big]^\omega.$$

Author's contributions

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