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Asymptotics of solutions of the heat equation in cones and dihedra under minimal assumptions on the boundary

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Abstract

In the first part of the paper, the authors obtain the asymptotics of Green's function of the first boundary value problem for the heat equation in an m -dimensional cone K . The second part deals with the first boundary value problem for the heat equation in the domain $K \times \mathbb{R}^{n-m}$. Here the right-hand side f of the heat equation is assumed to be an element of a weighted $L_{p,q}$ -space. The authors describe the behavior of the solution near the $(n-m)$ -dimensional edge of the domain.

Introduction

The paper is concerned with the first boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \mathcal{D} \times \mathbb{R}, \quad (1)$$

$$u = 0 \quad \text{on } (\partial \mathcal{D} \setminus M) \times \mathbb{R} \quad (2)$$

in the domain

$$\mathcal{D} = \{x = (x', x'') : x' \in K, x'' \in \mathbb{R}^{n-m}\},$$

where $K = \{x' = (x_1, \dots, x_m) : x'/|x'| \in \Omega\}$ is a cone in \mathbb{R}^m , $2 \leq m \leq n$, Ω denotes a subdomain of the unit sphere, and $M = \{x = (x', x'') : x' = 0\}$ is the $(n-m)$ -dimensional edge of \mathcal{D} . We are interested in the asymptotics of solutions in the class of the weighted Sobolev spaces $W_{p,q;\beta}^{2,l}(\mathcal{D} \times \mathbb{R})$. Here the space $W_{p,q;\beta}^{2,l}(\mathcal{D} \times \mathbb{R})$ is defined for an arbitrary integer $l \geq 0$ and real $p > 1$, $q > 1$, β as the set of all function $u(x, t)$ on $\mathcal{D} \times \mathbb{R}$ with the finite norm

$$\|u\|_{W_{p,q;\beta}^{2,l}(\mathcal{D} \times \mathbb{R})} = \left(\int_{\mathbb{R}} \left(\int_{\mathcal{D}} \sum_{|\alpha|+2k \leq 2l} |x'|^{p(\beta-2l+2k+|\alpha|)} |\partial_t^k \partial_x^\alpha u(x, t)|^p dx \right)^{q/p} dt \right)^{1/q}. \quad (3)$$

In the case $l = 0$, we write $W_{p,q;\beta}^{0,0} = L_{p,q;\beta}$. If, moreover, $\beta = 0$, then we write $L_{p,q;0} = L_{p,q}$.

For the case of smooth boundary $\partial \Omega$ (of class C^∞), the asymptotics of solutions was obtained in our previous paper [1]. For the particular case $p = q = 2$, $m = n$, we refer also to the paper [2] by Kozlov and Maz'ya, and for the case $p = q \neq 2$, $m = n = 2$, to the paper [3] by de Coster and Nicaise. The goal of the present paper is to describe the asymptotics

of solutions with a remainder in $W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$ under minimal smoothness assumptions on the boundary. Throughout the paper, we assume that $\partial\Omega \in C^{1,1}$.

The paper consists of two parts. The first part (Section 1) deals with the asymptotics of the Green function for the heat equation in the cone K . We obtain the same decomposition

$$G(x', y', t) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} \frac{\partial_t^k c_j(y', t) |x'|^{\lambda_j^+ + 2k} \phi_j(\omega_x)}{4^k k! (\sigma_j + k)_{(k)}} + R_\sigma(x', y', t)$$

as in [4, 5] (for the definition of λ_j^+ , ϕ_j , m_j , c_j and $\sigma_{(k)}$, see Section 1.1). However, the proof in [4, 5] does not work if $\partial\Omega$ is only of the class $C^{1,1}$. We give a new proof, which is completely different from that in [4, 5]. Our tools are estimates for solutions of the Dirichlet problem for the Laplace equation in a cone in weighted L_p Sobolev spaces and asymptotic formulas for solutions of this problem which were obtained in the papers [6, 7] by Maz'ya and Plamenevskii. Moreover, we use the estimates of the Green function in the recent paper [8] by Kozlov and Nazarov. In contrast to the case $\partial\Omega \in C^\infty$, the estimates for the second order x' - and y' -derivatives of the remainder R_σ contain an additional factor $(|x'|^{-1}d(x'))^{-\varepsilon}$ with a negative exponent $-\varepsilon$. Here, $d(x')$ is the distance from the boundary of ∂K .

In the second part of the paper (Section 2), we apply the results of Section 2 in order to obtain the asymptotics of solutions of the problem (1), (2) for $f \in L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})$. We show that, under a certain condition on β , there exists a solution of the form

$$u(x, t) = \sum_{\lambda_j^+ < 2-\beta-m/p} \sum_{k=0}^{m_j} \frac{(\partial_t - \Delta_{x''})^k H_j(x, t) |x'|^{\lambda_j^+ + 2k} \phi_j(\omega_x) + w(x, t)}{4^k k! (\sigma_j + k)_{(k)}}$$

with a remainder $w \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$. Here, H_j is an extension of the function

$$h_j(x'', t) = \int_{-\infty}^t \int_{\mathcal{D}} c_j(y', t - \tau) \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau,$$

Φ denotes the fundamental solution of the heat equation in \mathbb{R}^{n-m} . The proof of this result (Theorem 2.2) is essentially the same as in [1]. However, the proofs of some lemmas in [1] have to be modified under our weaker assumptions on $\partial\Omega$.

At the end of the paper, we show that the extensions of the functions h_j can be defined as

$$H_j(x, t) = (\mathcal{E}h_j)(x, t) = \int_0^\infty \int_{\mathbb{R}^{n-m}} T(\tau) R(z'') h_j(x'' - rz'', t - r^2\tau) dz'' d\tau,$$

where T and R are certain smooth functions on \mathbb{R}_+ and \mathbb{R}^{n-m} , respectively (see the beginning of Section 3 for their definition). This extends the result of [1, Corollary 4.5] to the case $p \neq q$.

1 The Green function of the heat equation in a cone

We start with the problem

$$\frac{\partial u}{\partial t} - \Delta_{x'} u = f \quad \text{in } K \times \mathbb{R}, \quad (4)$$

$$u = 0 \quad \text{on } (\partial K \setminus \{0\}) \times \mathbb{R}. \quad (5)$$

Let $G(x', y', t)$ be the Green function for the problem (4), (5). It is defined for every $y' \in K$ as the solution of the problem

$$\begin{aligned} \frac{\partial G(x', y', t)}{\partial t} - \Delta_{x'} G(x', y', t) &= \delta(x' - y') \delta(t) \quad \text{in } K \times \mathbb{R}, \\ G(x', y', t) &= 0 \quad \text{for } x' \in \partial K \setminus \{0\}, t \in \mathbb{R}, \quad G(x', y', t) = 0 \quad \text{for } t < 0. \end{aligned}$$

Furthermore, $(1 - \zeta)G(\cdot, y', \cdot) \in W_{2,\beta}^{2,1}(K \times \mathbb{R})$ if $\lambda_1^- < 2 - \beta - m/2 < \lambda_1^+$ (λ_1^\pm are defined below), and ζ is a function in $C_0^\infty(K \times \mathbb{R})$ equal to one in a neighborhood of the point $(x', t) = (y', 0)$. Here $W_{2,\beta}^{2,1}(K \times \mathbb{R})$ is the space of all functions $u = u(x', t)$ on $K \times \mathbb{R}$ such that $|x'|^{\beta-2+2k+|\alpha|} \partial_t^k \partial_{x'}^\alpha u \in L_2(K \times \mathbb{R})$ for $2k + |\alpha| \leq 2$. The goal of this section is to describe the behavior of the Green function for $|x'| < \sqrt{t}$.

