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Positive periodic solutions for a second-order functional differential equation

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Abstract

In this paper, the existence results of positive ω-periodic solutions are obtained for the second-order functional differential equation

$$\ddot{u}(t) = f(t, u(t), \dot{u}(t-\tau_1(t)), \ldots, \dot{u}(t-\tau_n(t))),$$

where $f : \mathbb{R} \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function which is ω-periodic in t, $\tau_i \in C(\mathbb{R}, [0,\infty))$ is a ω-periodic function, $i = 1, 2, \ldots, n$. Our discussion is based on the fixed point index theory in cones.

MSC: 34C25; 47H10

Keywords: functional differential equation; positive periodic solution; cone; fixed point index

1 Introduction

In this paper, we discuss the existence of positive ω-periodic solutions of the second-order functional differential equation with the delay terms of first-order derivative in nonlinearity,

$$\ddot{u}(t) = f\big(t, u(t), \dot{u}\big(t-\tau_1(t)\big), \ldots, \dot{u}\big(t-\tau_n(t)\big)\big), \quad t \in \mathbb{R}, \tag{1}$$

where $f : \mathbb{R} \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function which is ω-periodic in t and $\tau_i \in C(\mathbb{R}, [0,\infty))$ is a ω-periodic delay function, $i = 1, 2, \ldots, n$.

For the second-order differential equation without delay and the first-order derivative term in nonlinearity,

$$\ddot{u}(t) = f\big(t, u(t)\big), \quad t \in \mathbb{R}, \tag{2}$$

the existence problems of periodic solutions have attracted many authors' attention and concern. Many theorems and methods of nonlinear functional analysis have been applied to research the periodic problems of Equation (2), such as the upper and lower solutions method and monotone iterative technique [1–4], the continuation method of topological degree [5–7], variational method and critical point theory [8–10], the theory of the fixed point index in cones [11–16], *etc.*

In recent years, the existence of periodic solutions for the second-order delayed differential equations have also been researched by many authors; see [17–24] and the references

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therein. In some practice models, only positive periodic solutions are significant. In [20, 21, 23], the authors obtained the existence of positive periodic solutions for some delayed second-order differential equations as a special form of the following equation:

$$\ddot{u}(t) + b(t)\dot{u}(t) + a(t)u(t) = f\big(t, u\big(t-\tau_1(t)\big), \ldots, u\big(t-\tau_n(t)\big)\big), \quad t \in \mathbb{R}, \tag{3}$$

by using Krasnoselskii's fixed point theorem of cone mapping or the theory of the fixed point index in cones. In these works, the positivity of Green's function of the corresponding linear second-order periodic problems plays an important role. The positivity guarantees that the integral operators of the second-order periodic problems are cone-preserving in the cone

$$P = \big\{u \in C[0,\omega] \mid u(t) \geq \sigma\|u\|, t \in [0,\omega]\big\} \tag{4}$$

in the Banach space $C[0,\omega]$, where $\sigma > 0$ is a constant. Hence, the fixed point theorems of cone mapping can be applied to periodic problems of the second-order delay equation (3) as well as Equation (2) (for Equation (2), see [11–16]). However, few people consider the existence of positive periodic solutions of Equation (1). Since the nonlinearity of Equation (1) explicitly contains the delayed first-order derivative term, the corresponding integral operator has no definition on the cone P. Thus, the argument methods used in [20, 21, 23] are not applicable to Equation (1).

The purpose of this paper is to discuss the existence of positive periodic solutions of Equation (1). We will use a different method to treat Equation (1). Our main results will be given in Section 3. Some preliminaries to discuss Equation (1) are presented in Section 2.

2 Preliminaries

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic function $u(t)$ with the norm $\|u\|_C = \max_{0\leq t\leq\omega}|u(t)|$. Let $C^1_\omega(\mathbb{R})$ be the Banach space of all continuous differentiable ω-periodic function $u(t)$ with the norm

$$\|u\|_{C^1} = \|u\|_C + \|\dot{u}\|_C.$$

Generally, $C^n_\omega(\mathbb{R})$ denotes the nth-order continuous differentiable ω-periodic function space for $n \in \mathbb{N}$. Let $C^+_\omega(\mathbb{R})$ be the cone of all nonnegative functions in $C_\omega(\mathbb{R})$.