1.1 Asymptotics of Green's function

Let $\{\Lambda_j\}_{j=1}^\infty$ be the nondecreasing sequence of eigenvalues of the Beltrami operator $-\delta$ on Ω (with the Dirichlet boundary condition) counted with their multiplicities, and let $\{\phi_j\}_{j=1}^\infty$ be an orthonormal (in $L_2(\Omega)$) sequence of eigenfunctions corresponding to the eigenvalues Λ_j . Furthermore, we define

$$\lambda_j^\pm = \frac{2-m}{2} \pm \sqrt{(1-m/2)^2 + \Lambda_j} \quad \text{and} \quad \sigma_j = \lambda_j^+ - 1 + \frac{m}{2}.$$

This means that λ_j^\pm are the solutions of the quadratic equation $\lambda(m-2+\lambda) = \Lambda_j$. Obviously, $\lambda_j^+ > 0$ and $\lambda_j^- < 2-m$ for $j = 1, 2, \dots$

By [8, Theorem 3],

$$\begin{aligned} |\partial_t^k \partial_{x'}^\alpha \partial_{y'}^\gamma G(x', y', t)| &\leq ct^{-k-(m+|\alpha|+|\gamma|)/2} \left(\frac{|x'|}{|x'| + \sqrt{t}} \right)^{\lambda_1^+ - |\alpha| - \varepsilon} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \\ &\quad \times \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon_\alpha} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |x' - y'|^2}{t}\right) \end{aligned} \quad (6)$$

for $|\alpha| \leq 2$, $|\gamma| \leq 2$. Here $d(x')$ denotes the distance of the point x' from the boundary ∂K . Furthermore, ε_α is defined as zero for $|\alpha| \leq 1$, while ε_α is an arbitrarily small positive real number if $|\alpha| = 2$. Actually, the estimate (6) is proved in [8] only for $k = 0$, but for a more general class of operators, parabolic operators with discontinuous in time coefficients. If the coefficients in [8] do not depend on t , then one can use the same argument as in the proof of [8, Theorem 3] when treating the derivatives along the edge of the domain $\mathcal{D} = K \times \mathbb{R}^{n-m}$. This argument shows that the k th derivative with respect to t will bring only an additional factor t^{-k} to the right-hand side of (6).

The following lemma will be applied in the proof of Lemma 1.2. Here and in the sequel, we use the notation $r = |x'|$ and $\omega_x = x'/|x'|$.

Lemma 1.1 *Let $G(x', y', t)$ be the Green function introduced above, and let $G_j(r, \rho, t)$ denote the Green function of the initial-boundary value problem*

$$\begin{aligned} \partial_t U(r, t) - r^{-2}((r\partial_r)^2 + (m-2)r\partial_r - \Lambda_j)U(r, t) &= 0 \quad \text{for } r > 0, t > 0, \\ U(0, t) &= 0 \quad \text{for } t > 0, \quad U(r, 0) = \Phi(r) \quad \text{for } r > 0. \end{aligned}$$

Then

$$\int_{\Omega} G(x', y', t) \phi_j(\omega_x) d\omega_x = |y'|^{1-m} G_j(|x'|, |y'|, t) \phi_j(\omega_y). \quad (7)$$

Proof The solution of the problem

$$(\partial_t - \Delta_{x'}) u(x', t) = 0 \quad \text{for } x' \in K, t > 0, \quad (8)$$

$$u(x', t) = 0 \quad \text{for } x' \in \partial K, t > 0, \quad u(x', 0) = \phi(x') \quad (9)$$

is given by the formula

$$u(x', t) = \int_K G(x', y', t) \phi(y') dy'.$$

We define

$$U_j(r, t) = \int_{\Omega} u(x', t) \phi_j(\omega_x) d\omega_x.$$

Then it follows from (8) and (9) that

$$\begin{aligned} & \partial_t U_j(r, t) - r^{-2} ((r\partial_r)^2 + (m-2)r\partial_r - \Delta_j) U_j(r, t) \\ &= \int_{\Omega} (\partial_t - r^{-2} ((r\partial_r)^2 + (m-2)r\partial_r - \Delta_j)) u(x') \phi_j(\omega_x) d\omega_x \\ &= \int_{\Omega} (\partial_t - \Delta_{x'}) u(x') \phi_j(\omega_x) d\omega_x = 0. \end{aligned}$$

Furthermore,

$$U_j(r, 0) = \Phi_j(r) \stackrel{\text{def}}{=} \int_{\Omega} \phi(x') \phi_j(\omega_x) d\omega_x.$$

Therefore,

$$\begin{aligned} U_j(r, t) &= \int_0^\infty G_j(r, \rho, t) \Phi_j(\rho) d\rho = \int_0^\infty \int_{\Omega} G_j(r, \rho, t) \phi_j(\omega_y) \phi(y') d\omega_y d\rho \\ &= \int_K G_j(r, |y'|, t) \phi_j(\omega_y) \phi(y') |y'|^{1-m} dy'. \end{aligned}$$

Comparing this with the formula

$$U_j(r, t) = \int_{\Omega} u(x', t) \phi_j(\omega_x) d\omega_x = \int_K \int_{\Omega} G(x', y', t) \phi_j(\omega_x) d\omega_x \phi(y') dy',$$

we get (7). □

In the sequel, σ is an arbitrary real number satisfying the conditions

$$\sigma > \lambda_1^-, \quad \sigma \neq \lambda_j^+ \quad \text{for all } j. \quad (10)$$

We define $G_\sigma(x', y', t) = 0$ for $\sigma < \lambda_1^+$, while

$$G_\sigma(x', y', t) = \sum_{\lambda_j^+ < \sigma} u_j^{(m_j)}(x', \partial_t) c_j(y', t) \quad \text{for } \sigma > \lambda_1^+, \quad (11)$$

where

$$u_j^{(k)}(x', \partial_t) = r^{\lambda_j^+} \phi_j(\omega_x) \sum_{\mu=0}^k \frac{r^{2\mu} \partial_t^\mu}{4^\mu \mu! (\sigma_j + \mu)_{(\mu)}}, \quad (12)$$

$$c_j(y', t) = \frac{2}{\Gamma(1 + \sigma_j)} |y'|^{\lambda_j^- - 2} \left(\frac{|y'|^2}{4t} \right)^{\sigma_j + 1} \phi_j(\omega_y) \exp\left(-\frac{|y'|^2}{4t}\right), \quad (13)$$

and $m_j = \left\lfloor \frac{\sigma - \lambda_j^+}{2} \right\rfloor$. Here, we used the notation

$$\sigma_{(\mu)} = \sigma(\sigma - 1) \cdots (\sigma - \mu + 1) \quad \text{for } \mu = 1, 2, \dots \quad \text{and} \quad \sigma_{(0)} = 1.$$

We define $V_{p,\beta}^l(K)$ as the weighted Sobolev space with the norm

$$\|u\|_{V_{p,\beta}^l(K)} = \left(\int_K \sum_{|\alpha| \leq l} r^{p(\beta - l + |\alpha|)} |\partial_x^\alpha u(x)|^p dx \right)^{1/p}$$

for $1 < p < \infty$ and integer $l \geq 0$.

Lemma 1.2 Suppose that σ is a real number such that $\sigma > \lambda_1^-$ and $(\sigma - \lambda_j^+)/2$ is not integer for $\lambda_j^+ \leq \sigma$. Furthermore, let $1 < p < \infty$ and $\beta = 2 - \sigma - m/p$. Then

$$G(x', y', t) = G_\sigma(x', y', t) + R_\sigma(x', y', t),$$

where $\partial_t^k \partial_{y'}^\gamma R_\sigma(\cdot, y', t) \in V_{p,\beta}^2(K)$ for $y' \in K$, $t > 0$, $|\gamma| \leq 2$.

Proof We prove the lemma by induction in $m_1 = \lfloor (\sigma - \lambda_1^+)/2 \rfloor$.

First, let $\lambda_1^- < \sigma < \lambda_1^+$. Then it follows from [7, Corollary 4.1 and Theorem 4.2] (see also [6, Theorem 3.2]) that $\partial_t^k \partial_{y'}^\gamma G(\cdot, y', t) \in V_{p,\beta}^2(K)$ for all $y' \in K$, $t > 0$, $|\gamma| \leq 2$, where $\beta = 2 - \sigma - m/p$. Thus, the assertion of the lemma is true for $\sigma < \lambda_1^+$.