Let $M \in (0, \frac{\pi^2}{\omega^2})$ be a constant. For $h \in C_\omega(\mathbb{R})$, we consider the linear second-order differential equation

$$\ddot{u}(t) + Mu(t) = h(t), \quad t \in \mathbb{R}. \tag{5}$$

The ω-periodic solutions of Equation (5) are closely related to the linear second-order boundary value problem

$$\begin{cases} \ddot{u}(t) + Mu(t) = 0, & 0 \leq t \leq \omega, \\ u(0) - u(\omega) = 0, & \dot{u}(0) - \dot{u}(\omega) = 1, \end{cases} \tag{6}$$

see [14]. It is easy to see that problem (6) has a unique solution which is explicitly given by

$$\mathrm{U}(t) = \frac{\cos\beta(t-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}, \quad 0 \le t \le \omega, \tag{7}$$

where $\beta = \sqrt{M}$. By [14, Lemma 1], we have

Lemma 2.1 *Let $M \in (0, \frac{\pi^2}{\omega^2})$. Then, for every $h \in C_\omega(\mathbb{R})$, the linear equation* (5) *has a unique ω-periodic solution $u(t)$ which is given by*

$$u(t) = \int_{t-\omega}^{t} \mathrm{U}(t-s)h(s)\,ds := Sh(t), \quad t \in \mathbb{R}. \tag{8}$$

Moreover, $S : C_\omega(\mathbb{R}) \to C^1_\omega(\mathbb{R})$ is a completely continuous linear operator.

Since $\mathrm{U}(t) > 0$, for every $t \in [0,\omega]$, by (8), if $h \in C^+_\omega(\mathbb{R})$ and $h(t) \not\equiv 0$, then the ω-periodic solution of Equation (5) $u(t) > 0$ for every $t \in \mathbb{R}$, and we term it the positive ω-periodic solution. Let

$$\begin{aligned}
&\overline{\mathrm{U}} = \max_{0\le t\le\omega} \mathrm{U}(t) = \frac{1}{2\beta\sin\frac{\beta\omega}{2}}, \qquad \underline{\mathrm{U}} = \min_{0\le t\le\omega} \mathrm{U}(t) = \frac{\cos\frac{\beta\omega}{2}}{2\beta\sin\frac{\beta\omega}{2}},\\
&\overline{\mathrm{U}}_1 = \max_{0\le t\le\omega} \left|\dot{\mathrm{U}}(t)\right| = \max_{0\le t\le\omega} \frac{|\sin\beta(t-\frac{\omega}{2})|}{2\sin\frac{\beta\omega}{2}} = \frac{1}{2},\\
&\sigma = \frac{\underline{\mathrm{U}}}{\overline{\mathrm{U}}} = \cos\frac{\beta\omega}{2}, \qquad C_0 = \frac{\overline{\mathrm{U}}_1}{\underline{\mathrm{U}}} = \beta\tan\frac{\beta\omega}{2}.
\end{aligned} \tag{9}$$

Define a set K in $C^1_\omega(\mathbb{R})$ by

$$K = \left\{u \in C^1_\omega(\mathbb{R}) \mid u(t) \ge \sigma\|u\|_C, \left|\dot{u}(\tau)\right| \le C_0 u(t), \tau, t \in \mathbb{R}\right\}. \tag{10}$$

It is easy to verify that K is a closed convex cone in $C^1_\omega(\mathbb{R})$.