Suppose the assertion is proved for $\sigma < \lambda_1^+ + 2l$. Now let $\lambda_1^+ + 2l < \sigma < \lambda_1^+ + 2(l+1)$. We set $\sigma' = \sigma - 2$ if $l > 0$ and $\sigma' = \lambda_1^+ - \varepsilon$ if $l = 0$, where ε is a sufficiently small positive number. Then

$$\left\lfloor \frac{\sigma' - \lambda_j^+}{2} \right\rfloor = \left\lfloor \frac{\sigma - \lambda_j^+}{2} \right\rfloor - 1 = m_j - 1 \quad \text{for } \lambda_j^+ < \sigma'.$$

By the induction hypothesis, we have

$$G(x', y', t) = G_{\sigma'}(x', y', t) + R_{\sigma'}(x', y', t),$$

where $G_{\sigma'}$ is given by (11) (with σ' instead of σ and $m_j - 1$ instead of m_j), $\partial_t^k \partial_{y'}^\gamma R_{\sigma'}(\cdot, y', t) \in V_{p,\beta'}^2(K)$, $\beta' = 2 - \sigma' - m/p$. The coefficients $c_j(y', t)$ in $G_{\sigma'}$ are given by (13) and satisfy the

equation $(\partial_t - \Delta_{y'})c_j(y', t) = 0$. Therefore,

$$(\partial_t - \Delta_{y'})R_{\sigma'}(x', y', t) = 0$$

for $x', y' \in K$, $t > 0$. Obviously, $G_{\sigma'}(ax', ay', a^2t) = a^{-m}G_{\sigma'}(x', y', t)$ for $a > 0$. Using the same equality for the Green function $G(x', y', t)$, we obtain

$$R_{\sigma'}(ax', ay', a^2t) = a^{-m}R_{\sigma'}(x', y', t) \quad \text{for } a > 0.$$

Furthermore,

$$\begin{aligned} \Delta_{x'}R_{\sigma'}(x', y', t) &= \Delta_{x'}G(x', y', t) - \Delta_{x'}G_{\sigma'}(x', y', t) \\ &= (\partial_t - \Delta_{x'})G_{\sigma'}(x', y', t) + \partial_t R_{\sigma'}(x', y', t) \\ &= (\partial_t - \Delta_{x'}) \sum_{\lambda_j^+ < \sigma'} \sum_{k=0}^{m_j-1} \frac{\partial_t^k c_j(y', t)}{4^k k! (\sigma_j + k)_{(k)}} r^{\lambda_j^+ + 2k} \phi_j(\omega_x) + \partial_t R_{\sigma'}(x', y', t). \end{aligned}$$

Using the formula

$$\Delta_{x'} r^{\lambda_j^+ + 2k} \phi_j(\omega) = 4k(\sigma_j + k) r^{\lambda_j^+ + 2k-2} \phi_j(\omega_x),$$

we get

$$\begin{aligned} \Delta_{x'}R_{\sigma'}(x', y', t) &= \sum_{\lambda_j^+ < \sigma'} \frac{\partial_t^{m_j} c_j(y', t) r^{\lambda_j^+ + 2m_j-2} \phi_j(\omega_x)}{4^{m_j-1} (m_j-1)! (\sigma_j + m_j - 1)_{(m_j-1)}} + \partial_t R_{\sigma'}(x', y', t) \\ &= \Delta_{x'} \Sigma' + \partial_t R_{\sigma'}(x', y', t), \end{aligned} \quad (14)$$

where

$$\Sigma' = \sum_{\lambda_j^+ < \sigma'} \frac{\partial_t^{m_j} c_j(y', t)}{4^{m_j} m_j! (\sigma_j + m_j)_{(m_j)}} r^{\lambda_j^+ + 2m_j} \phi_j(\omega_x)$$

($\Sigma' = 0$ for $l = 0$). Let χ be a smooth function with compact support on $[0, \infty)$ such that $\chi(r) = 1$ for $r < 1$. Using the notation $r = |x'|$, the function χ can be also considered as a function in K . Since $\sigma' < \lambda_j^+ + 2m_j < \sigma$ for $\lambda_j^+ < \sigma'$, we have $\chi \partial_t^k \partial_{y'}^{\gamma} \Sigma'(\cdot, y', t) \in V_{p, \beta'}^2(K)$ and $(1 - \chi) \partial_t^k \partial_{y'}^{\gamma} \Sigma'(\cdot, y', t) \in V_{p, \beta'}^2(K)$ for all $y' \in K$, $t > 0$. Consequently, $\partial_t^k \partial_{y'}^{\gamma} (R_{\sigma'}(\cdot, y', t) - \chi \Sigma'(\cdot, y', t)) \in V_{p, \beta'}^2(K)$ and

$$\begin{aligned} \Delta_{x'} \partial_t^k \partial_{y'}^{\gamma} (R_{\sigma'}(\cdot, y', t) - \chi \Sigma'(\cdot, y', t)) \\ = \partial_t^{k+1} \partial_{y'}^{\gamma} R_{\sigma'}(\cdot, y', t) + \Delta_{x'} \partial_t^k \partial_{y'}^{\gamma} (1 - \chi) \Sigma'(\cdot, y', t) \in V_{p, \beta}^0(K). \end{aligned}$$

Applying [7, Theorem 4.2], we obtain

$$\begin{aligned} \partial_t^k \partial_{y'}^{\gamma} (R_{\sigma'}(x', y', t) - \chi(r) \Sigma'(x', y', t)) \\ = \sum_{\sigma' < \lambda_{\mu}^+ < \sigma} c_{\mu, k, \gamma}(y', t) r^{\lambda_{\mu}^+} \phi_{\mu}(\omega) + v_{k, \gamma}(x', y', t), \end{aligned} \quad (15)$$

where $v_{k,\gamma}(\cdot, y', t) \in V_{p,\beta}^2(K)$. The coefficients $c_{\mu,k,\gamma}$ are given by the formula

$$c_{\mu,k,\gamma}(y', t) = \int_K \partial_t^k \partial_{y'}^\gamma (\partial_t R_{\sigma'}(x', y', t) + \Delta_{x'}(1 - \chi)\Sigma'(x', y', t)) v_\mu(x') dx', \quad (16)$$

where $v_\mu(x') = -\frac{1}{2\sigma_\mu} r^{\lambda_\mu^-} \phi_\mu(\omega_x)$. The integral in (16) is well defined, since

$$\partial_t^k \partial_{y'}^\gamma (\partial_t R_{\sigma'}(\cdot, y', t) + \Delta_{x'}(1 - \chi)\Sigma'(\cdot, y', t)) \in V_{p,\beta}^0(K) \cap V_{p,\beta'}^0(K)$$

and $v_\mu \in V_{p',-\beta}^0(K) + V_{p',-\beta'}^0(K)$, $p' = p/(p-1)$, for $\sigma' < \lambda_\mu^+ < \sigma$. The remainder $v_{k,\gamma}$ and the coefficients $c_{\mu,k,\gamma}$ in (15) satisfy the estimate

$$\begin{aligned} & \|v_{k,\gamma}(\cdot, y', t)\|_{V_{p,\beta}^2(K)} + \sum_{\sigma' < \lambda_\mu^+ < \sigma} |c_{\mu,k,\gamma}(y', t)| \\ & \leq c \|\partial_t^k \partial_{y'}^\gamma (\partial_t R_{\sigma'}(\cdot, y', t) + \Delta_{x'}(1 - \chi)\Sigma'(\cdot, y', t))\|_{V_{p,\beta}^0(K) \cap V_{p,\beta'}^0(K)}. \end{aligned} \quad (17)$$

Obviously, $c_{\mu,k,\gamma}(y', t) = \partial_t^k \partial_{y'}^\gamma c_\mu(y', t) = \partial_t^k \partial_{y'}^\gamma c_{\mu,0,0}(y', t)$. This means that

$$R_{\sigma'}(x', y', t) - \chi(r)\Sigma'(x', y', t) = \sum_{\sigma' < \lambda_\mu^+ < \sigma} c_\mu(y', t) r^{\lambda_\mu^+} \phi_\mu(\omega_x) + v(x', y', t),$$

where $\partial_t^k \partial_{y'}^\gamma v(\cdot, y', t) = v_{k,\gamma}(\cdot, y', t) \in V_{p,\beta}^2(K)$. Consequently,

$$R_{\sigma'}(x', y', t) = \Sigma(x', y', t) + R_\sigma(x', y', t), \quad (18)$$

where

$$\Sigma(x', y', t) = \Sigma'(x', y', t) + \sum_{\sigma' < \lambda_\mu^+ < \sigma} c_\mu(y', t) r^{\lambda_\mu^+} \phi_\mu(\omega_x) = \sum_{\lambda_j^+ < \sigma} \frac{\partial_t^{m_j} c_j(y', t) r^{\lambda_j^+ + 2m_j} \phi_j(\omega_x)}{4^{m_j} m_j! (\sigma_j + m_j)_{(m_j)}}$$

and $R_\sigma(x', y', t) = v(x', y', t) + (\chi - 1)\Sigma'(x', y', t)$. Obviously, $\partial_t^k \partial_{y'}^\gamma R_\sigma(\cdot, y', t) \in V_{p,\beta}^2(K)$ for $|\gamma| \leq 2$. Using (18) and the equality

$$G_{\sigma'}(x', y', t) + \Sigma(x', y', t) = G_\sigma(x', y', t),$$

we conclude that

$$G(x', y', t) = G_{\sigma'}(x', y', t) + R_{\sigma'}(x', y', t) = G_\sigma(x', y', t) + R_\sigma(x', y', t).$$

It remains to show that the coefficients

$$\begin{aligned} & c_\mu(y', t) \\ & = -\frac{1}{2\sigma_\mu} \int_0^\infty \int_\Omega (\partial_t R_{\sigma'}(x', y', t) + \Delta_{x'}(1 - \chi)\Sigma'(x', y', t)) \phi_\mu(\omega_x) d\omega_x r^{\lambda_\mu^- + m - 1} dr \end{aligned} \quad (19)$$

in (15) have the form (13) for $\sigma' < \lambda_\mu^+ < \sigma$. First, note that

$$(\partial_t - \Delta_{y'})c_\mu(y', t) = 0 \quad \text{for } y' \in K, t > 0,$$

since $(\partial_t - \Delta_{y'})R_{\sigma'}(x, y, t) = 0$ and $(\partial_t - \Delta_{y'})\Sigma'(x, y, t) = 0$.