Lemma 2.2 *Let $M \in (0, \frac{\pi^2}{\omega^2})$. Then, for every $h \in C^+_\omega(\mathbb{R})$, the positive ω-periodic solution of Equation* (5) *$u = Sh \in K$. Namely, $S(C^+_\omega(\mathbb{R})) \subset K$.*

Proof Let $h \in C^+_\omega(\mathbb{R})$, $u = Sh$. For every $t \in \mathbb{R}$, from (8) it follows that

$$u(t) = \int_{t-\omega}^{t} \mathrm{U}(t-s)h(s)\,ds \le \overline{\mathrm{U}}\int_{t-\omega}^{t} h(s)\,ds = \overline{\mathrm{U}}\int_0^\omega h(s)\,ds,$$

and therefore,

$$\|u\|_C \le \overline{\mathrm{U}}\int_0^\omega h(s)\,ds.$$

Using (8), we obtain that

$$u(t) = \int_{t-\omega}^{t} \mathrm{U}(t-s)h(s)\,ds \ge \underline{\mathrm{U}}\int_{t-\omega}^{t} h(s)\,ds = \underline{\mathrm{U}}\int_0^\omega h(s)\,ds \ge \sigma\|u\|_C.$$

For every $\tau \in \mathbb{R}$, since

$$\dot{u}(\tau) = \int_{\tau-\omega}^{\tau} \dot{\mathrm{U}}(\tau - s)h(s)\,ds,$$

we have

$$\begin{aligned}\left|\dot{u}(\tau)\right| &\leq \int_{\tau-\omega}^{\tau} \left|\dot{\mathrm{U}}(\tau - s)\right| h(s)\,ds \leq \overline{\mathrm{U}}_1 \int_{\tau-\omega}^{\tau} h(s)\,ds \\ &= \overline{\mathrm{U}}_1 \int_0^{\omega} h(s)\,ds = C_0 \underline{\mathrm{U}} \int_0^{\omega} h(s)\,ds \leq C_0 u(t).\end{aligned}$$

Hence, $u \in K$. □

Now we consider the nonlinear delay equation (1). Hereafter, we assume that the nonlinearity f satisfies the condition

(F0) There exists $M \in (0, \frac{\pi^2}{\omega^2})$ such that

$$f(t, x, y_1, \ldots, y_n) + Mx \geq 0, \quad x \geq 0, t \in \mathbb{R}, (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

Let $f_1(t, x, y_1, \ldots, y_n) = f(t, x, y_1, \ldots, y_n) + Mx$, then $f_1(t, x, y_1, \ldots, y_n) \geq 0$ for $x \geq 0$, $t \in \mathbb{R}$, $(y_1, \ldots, y_n) \in \mathbb{R}^n$, and Equation (1) is rewritten to

$$\ddot{u}(t) + Mu(t) = f_1\big(t, u(t), \dot{u}\big(t - \tau_1(t)\big), \ldots, \dot{u}\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{11}$$

For every $u \in K$, set

$$F(u)(t) := f_1\big(t, u(t), \dot{u}\big(t - \tau_1(t)\big), \ldots, \dot{u}\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{12}$$

Then $F : K \to C_\omega^+(\mathbb{R})$ is continuous. We define an integral operator $A : K \to C_\omega^1(\mathbb{R})$ by

$$Au(t) = \int_{t-\omega}^{t} \mathrm{U}(t - s)F(u)(s)\,ds = (S \circ F)(t). \tag{13}$$

By the definition of the operator S, the positive ω-periodic solution of Equation (1) is equivalent to the nontrivial fixed point of A. From assumption (F0), Lemma 2.1 and Lemma 2.2, we easily see that

Lemma 2.3 *$A(K) \subset K$ and $A : K \to K$ is completely continuous.*

We will find the non-zero fixed point of A by using the fixed point index theory in cones. We recall some concepts and conclusions on the fixed point index in [25, 26]. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with the boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i\,(A, K \cap \Omega, K)$ has a definition. One important fact is that if $i\,(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas are needed in our argument.