Obviously, the functions $\partial_t G_{\sigma'}(x', y', t)$ and

$$\begin{aligned} & \Delta_x(1 - \chi)\Sigma'(x', y', t) \\ &= r^{-2}((r\partial_r)^2(1 - \chi)\Sigma'(x', y', t) + (m - 2)\partial_r(1 - \chi)\Sigma'(x', y', t) + (1 - \chi)\delta_\omega\Sigma') \end{aligned}$$

contain only functions $\phi_j(\omega_x)$ with $\lambda_j^+ < \sigma'$. Thus, the orthogonality of the functions ϕ_j implies

$$\begin{aligned} & \int_{\Omega} (\partial_t R_{\sigma'}(x', y', t) + \Delta_{x'}(1 - \chi)\Sigma'(x', y', t))\phi_\mu(\omega_x) d\omega_x \\ &= \int_{\Omega} \partial_t G(x', y', t)\phi_\mu(\omega_x) d\omega_x \end{aligned} \quad (20)$$

for $\lambda_\mu^+ > \sigma'$. Applying Lemma 1.1, we conclude that $c_\mu(y', t)$ has the form

$$c_\mu(y', t) = \rho^{1-m}\phi_\mu(\omega_y)f_\mu(\rho, t), \quad (21)$$

where $\rho = |y'|$. Since $R_{\sigma'}(ax', ay', a^2t) = a^{-m}R_{\sigma'}(x', y', t)$ and $\Sigma'(ax', ay', a^2t) = a^{-m}\Sigma'(x', y', t)$ for all $a > 0$, it follows from (18) that

$$\sum_{\sigma' < \lambda_\mu^+ < \sigma} (a^{\lambda_\mu^+}c_\mu(ay', a^2t) - a^{-m}c_\mu(y', t))r^{\lambda_\mu^+}\phi_\mu(\omega_x) = a^{-m}R_\sigma(x', y', t) - R_\sigma(ax', ay', a^2t).$$

The function on the right-hand side belongs to $V_{p,\beta}^2(K)$ for all $y' \in K, t > 0, a > 0$, while the left-hand side belongs only to $V_{p,\beta}^2(K)$ if

$$c_\mu(ay', a^2t) = a^{-m-\lambda_\mu^+}c_\mu(y', t).$$

Combining the last equality with (21), we get the representation

$$c_\mu(y', t) = \rho^{-m-\lambda_\mu^+}\phi_\mu(\omega_y)h_\mu\left(\frac{\rho^2}{4t}\right) = \rho^{\lambda_\mu^+-2}\phi_\mu(\omega_y)h_\mu\left(\frac{\rho^2}{4t}\right).$$

Inserting this into the equation $(\partial_t - \Delta_{y'})c_\mu(y', t) = 0$, we obtain

$$r^2h_\mu''(r) + (r - \sigma_\mu - 1)rh_\mu'(r) + (\sigma_\mu + 1)h_\mu(r) = 0.$$

The substitution $h_\mu(r) = e^{-r}r^{\sigma_\mu+1}u(r)$ leads to the differential equation

$$r^2u''(r) + (\sigma_\mu + 1 - r)ru'(r) = 0$$

which has the solution

$$u(r) = d_1 + d_2 \int_r^1 s^{-\sigma_\mu-1} e^s ds$$

with arbitrary constants d_1 and d_2 . Consequently,

$$c_\mu(y', t) = \rho^{\lambda_\mu-2} \phi_\mu(\omega_y) \left(\frac{\rho^2}{4t} \right)^{\sigma_\mu+1} \exp\left(-\frac{\rho^2}{4t}\right) \left(d_1 + d_2 \int_{\rho^2/(4t)}^1 s^{-\sigma_\mu-1} e^s ds \right). \quad (22)$$

Using (6) and (17), one gets the estimate

$$|\partial_t^k c_\mu(y', t)| \leq C_k(t) \rho^{\lambda_1^+-\varepsilon}$$

with certain functions C_k for $\rho = |y| < \sqrt{t}$. Thus, the constant d_2 in (22) must be zero. Integrating (19), we get

$$\int_0^\infty c_\mu(y', t) dt = -v_\mu(y') = \frac{1}{2\sigma_\mu} \rho^{\lambda_\mu} \phi_\mu(\omega_y)$$

by means of (20). Hence,

$$d_1 \rho^{\lambda_\mu-2} \phi_\mu(\omega_y) \int_0^\infty \left(\frac{\rho^2}{4t} \right)^{\sigma_\mu+1} \exp\left(-\frac{\rho^2}{4t}\right) dt = \frac{1}{2\sigma_\mu} \rho^{\lambda_\mu} \phi_\mu(\omega_y).$$

The integral on the left-hand side is equal to $\frac{1}{4} \rho^2 \Gamma(\sigma_\mu)$. Thus, we get $u(r) = d_1 = 2/\Gamma(\sigma_\mu + 1)$ and

$$h_\mu(r) = \frac{2}{\Gamma(\sigma_\mu + 1)} r^{\sigma_\mu+1} e^{-r}.$$

This means that the formula (13) is valid for the coefficients c_j if $\sigma' < \lambda_j^+ < \sigma$. The proof of the lemma is complete. \square

1.2 Point estimates for the remainder in the asymptotics of Green's function

We are interested in point estimates for the remainder $R_\sigma(x', y', t)$ in Lemma 1.2 in the case $|x'| < \sqrt{t}$. For this, we need the following lemma.

Lemma 1.3 Suppose that $u \in L_{p,\beta}(K)$ and $d\nabla u \in L_{p,\beta}(K)$, where $p > m$. Then

$$\sup_{x \in K} d(x)^{m/p} r(x)^\beta |u(x')| \leq c \left(\int_K r^{p\beta} (|d(x') \nabla u(x')|^p + |u(x')|^p) dx' \right)^{1/p}$$

with a constant c independent of u .

Proof Let x'_0 be a point in K , and let B_0 be a ball centered at x'_0 with radius $d_0/2 = d(x'_0)/2$. We introduce the new coordinates $y' = d_0^{-1}x'$ and set $v(y') = u(d_0 y') = u(x')$. Obviously, the point $y'_0 = d_0^{-1}x'_0$ has the distance 1 from ∂K . Hence,

$$|v(y'_0)|^p \leq c \int_{|y'-y'_0| < 1/2} (|\nabla_{y'} v(y')|^p + |v(y')|^p) dy'.$$

This implies

$$|u(x'_0)|^p \leq cd_0^{-m} \int_{B_0} (|d_0 \nabla_{x'} u(x')|^p + |u(x')|^p) dx'.$$

Since $d_0/2 < d(x') < 3d_0/2$ and $r(x'_0)/2 < r(x') < 3r(x'_0)/2$ for $x' \in B_0$, we obtain

$$d_0^m r(x'_0)^{p\beta} |u(x'_0)|^p \leq c \int_{B_0} r^{p\beta} (|d(x') \nabla_{x'} u(x')|^p + |u(x')|^p) dx'.$$

The result follows. \square

Using the last two lemmas, we can prove the following theorem.

Theorem 1.1 Suppose that σ is a real number satisfying (10). Then

$$G(x', y', t) = G_\sigma(x', y', t) + R_\sigma(x', y', t),$$

where

$$\begin{aligned} |\partial_t^k \partial_{x'}^\alpha \partial_{y'}^\gamma R_\sigma(x', y', t)| &\leq ct^{-k-(m+|\alpha|+|\gamma|)/2} \left(\frac{|x'|}{\sqrt{t}}\right)^{\sigma-|\alpha|} \left(\frac{|y'|}{|y'|+\sqrt{t}}\right)^{\lambda_1^+-|\gamma|-\varepsilon} \\ &\quad \times \left(\frac{d(x')}{|x'|}\right)^{-\varepsilon_\alpha} \left(\frac{d(y')}{|y'|}\right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa|y'|^2}{t}\right) \end{aligned} \quad (23)$$

for $|x'| < \sqrt{t}$, $|\alpha| \leq 2$, $|\gamma| \leq 2$. Here $\varepsilon_\alpha = 0$ for $|\alpha| \leq 1$, while ε_α is an arbitrarily small positive real number if $|\alpha| = 2$.

Proof Since $G_\sigma = G_{\sigma+\varepsilon}$ for small positive ε , we may assume, without loss of generality, that $(\sigma - \lambda_j^+)/2$ is not integer for $\lambda_j^+ < \sigma$. We prove the theorem by induction in $m_1 = [(\sigma - \lambda_1^+)/2]$.

If $\lambda_1^- < \sigma < \lambda_1^+$, then the assertion of the theorem follows from [8, Theorem 3]. Suppose that $\lambda_1^+ + 2l < \sigma < \lambda_1^+ + 2(l+1)$, $l \geq 0$, and that the theorem is proved for $\sigma < \lambda_1^+ + 2l$. We set $\sigma' = \sigma - 2$ if $l > 0$. In the case $l = 0$, let σ' be an arbitrary real number satisfying the inequalities $\lambda_1^- < \sigma' < \lambda_1^+$ and $\sigma' \geq \sigma - 2$. By the induction hypothesis, we have

$$G(x', y', t) = G_{\sigma'}(x', y', t) + R_{\sigma'}(x', y', t),$$

where $G_{\sigma'}$ is given by (11) (with σ' instead of σ and $m_j - 1$ instead of m_j). Since $G_{\sigma'} = G_{\sigma'+\delta}$ for sufficiently small δ , it follows from the induction hypothesis that

$$\begin{aligned} |\partial_t^k \partial_{x'}^\alpha \partial_{y'}^\gamma R_{\sigma'}(x', y', t)| &\leq ct^{-k-(m+|\alpha|+|\gamma|)/2} \left(\frac{|x'|}{\sqrt{t}}\right)^{\sigma'+\delta-|\alpha|} \left(\frac{|y'|}{|y'|+\sqrt{t}}\right)^{\lambda_1^+-|\gamma|-\varepsilon} \\ &\quad \times \left(\frac{d(x')}{|x'|}\right)^{-\varepsilon_\alpha} \left(\frac{d(y')}{|y'|}\right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa|y'|^2}{t}\right) \end{aligned} \quad (24)$$

for $|x'| < 2\sqrt{t}$, $|\alpha| \leq 2$, $|\gamma| \leq 2$. As was shown in the proof of Lemma 1.2, the remainder $R_{\sigma'}$ admits the decomposition

$$R_{\sigma'}(x', y', t) = \Sigma(x', y', t) + R_\sigma(x', y', t),$$

where

$$\Sigma(x', y', t) = \sum_{\lambda_j^+ < \sigma} \frac{r^{\lambda_j^+ + 2m_j} \phi_j(\omega_x) \partial_t^{m_j} c_j(y', t)}{4^{m_j} m_j! (\sigma_j + m_j)_{(m_j)}}$$

and $\partial_t^k \partial_{y'}^\gamma R_\sigma(\cdot, y', t) \in V_{p, \beta}^2(K)$ for $t > 0$, $y' \in K$, $|\gamma| \leq 2$. Here $\beta = 2 - \sigma - m/p$. Furthermore (cf. (14)),

$$\begin{aligned} \Delta_{x'} R_\sigma(x', y', t) &= \Delta_{x'} (R_{\sigma'}(x', y', t) - \Sigma(x', y', t)) = \Delta_{x'} (R_{\sigma'}(x', y', t) - \Sigma'(x', y', t)) \\ &= \partial_t R_{\sigma'}(x', y', t). \end{aligned}$$

Let χ be a smooth cut-off function on the interval $[0, \infty)$, $\chi = 1$ in $[0, 1)$ and $\chi = 0$ on $(2, \infty)$. We define $\chi_1(x', t) = \chi(t^{-1/2}|x'|)$ for $x' \in K$, $t > 0$. Then

$$\Delta_{x'} (\chi_1(x', t) \partial_{y'}^\gamma \partial_t^k R_\sigma(x', y', t)) = f(x', y', t),$$

where

$$f = \chi_1 \partial_{y'}^\gamma \partial_t^{k+1} R_{\sigma'} + 2 \nabla_{x'} \chi_1 \cdot \nabla_{x'} \partial_{y'}^\gamma \partial_t^k (R_{\sigma'} - \Sigma) + (\Delta_{x'} \chi_1) \partial_{y'}^\gamma \partial_t^k (R_{\sigma'} - \Sigma).$$

Thus, by [7, Theorem 4.1], there exists a constant c such that

$$\|\chi_1(\cdot, t) \partial_{y'}^\gamma \partial_t^k R_\sigma(\cdot, y', t)\|_{V_{p, \beta}^2(K)} \leq c \|f(\cdot, y', t)\|_{V_{p, \beta}^0(K)} \quad (25)$$

for all $y' \in K$, $t > 0$, $|\gamma| \leq 2$. We estimate the norm of f . Using (24), we get

$$\begin{aligned} \|\chi_1 \partial_t^{k+1} \partial_{y'}^\gamma R_{\sigma'}(\cdot, y', t)\|_{V_{p, \beta}^0(K)} &\leq c t^{-k-1-(m+|\gamma|+\sigma'+\delta)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \exp\left(-\frac{\kappa |y'|^2}{t}\right) \\ &\quad \times \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \left(\int_{|x'| < 2\sqrt{t}} |x'|^{p(\beta+\sigma'+\delta)} dx' \right)^{1/p}. \end{aligned}$$

Here, $p(\beta + \sigma' + \delta) > -m$. Thus,

$$\begin{aligned} &\|\chi_1 \partial_t^{k+1} \partial_{y'}^\gamma R_{\sigma'}(\cdot, y', t)\|_{V_{p, \beta}^0(K)} \\ &\leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right). \end{aligned}$$

Since $\nabla_{x'} \chi_1$ vanishes outside the region $\sqrt{t} < |x'| < 2\sqrt{t}$ and $|\partial_{x'}^\alpha \chi_1(x', t)| \leq c t^{-|\alpha|/2}$, the estimate (24) also yields

$$\begin{aligned} &\|\nabla_{x'} \chi_1 \cdot \nabla_{x'} \partial_{y'}^\gamma \partial_t^k R_{\sigma'}(\cdot, y', t)\|_{V_{p, \beta}^0(K)} + \|(\Delta_{x'} \chi_1) \partial_{y'}^\gamma \partial_t^k R_{\sigma'}(\cdot, y', t)\|_{V_{p, \beta}^0(K)} \\ &\leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right). \end{aligned}$$

Finally, it follows from the inequality

$$|\partial_{y'}^j \partial_t^k c_\mu(y', t)| \leq ct^{-k-(m+|\gamma|+\lambda_\mu^+)/2} \left(\frac{|y'|}{\sqrt{t}}\right)^{\lambda_\mu^+ - |\gamma|} \exp\left(-\frac{|y'|^2}{6t}\right)$$

that

$$\begin{aligned} & \|\nabla_{x'} \chi_1 \cdot \nabla_{x'} \partial_{y'}^j \partial_t^k \Sigma(\cdot, y', t)\|_{V_{p,\beta}^0(K)} + \|(\Delta_{x'} \chi_1) \partial_{y'}^j \partial_t^k \Sigma(\cdot, y', t)\|_{V_{p,\beta}^0(K)} \\ & \leq c \sum_{\lambda_j^+ < \sigma} t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{\sqrt{t}}\right)^{\lambda_j^+ - |\gamma|} \exp\left(-\frac{|y'|^2}{6t}\right) \\ & \leq ct^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}}\right)^{\lambda_1^+ - |\gamma|} \exp\left(-\frac{|y'|^2}{8t}\right). \end{aligned}$$

Consequently, by (25),

$$\begin{aligned} \|\chi_1(\cdot, t) \partial_{y'}^j \partial_t^k R_\sigma(\cdot, y', t)\|_{V_{p,\beta}^2(K)} & \leq ct^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}}\right)^{\lambda_1^+ - |\gamma| - \varepsilon} \\ & \quad \times \left(\frac{d(y')}{|y'|}\right)^{-\varepsilon \gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right) \end{aligned} \quad (26)$$

with a positive constant κ . Applying the estimate

$$\sum_{|\alpha| \leq 1} |x'|^{\beta-2+|\alpha|+m/p} |\partial_{x'}^\alpha \chi_1(x', t) \partial_{y'}^j \partial_t^k R_\sigma(x', y', t)| \leq c \|\chi_1 \partial_{y'}^j \partial_t^k R_\sigma(\cdot, y', t)\|_{V_{p,\beta}^2(K)}$$

for $p > m$ (cf. [9, Lemma 1.2.3]), we obtain (23) for $|\alpha| \leq 1$.