Lemma 2.4 ([26]) *Let Ω be a bounded open subset of E with $\theta \in \Omega$ and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leq 1$, then $i\,(A, K \cap \Omega, K) = 1$.*

Lemma 2.5 ([26]) *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

In the next section, we will use Lemma 2.4 and Lemma 2.5 to discuss the existence of positive ω-periodic solutions of Equation (1).

3 Main results

We consider the existence of positive ω-periodic solutions of the functional differential equation (1). Let $f \in C(\mathbb{R} \times [0,\infty) \times \mathbb{R}^n)$ satisfy assumption (F0) and $f(t,x,y_1,\ldots,y_n)$ be ω-periodic in t. Let C_0 be the constant defined by (9) and $I = [0,\omega]$. For convenience, we introduce the notations

$$f_0 = \liminf_{x\to 0^+} \min_{t\in I, |y_i|\leq C_0|x|, i=1,\ldots,n} \bigl(f(t,x,y_1,\ldots,y_n)/x\bigr),$$

$$f^0 = \limsup_{x\to 0^+} \max_{t\in I, |y_i|\leq C_0|x|, i=1,\ldots,n} \bigl(f(t,x,y_1,\ldots,y_n)/x\bigr),$$

$$f_\infty = \liminf_{x\to +\infty} \min_{t\in I, |y_i|\leq C_0|x|, i=1,\ldots,n} \bigl(f(t,x,y_1,\ldots,y_n)/x\bigr),$$

$$f^\infty = \limsup_{x\to +\infty} \max_{t\in I, |y_i|\leq C_0|x|, i=1,\ldots,n} \bigl(f(t,x,y_1,\ldots,y_n)/x\bigr).$$

Our main results are as follows.

Theorem 3.1 *Let $f \in C(R \times [0,\infty) \times \mathbb{R}^n)$ and $f(t,x,y_1,\ldots,y_n)$ be ω-periodic in t, $\tau_1,\ldots,\tau_n \in C^+_\omega(\mathbb{R})$. If f satisfies assumption* (F0) *and the condition*

(F1) $f^0 < 0, f_\infty > 0$,

then Equation (1) *has at least one positive ω-periodic solution.*

Theorem 3.2 *Let $f \in C(R \times [0,\infty) \times \mathbb{R}^n)$ and $f(t,x,y_1,\ldots,y_n)$ be ω-periodic in t, $\tau_1,\ldots,\tau_n \in C^+_\omega(\mathbb{R})$. If f satisfies assumption* (F0) *and the conditions*

(F2) $f_0 > 0, f^\infty < 0$,

then Equation (1) *has at least one positive ω-periodic solution.*

In Theorem 3.1, the condition (F1) allows $f(t,x,y_1,\ldots,y_n)$ to be superlinear growth on x and $y_1,\ldots,y_n$. For example,

$$f(t,x,y_1,\ldots,y_n) = x^2 + y_1^2 + \cdots + y_n^2 - \frac{1}{4}\frac{\pi^2}{\omega^2}\left(2 + \sin\frac{2\pi t}{\omega}\right)x$$

satisfies (F0) with $M = \frac{3}{4}\frac{\pi^2}{\omega^2}$ and (F1) with $f^0 = -\frac{1}{4}\frac{\pi^2}{\omega^2}$ and $f_\infty = +\infty$.

In Theorem 3.2, the condition (F2) allows $f(t,x,y_1,\ldots,y_n)$ to be sublinear growth on x and $y_1,\ldots,y_n$. For example,

$$f(t,x,y_1,\ldots,y_n) = \sqrt{x} + \sqrt{|y_1|} + \cdots + \sqrt{|y_n|} - \frac{1}{4}\frac{\pi^2}{\omega^2}\left(2 + \sin\frac{2\pi t}{\omega}\right)x$$

satisfies (F0) with $M = \frac{3}{4}\frac{\pi^2}{\omega^2}$ and (F2) with $f_0 = +\infty$ and $f^\infty = -\frac{1}{4}\frac{\pi^2}{\omega^2}$.