It remains to prove the estimate (23) for $|\alpha| = 2$. Let $\rho(x')$ be the “regularized distance” of the point x' to the boundary ∂K , i.e., ρ is a smooth function in K satisfying the inequalities

$$c_1 d(x') \leq \rho(x') \leq c_2 d(x')$$

with positive constants c_1 and c_2 (cf. [10, Chapter VI, § 2.1]). Moreover, ρ satisfies the inequality

$$|\partial_{x'}^\alpha \rho(x')| \leq cr(x')^{1-|\alpha|}. \quad (27)$$

We consider the function

$$\nu(x', y', t) = \chi_1(x', t) \rho(x') \partial_{x_j} \partial_{y'}^j \partial_t^k R_\sigma(x', y', t)$$

for $1 \leq j \leq m$. It follows from the equation $\Delta_{x'} R_\sigma = \partial_t R_{\sigma'}$ that

$$\Delta_{x'} \nu = f_1 + f_2 + f_3,$$

where $f_1 = \chi_1 \rho \partial_{x_j} \partial_y^\gamma \partial_t^{k+1} R_\sigma$, $f_2 = (\Delta_{x'}(\chi_1 \rho)) \partial_{x_j} \partial_y^\gamma \partial_t^k R_\sigma$ and $f_3 = 2 \nabla_{x'}(\chi_1 \rho) \cdot \nabla_{x'} \partial_{x_j} \partial_y^\gamma \partial_t^k R_\sigma$. Using (24) and (27), we obtain

$$\|f_1(\cdot, y', t)\|_{V_{p,\beta}^0(K)} \leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right).$$

Let $\chi_2(x', t) = \chi(|x'|/(2\sqrt{t}))$. The inequalities $|\Delta_{x'}(\chi_1 \rho)| \leq c r^{-1}$ and $|\nabla_{x'}(\chi_1 \rho)| \leq c$ yield

$$\begin{aligned} & \|f_2(\cdot, y', t)\|_{V_{p,\beta}^0(K)} + \|f_3(\cdot, y', t)\|_{V_{p,\beta}^0(K)} \\ & \leq c \|\chi_2(\cdot, t) \partial_y^\gamma \partial_t^k R_\sigma(\cdot, y', t)\|_{V_{p,\beta}^2(K)} \\ & \leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right) \end{aligned}$$

(see (26)). Consequently by [7, Theorem 4.1], the function $v = \chi_1 \rho \partial_{x_j} \partial_y^\gamma \partial_t^k R_\sigma$ satisfies the estimate

$$\begin{aligned} \|v(\cdot, y', t)\|_{V_{p,\beta}^2(K)} & \leq c \|f_1 + f_2 + f_3\|_{V_{p,\beta}^0(K)} \\ & \leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right). \end{aligned}$$

Applying Lemma 1.3 to the function $u(x', y', t) = \chi_1(x', t) \partial_{x'}^\alpha \partial_y^\gamma \partial_t^k R_\sigma(x', y', t)$ with an arbitrary multi-index α with length $|\alpha| = 2$, we get

$$\begin{aligned} & \sup_{x' \in K} d(x')^{m/p} |x'|^\beta |\chi_1(x', t) \partial_{x'}^\alpha \partial_y^\gamma \partial_t^k R_\sigma(x', y', t)| \\ & \leq c \left(\int_K r^{p\beta} (|\rho \nabla_{x'} \chi_1 \partial_{x'}^\alpha \partial_y^\gamma \partial_t^k R_\sigma(x', y', t)|^p + |(\chi_1 \partial_{x'}^\alpha \partial_y^\gamma \partial_t^k R_\sigma)(x', y', t)|^p) dx' \right)^{1/p} \\ & \leq c (\|\chi_1 \rho \nabla_{x'} \partial_y^\gamma \partial_t^k R_\sigma(\cdot, y', t)\|_{V_{p,\beta}^2(K)} + \|(\chi_1 \partial_{x'}^\alpha \partial_y^\gamma \partial_t^k R_\sigma)(\cdot, y', t)\|_{V_{p,\beta}^2(K)}) \\ & \leq c t^{-k-(m+|\gamma|+\sigma)/2} \left(\frac{|y'|}{|y'| + \sqrt{t}} \right)^{\lambda_1^+ - |\gamma| - \varepsilon} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_\gamma} \exp\left(-\frac{\kappa |y'|^2}{t}\right) \end{aligned}$$

for $|\alpha| = 2$, $|\gamma| \leq 2$, $p > m$. Since p can be chosen arbitrarily large, the estimate (23) holds in the case $|\alpha| = 2$. The proof is complete. \square

2 Asymptotics of solutions of the problem in \mathcal{D}

Now we consider the problem (1), (2) in the domain \mathcal{D} . Throughout this section, it is assumed that $f \in L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})$, where p and β satisfy the inequalities

$$2 - \beta - m/p > \lambda_1^- = 2 - m - \lambda_1^+ \quad \text{and} \quad 2 - \beta - m/p \neq \lambda_j^+ \quad \text{for } j = 1, 2, \dots, \quad (28)$$

and q is an arbitrary real number > 1 . Let $G(x', y', t)$ be the Green function of the problem (4), (5). Furthermore, let

$$\Phi(x'', y'', t) = (4\pi t)^{(m-n)/2} \exp\left(-\frac{|x'' - y''|^2}{4t}\right)$$

be the fundamental solution of the heat equation in \mathbb{R}^{n-m} . Then

$$\mathcal{G}(x, y, t) = G(x', y', t) \Phi(x'', y'', t)$$

is the Green function of the problem (1), (2). We consider the solution

$$u(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \mathcal{G}(x, y, t - \tau) f(y, \tau) dy d\tau \quad (29)$$

of the problem (1), (2).

We again denote by $G_{\sigma}(x', y', t)$ the function (11) introduced in Section 1. In the sequel, σ is an arbitrary real number such that

$$\sigma > 2 - \beta - m/p, \quad \lambda_j^+ \notin [2 - \beta - m/p, \sigma] \quad \text{for all } j \quad (30)$$

and

$$m_j = \left\lceil \frac{\sigma - \lambda_j^+}{2} \right\rceil = \left\lceil \frac{2 - \beta - \lambda_j^+ - m/p}{2} \right\rceil \quad \text{for } \lambda_j^+ < 2 - \beta - m/p. \quad (31)$$

Then $G_{\sigma}(x', y', t) = G_{2-\beta-m/p}(x', y', t)$. Let χ be an infinitely differentiable function on \mathbb{R}_+ = $(0, \infty)$ equal to one on the interval $(0, 1)$ and vanishing on $(2, \infty)$. We define

$$\chi_1(x', y') = \chi\left(\frac{|x'|}{|y'|}\right), \quad \chi_2(x', t, \tau) = \chi\left(\frac{|x'|}{\sqrt{t - \tau}}\right).$$

Obviously,

$$u = \Sigma + v,$$

where

$$\Sigma(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \chi_1 \chi_2 G_{\sigma}(x', y', t - \tau) \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau, \quad (32)$$

$$\begin{aligned} v(x, t) = & \int_{-\infty}^t \int_{\mathcal{D}} (G(x', y', t - \tau) - \chi_1 \chi_2 G_{\sigma}(x', y', t - \tau)) \\ & \times \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau. \end{aligned} \quad (33)$$

We also consider the decomposition

$$u = \Sigma' + w,$$

where

$$\Sigma' = \sum_{\lambda_j^+ < 2 - \beta - m/p} u_j^{(m_j)}(x', \partial_t - \Delta_{x''}) H_j(x, t) \quad (34)$$

and

$$H_j(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \chi_1(x', y') \chi_2(x', t, \tau) c_j(y', t - \tau) \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau \quad (35)$$

is an extension of the function

$$h_j(x'', t) = \int_{-\infty}^t \int_{\mathcal{D}} c_j(y', t - \tau) \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau \quad (36)$$

with c_j defined by (13). Our goal is to show that both remainders v and w are elements of the space $W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$. We start with the case $p = q$.

2.1 Estimates in weighted L_p Sobolev spaces

Let $W_{p,q;\beta}^{2,l,l}(\mathcal{D} \times \mathbb{R})$ be the weighted Sobolev space with the norm (3). Furthermore, let

$$W_{p;\beta}^{2,l,l}(\mathcal{D} \times \mathbb{R}) = W_{p,p;\beta}^{2,l,l}(\mathcal{D} \times \mathbb{R}), \quad L_{p;\beta}(\mathcal{D} \times \mathbb{R}) = W_{p;\beta}^{0,0}(\mathcal{D} \times \mathbb{R}).$$

In this subsection, we assume that $f \in L_{p;\beta}(\mathcal{D} \times \mathbb{R})$, where p and β satisfy (28). First, we prove that $\Sigma - \Sigma' \in W_{p;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$. This was shown in [1, Corollary 2.3] for the case $\partial\Omega \in C^\infty$. In the case $\partial\Omega \in C^{1,1}$, we must keep in mind that the second-order derivatives of the eigenfunctions ϕ_j must not be bounded. Then we have the estimate

$$|\partial_x^\alpha \phi_j(\omega_x)| \leq c |x'|^{-|\alpha|} \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon_\alpha} \quad (37)$$

for $|\alpha| \leq 2$, where $\varepsilon_\alpha = 0$ for $|\alpha| \leq 1$ and ε_α is an arbitrarily small positive real number if $|\alpha| \leq 1$. However, this requires only a small modification of the proof in [1].

Lemma 2.1 *Suppose that $f \in L_{p;\beta}(\mathcal{D} \times \mathbb{R})$. Then $\partial_x^\alpha \partial_t^k (\Sigma - \Sigma') \in L_{p;\beta-2+|\alpha|+2k}(\mathcal{D} \times \mathbb{R})$ and*

$$\|\partial_x^\alpha \partial_t^k (\Sigma - \Sigma')\|_{L_{p;\beta-2+|\alpha|+2k}(\mathcal{D} \times \mathbb{R})} \leq c \|f\|_{L_{p;\beta}(\mathcal{D} \times \mathbb{R})}$$

for $|\alpha| \leq 2$ and all k .

Proof A simple calculation (see the proof of [1, Corollary 1]) yields

$$\begin{aligned} \Sigma - \Sigma' &= - \sum_{\lambda_j^+ < \sigma} \int_{-\infty}^t \int_{\mathcal{D}} \chi_1(x', y') ([u_j^{(m_j)}(x', \partial_t), \chi_2] c_j(y', t - \tau)) \\ &\quad \times \Phi(x'', y'', t - \tau) f(y, \tau) dy d\tau, \end{aligned}$$

where $[u_j^{(m_j)}(x', \partial_t), \chi_2] = u_j^{(m_j)}(x', \partial_t) \chi_2 - \chi_2 u_j^{(m_j)}(x', \partial_t)$ denotes the commutator of $u_j^{(m_j)}(x', \partial_t)$ and χ_2 . Obviously, the inequalities

$$|x'| \leq 2|y'| \quad \text{and} \quad \sqrt{t - \tau} \leq |x'| \leq 2\sqrt{t - \tau}$$

are satisfied on the support of the kernel

$$K_j(x, y, t, \tau) = \chi_1(x', y') ([u_j^{(m_j)}(x', \partial_t), \chi_2] c_j(y', t - \tau)) \Phi(x'', y'', t - \tau). \quad (38)$$

Since, moreover, the eigenfunctions ϕ_j satisfy the inequality (37) for $|\alpha| \leq 2$, we obtain

$$|\partial_x^\alpha \partial_t^k K_j(x, y, t, \tau)| \leq c (t - \tau)^{-n/2} \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} |x'|^{-|\alpha|-2k-\sigma} |y'|^\sigma \exp\left(-\frac{|y'|^2 + |x'' - y''|^2}{8(t - \tau)}\right)$$

for $|\alpha| \leq 2$. Using Hölder's inequality, we obtain

$$|\partial_x^\alpha \partial_t^k (\Sigma - \Sigma')(x, t)| \leq c \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} |x'|^{-|\alpha|-2k-\sigma} A^{1/p} B^{1/p'},$$

where

$$A = \int_{t-|x'|^2}^{t-|x'|^2/4} \int_{\mathcal{D}} (t-\tau)^{-n/2} |y'|^{p\beta} |f(y, \tau)|^p \exp\left(-\frac{|y'|^2 + |x'' - y''|^2}{8(t-\tau)}\right) dy d\tau$$

and

$$B = \int_{t-|x'|^2}^{t-|x'|^2/4} \int_{\substack{\mathcal{D} \\ |y'| > |x'|/2}} (t-\tau)^{-n/2} |y'|^{p'(\sigma-\beta)} \exp\left(-\frac{|y'|^2 + |x'' - y''|^2}{8(t-\tau)}\right) dy d\tau.$$

The substitution $y' = z'\sqrt{t-\tau}$, $y'' = x'' + z''\sqrt{t-\tau}$ yields

$$\begin{aligned} B &\leq c \int_{t-|x'|^2}^{t-|x'|^2/4} (t-\tau)^{p'(\sigma-\beta)/2} d\tau \int_{|z'| > 1/2} |z'|^{p'(\sigma-\beta)} \exp\left(-\frac{|z'|^2}{8}\right) dz' \\ &\quad \times \int_{\mathbb{R}^{n-m}} \exp\left(-\frac{|z''|^2}{8}\right) dz'', \end{aligned}$$

i.e., $B \leq c |x'|^{p'(\sigma-\beta)+2}$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathcal{D}} |x'|^{p(\beta-2+|\alpha|+2k)} |\partial_x^\alpha \partial_t^k (\Sigma - \Sigma')(x, t)|^p dx dt \\ &\leq c \int_{\mathbb{R}} \int_{\mathcal{D}} |x'|^{-2} \left(\frac{d(x')}{|x'|} \right)^{-p\varepsilon} |A(x, t)| dx dt \\ &\leq c \int_{\mathbb{R}} \int_{\mathcal{D}} |y'|^{p\beta} |f(y, \tau)|^p D(y, \tau) dy d\tau, \end{aligned}$$

where

$$\begin{aligned} D(y, \tau) &= \int_{\tau}^{\tau+|y'|^2} \int_{\substack{\mathcal{D} \\ \sqrt{t-\tau} < |x'| < 2\sqrt{t-\tau}}} |x'|^{-2} \left(\frac{d(x')}{|x'|} \right)^{-p\varepsilon} (t-\tau)^{-n/2} \\ &\quad \times \exp\left(-\frac{|y'|^2 + |x'' - y''|^2}{8(t-\tau)}\right) dx dt. \end{aligned}$$

Substituting $x' = z'\sqrt{t-\tau}$ and $x'' = y'' + z''\sqrt{t-\tau}$, we obtain

$$D(y, \tau) = \int_{\tau}^{\tau+|y'|^2} (t-\tau)^{-1} \exp\left(-\frac{|y'|^2}{8(t-\tau)}\right) dt \int_{\substack{K \\ 1 < |z'| < 2}} |z'|^{-2} \left(\frac{d(z')}{|z'|} \right)^{-p\varepsilon} dz'.$$

This means that $D(y, \tau)$ is a constant. This proves the lemma. \square

Next, we estimate the first-order x -derivatives of the remainder v . For this, we employ the following lemma (cf. [11, Lemma A.1]).

Lemma 2.2 *Let \mathcal{K} be the integral operator*

$$(\mathcal{K}f)(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^n} K(x, y, t, \tau) f(y, \tau) dy d\tau \quad (39)$$

with a kernel $K(x, y, t, \tau)$ satisfying the estimate

$$|K| \leq c(t - \tau)^{-(n+2-r)/2} \left(\frac{|x'|}{|x'| + \sqrt{t - \tau}} \right)^{a+r} \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^b \frac{|x'|^{\mu-r}}{|y'|^\mu} \exp\left(\frac{-\kappa|x - y|^2}{t - \tau}\right),$$

where $\kappa > 0$, $0 < r \leq 2$, $a + b > -m$, $-\frac{m}{p} - a < \mu < m - \frac{m}{p} + b$. Then \mathcal{K} is bounded on $L_p(\mathbb{R}^n \times \mathbb{R})$.

In the proof of the following assertion, we use another decomposition of the remainder v as in [1, Lemma 2.4]. This allows us to apply directly the estimate in Theorem 1.1.