Proof of Theorem 3.1 Choose the working space $E = C^1_\omega(\mathbb{R})$. Let $K \subset C^1_\omega(\mathbb{R})$ be the closed convex cone in $C^1_\omega(\mathbb{R})$ defined by (10) and $A : K \to K$ be the operator defined by (13). Then the positive ω-periodic solution of Equation (1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in C^1_\omega(\mathbb{R}) \mid \|u\|_{C^1} < r\}, \qquad \Omega_2 = \{u \in C^1_\omega(\mathbb{R}) \mid \|u\|_{C^1} < R\}. \tag{14}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R is large enough.

By $f^0 < 0$ and the definition of f^0, there exist $\varepsilon \in (0, M)$ and $\delta > 0$ such that

$$f(t, x, y_1, \dots, y_n) \le -\varepsilon x, \quad t \in I, 0 \le x \le \delta, |y_i| \le C_0 x, i = 1, \dots, n. \tag{15}$$

Let $r \in (0, \delta)$. We now prove that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$, namely $\lambda Au \ne u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \le 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \le 1$ such that $\lambda_0 Au_0 = u_0$, then by the definition of A and Lemma 2.1, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the delay differential equation

$$\ddot{u}_0(t) + Mu_0(t) = \lambda_0 f_1\big(t, u_0(t), \dot{u}_0\big(t - \tau_1(t)\big), \dots, \dot{u}_0\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{16}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$\begin{aligned} &0 \le \sigma\|u_0\|_C \le u_0(t) \le \|u_0\|_C \le \|u_0\|_{C^1} = r < \delta, \\ &\big|\dot{u}_0\big(t - \tau_i(t)\big)\big| \le C_0 u_0(t), \quad i = 1, \dots, n, t \in \mathbb{R}. \end{aligned} \tag{17}$$

Hence, from (15) it follows that

$$f\big(t, u_0(t), \dot{u}_0\big(t - \tau_1(t)\big), \dots, \dot{u}_0\big(t - \tau_n(t)\big)\big) \le -\varepsilon u_0(t), \quad t \in \mathbb{R}.$$

By this, (16) and the definition of f_1, we have

$$\ddot{u}_0(t) + Mu_0(t) \le \lambda_0\big(Mu_0(t) - \varepsilon u_0(t)\big) \le (M - \varepsilon)u_0(t), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_0, we obtain that

$$M \int_0^\omega u_0(t)\,dt \le (M - \varepsilon) \int_0^\omega u_0(t)\,dt.$$

Since $\int_0^\omega u_0(t)\,dt \ge \omega\sigma\|u_0\|_C > 0$, it follows that $M \le M - \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$i\,(A, K \cap \Omega_1, K) = 1. \tag{18}$$

On the other hand, since $f_\infty > 0$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$f(t, x, y_0, \dots, y_n) \ge \varepsilon_1 x, \quad t \in I, x \ge H, |y_i| \le C_0 x, i = 1, \dots, n. \tag{19}$$

Choose $R > \max\{\frac{1+C_0}{\sigma}H, \delta\}$ and let $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$, namely $u - Au \neq \mu v$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$, since $u_1 - \mu_1 e = Au_1$, by the definition of A and Lemma 2.1, $u_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$\ddot{u}_1(t) + M\big(u_1(t) - \mu_1\big) = f_1\big(t, u_1(t), \dot{u}_1\big(t - \tau_1(t)\big), \ldots, \dot{u}_1\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{20}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t) \geq \sigma \|u_1\|_C, \qquad \big|\dot{u}_1(\tau)\big| \leq C_0 u_1(t), \quad \tau, t \in I. \tag{21}$$

By the latter inequality of (21), we have that $\|\dot{u}_1\|_C \leq C_0\|u_1\|_C$. This implies that $\|u_1\|_{C^1} = \|u_1\|_C + \|\dot{u}_1\|_C \leq (1 + C_0)\|u_1\|_C$. Consequently,

$$\|u_1\|_C \geq \frac{1}{1 + C_0}\|u_1\|_{C^1}. \tag{22}$$

By (22) and the former inequality of (21), we have

$$u_1(t) \geq \sigma\|u_1\|_C \geq \frac{\sigma}{1 + C_0}\|u_1\|_{C^1} = \frac{\sigma R}{1 + C_0} > H, \quad t \in I.$$

From this, the latter inequality of (21) and (19), it follows that

$$f\big(t, u_1(t), \dot{u}_1\big(t - \tau_1(t)\big), \ldots, \dot{u}_1\big(t - \tau_n(t)\big)\big) \geq \varepsilon_1 u_1(t), \quad t \in I.$$

By this inequality, (20) and the definition of f_1, we have

$$u_1{}^{(n)}(t) + M\big(u_1(t) - \mu_1\big) \geq (M + \varepsilon_1)u_1(t), \quad t \in I.$$

Integrating this inequality on I and using the periodicity of u_1, we get that

$$M\int_0^\omega u_1(t)\,dt - \omega M\mu_1 \geq (M + \varepsilon_1)\int_0^\omega u_1(t)\,dt.$$

Since $\int_0^\omega u_1(t)\,dt \geq \omega\sigma\|u_1\|_C > 0$, from this inequality it follows that $M \geq M + \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5,

$$i\,(A, K \cap \Omega_2, K) = 0. \tag{23}$$

Now, by the additivity of fixed point index, (18) and (23), we have

$$i\,\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i\,(A, K \cap \Omega_2, K) - i\,(A, K \cap \Omega_1, K) = -1.$$

Hence, A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of Equation (1). $\square$

Proof of Theorem 3.2 Let $\Omega_1, \Omega_2 \subset C^1_\omega(\mathbb{R})$ be defined by (14). We prove that the operator A defined by (13) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R is large enough.

By $f_0 > 0$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$f(t,x,y_1,\dots,y_n) \geq \varepsilon x, \quad t \in I, 0 < x \leq \delta, |y_i| \leq C_0 x, i = 1,\dots,n. \tag{24}$$

Let $r \in (0,\delta)$ and $e(t) \equiv 1$. We prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$, namely $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_1$ and $\mu \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0 e$, since $u_0 - \mu_0 e = Au_0$, by the definition of A and Lemma 2.1, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the delay differential equation

$$\ddot{u}_0(t) + M\big(u_0(t) - \mu_0\big) = f_1\big(t, u_0(t), \dot{u}_0\big(t - \tau_1(t)\big), \dots, \dot{u}_0\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{25}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (17). From (17) and (24) it follows that

$$f_1\big(t, u_0(t), \dot{u}_0\big(t - \tau_1(t)\big), \dots, \dot{u}_0\big(t - \tau_n(t)\big)\big) \geq \varepsilon u_0(t), \quad t \in \mathbb{R}.$$

By this, (25) and the definition of f_1, we have

$$\begin{aligned} u_0''(t) + Mu_0(t) &= f_1\big(t, u_0(t), \dot{u}_0\big(t - \tau_1(t)\big), \dots, \dot{u}_0\big(t - \tau_n(t)\big)\big) + M\mu_0 \\ &\geq (M + \varepsilon) u_0(t), \quad t \in \mathbb{R}. \end{aligned}$$

Integrating this inequality on $[0,\omega]$ and using the periodicity of $u_0(t)$, we obtain that

$$M \int_0^\omega u_0(t)\,dt \geq (M + \varepsilon) \int_0^\omega u_0(t)\,dt.$$

Since $\int_0^\omega u_0(t)\,dt \geq \omega\sigma \|u_0\|_C > 0$, from this inequality it follows that $M \geq M + \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$. By Lemma 2.5, we have

$$i\,(A, K \cap \Omega_1, K) = 0. \tag{26}$$

Since $f^\infty < 0$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, M)$ and $H > 0$ such that

$$f(t,x,y_1,\dots,y_n) \leq -\varepsilon_1 x, \quad t \in I, x \geq H, |y_i| \leq C_0 x, i = 1,\dots,n. \tag{27}$$

Choosing $R > \max\{\frac{1+C_0}{\sigma} H, \delta\}$, we show that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$, namely $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 A u_1 = u_1$, then by the definition of A and Lemma 2.1, $u_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$\ddot{u}_1(t) + Mu_1(t) = \lambda_1 f_1\big(t, u_1(t), \dot{u}_1\big(t - \tau_1(t)\big), \dots, \dot{u}_1\big(t - \tau_n(t)\big)\big), \quad t \in \mathbb{R}. \tag{28}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (21). From the second inequality of (21), it follows that (22) holds. By (22) and the first inequality of (21), we have

$$u_1(t) \geq \sigma \|u_1\|_C \geq \frac{\sigma}{(1+C_0)} \|u_1\|_{C^1} = \frac{\sigma R}{(1+C_0)} > H, \quad t \in \mathbb{R}.$$

From this, the second inequality of (21) and (27), it follows that

$$f\big(t, u_1(t), \dot{u}_1\big(t-\tau_1(t)\big), \ldots, \dot{u}_1\big(t-\tau_n(t)\big)\big) \leq -\varepsilon_1 u_1(t), \quad t \in \mathbb{R}.$$

By this and (28), we have

$$\ddot{u}_1(t) + Mu_1(t) \leq \lambda_1\big(Mu_1(t) - \varepsilon_1 u_1(t)\big) \leq (M-\varepsilon_1)u_1(t), \quad t \in \mathbb{R}.$$

Integrating this inequality on $[0,\omega]$ and using the periodicity of $u_1(t)$, we obtain that

$$M\int_0^\omega u_1(t)\,dt \leq (M-\varepsilon_1)\int_0^\omega u_1(t)\,dt.$$

Since $\int_0^\omega u_1(t)\,dt \geq \omega\sigma\|u_1\|_C > 0$, from this inequality it follows that $M \leq M - \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$. By Lemma 2.4,

$$i\,(A, K \cap \Omega_2, K) = 1. \tag{29}$$

Now, from (26) and (29), it follows that

$$i\,\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i\,(A, K \cap \Omega_2, K) - i\,(A, K \cap \Omega_1, K) = 1.$$

Hence, A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of Equation (1). □

Example 1 Consider the following second-order differential equation with delay:

$$\ddot{u}(t) = a_1(t)u(t) + a_2(t)u^2(t) + a_3(t)\dot{u}^2(t-\omega/2), \quad t \in \mathbb{R}, \tag{30}$$

where $a_i(t) \in C_\omega(\mathbb{R})$, $i = 1,2,3$. If $-\frac{\pi^2}{\omega^2} < a_1(t) < 0$ and $a_2(t), a_3(t) > 0$ for $t \in [0,\omega]$, we can verify that

$$f(t,x,y) = a_1(t)x + a_2(t)x^2 + a_3(t)y^2$$

satisfies the conditions (F0) and (F1) for $n = 1$. By Theorem 3.1, the delay equation (30) has at least one positive ω-periodic solution.

Example 2 Consider the functional differential equation

$$\ddot{u}(t) = c_1(t)u(t) + c_2(t)\sqrt[3]{u^2(t)} + c_3(t)\sqrt[3]{\dot{u}^2\big(t-\tau(t)\big)}, \quad t \in \mathbb{R}, \tag{31}$$

where $c_i(t) \in C_\omega(\mathbb{R})$, $i = 1,2,3$, and $\tau \in C_\omega^+(\mathbb{R})$. If $-\frac{\pi^2}{\omega^2} < c_1(t) < 0$ and $c_2(t), c_3(t) > 0$ for $t \in [0,\omega]$. We easily see that

$$f(t,x,y) = c_1(t)x + c_2(t)|x|^{2/3} + c_3(t)|y|^{2/3}$$

satisfies the conditions (F0) and (F2) for $n = 1$. By Theorem 3.2, the functional differential equation (31) has a positive ω-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL carried out the main part of this article. All authors read and approved the final manuscript.

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