Lemma 2.3 *Let p and β satisfy the condition (28). Furthermore, let v be the function (33), where $f \in L_{p,\beta}(\mathcal{D} \times \mathbb{R})$, $1 < p < \infty$. Then $\partial_x^\alpha v \in L_{p,\beta-2+|\alpha|}(\mathcal{D} \times \mathbb{R})$ for $|\alpha| \leq 1$ and*

$$\sum_{|\alpha| \leq 1} \|\partial_x^\alpha v\|_{L_{p,\beta-2+|\alpha|}(\mathcal{D} \times \mathbb{R})} \leq c \|f\|_{L_{p,\beta}(\mathcal{D} \times \mathbb{R})}$$

with a constant c independent of f . The same is true for the function w .

Proof Obviously,

$$v = \sum_{j=1}^3 \int_{-\infty}^t \int_{\mathcal{D}} \Psi_j(x, y, t, \tau) f(y, \tau) dy d\tau,$$

where

$$\begin{aligned} \Psi_1(x, y, t, \tau) &= \chi_2(x', t, \tau) (G - G_\sigma)(x', y', t - \tau) \Phi(x'', y'', t - \tau), \\ \Psi_2(x, y, t, \tau) &= (1 - \chi_2(x', t, \tau)) G(x', y', t - \tau) \Phi(x'', y'', t - \tau) \end{aligned}$$

and

$$\Psi_3(x, y, t, \tau) = (1 - \chi_1(x', y')) \chi_2(x', t, \tau) G_\sigma(x', y', t - \tau) \Phi(x'', y'', t - \tau).$$

We show that the integral operators with the kernels

$$K_j^{(\alpha)}(x, y, t, \tau) = |x'|^{\beta-2+|\alpha|} |y'|^{-\beta} \partial_x^\alpha \Psi_j(x, y, t, \tau)$$

are bounded in $L_p(\mathcal{D} \times \mathbb{R})$ for $j = 1, 2, 3$ and $|\alpha| \leq 1$. Using Theorem 1.1, we get

$$\begin{aligned} |K_1^{(\alpha)}(x, y, t, \tau)| &\leq c \frac{|x'|^{\beta-2+|\alpha|}}{|y'|^\beta} (t - \tau)^{-(n+|\alpha|)/2} \left(\frac{|x'|}{\sqrt{t - \tau}} \right)^{\sigma-|\alpha|} \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^{\lambda_1^+ - \varepsilon} \\ &\quad \times \exp\left(-\frac{\kappa|x - y|^2}{t - \tau}\right), \end{aligned}$$

where ε is an arbitrarily small positive number. Applying Lemma 2.2 with $r = 2 - |\alpha|$, $\mu = \beta$, $a = \sigma - 2$, $b = \lambda_1^+ - \varepsilon$, we conclude that the integral operator with the kernel $K_1^{(\alpha)}(x, y, t, \tau)$ is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ for $|\alpha| \leq 1$.

Since $|x'| \leq |x'| + \sqrt{t - \tau} \leq 2|x'|$ on the support of $K_2^{(\alpha)}$, the estimate (6) implies

$$|K_2^{(\alpha)}(x, y, t, \tau)| \leq c \frac{|x'|^{\beta-2+|\alpha|}}{|y'|^\beta} (t - \tau)^{-(n+|\alpha|)/2} \left(\frac{|x'|}{|x'| + \sqrt{t - \tau}} \right)^a \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^{\lambda_1^+ - \varepsilon} \times \exp\left(-\frac{\kappa|x - y|^2}{t - \tau}\right)$$

with arbitrary real a . Thus, by Lemma 2.2, the integral operator with the kernel $K_2(x, y, t, \tau)$ is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ for $|\alpha| \leq 1$.

We consider the kernel $K_3^{(\alpha)}$. Since $G_\sigma(x', y', t)$ has the form

$$G_\sigma(x', y', t) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} c_{j,k} |x'|^{\lambda_j^+ + 2k} |y'|^{\lambda_j^+} \phi_j(\omega_x) \phi_j(\omega_y) \partial_t^k t^{-\lambda_j^+ - m/2} \exp\left(-\frac{|y'|^2}{4t}\right),$$

we get the representation

$$K_3^{(\alpha)}(x', y', t, \tau) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} K_{j,k}(x, y, t, \tau),$$

where

$$|K_{j,k}(x, y, t, \tau)| \leq c \frac{|x'|^{\beta-2+|\alpha|}}{|y'|^\beta} |x'|^{\lambda_j^+ + 2k - |\alpha|} |y'|^{\lambda_j^+} (t - \tau)^{-k - \lambda_j^+ - n/2} \exp\left(-\frac{\kappa|x - y|^2}{t - \tau}\right).$$

Here we used the fact that $|y'| \leq |x'| \leq 2\sqrt{t - \tau}$ on the support of the function $(1 - \chi_1)\chi_2$. The inequalities $|y'| \leq |x'| \leq 2\sqrt{t - \tau}$ and $\lambda_j^+ + 2k \leq \sigma$ imply

$$|K_{j,k}(x, y, t, \tau)| \leq c \frac{|x'|^{\beta-2+|\alpha|}}{|y'|^\beta} (t - \tau)^{-(n+|\alpha|)/2} \left(\frac{|x'|}{\sqrt{t - \tau}} \right)^{\sigma - |\alpha|} \left(\frac{|y'|}{\sqrt{t - \tau}} \right)^{2\lambda_1^+ - \sigma} \times \exp\left(-\frac{\kappa|x - y|^2}{t - \tau}\right).$$

It is no restriction to assume that $\sigma < 2\lambda_1^+ + m - \beta - m/p$ in addition to (30) and (31). Therefore, we can apply Lemma 2.2 with $r = 2 - |\alpha|$, $a = \sigma - 2$ and $b = 2\lambda_1^+ - \sigma$ to the integral operator with the kernel $K_{j,k}$. It follows that the integral operator with the kernel $K_3^{(\alpha)}(x, y, t, \tau)$ is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ for $|\alpha| \leq 1$. Consequently, the integral operator with the kernel

$$K^{(\alpha)}(x, y, t, \tau) = \sum_{j=1}^3 K_j^{(\alpha)}(x, y, t, \tau) = \frac{|x'|^{\beta-2+|\alpha|}}{|y'|^\beta} \sum_{j=1}^3 \partial_x^\alpha \Psi_j(x, y, t, \tau)$$

is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ for $|\alpha| \leq 1$. This proves the lemma. \square

Furthermore, the assertions of [1, Lemmas 2.5, 2.6, Theorem 2.7] are also valid if $\partial\Omega$ is only of the class $C^{1,1}$. The proof under this weaker assumption on Ω does not require any modifications of the method in [1]. We give here only the formulation of [1, Theorem 2.7].

Theorem 2.1 *Let $f \in L_{p,\beta}(\mathcal{D} \times \mathbb{R})$, where p and β satisfy the condition (28). Then there exists a solution of the problem (1), (2) which has the form*

$$u = \sum_{\lambda_j^+ < 2-\beta-m/p} u_j^{(m_j)}(x', \partial_t - \Delta_{x''}) H_j(x, t) + w,$$

where $w \in W_{p,\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$ and $u_j^{(k)}$, m_j , H_j are given by (12), (31) and (35), respectively. The functions H_j depend only on $|x'|$, x'' and t and satisfy the estimates

$$\|\partial_t^k \partial_{x''}^\gamma H_j\|_{L_{p,\beta+\lambda_j^++2k+|\gamma|-2}(\mathcal{D} \times \mathbb{R})} \leq c_{k,\gamma} \|f\|_{L_{p,\beta}(\mathcal{D} \times \mathbb{R})} \quad (40)$$

for $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$ and

$$\|\partial_t^k \partial_{x''}^\alpha \partial_{x''}^\gamma H_j\|_{L_{p,\beta+\lambda_j^++2k+|\alpha|+|\gamma|-2}(\mathcal{D} \times \mathbb{R})} \leq c_{k,\alpha,\gamma} \|f\|_{L_{p,\beta}(\mathcal{D} \times \mathbb{R})} \quad (41)$$

for all k, α, γ , $|\alpha| \geq 1$.

2.2 Weighted $L_{p,q}$ estimates for the remainder

We assume now that $f \in L_{p,q,\beta}(\mathcal{D} \times \mathbb{R})$ and consider the decomposition

$$u = \Sigma' + w$$

of the solution (29), where Σ' is defined by (34). Our goal is to show that $w \in W_{p,q,\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$ if p and β satisfy the condition (28). For the proof, we will use the next lemma which follows directly from [12, Theorem 3.8].

Lemma 2.4 *Suppose that \mathcal{K} is a linear operator on $L_p(\mathbb{R}^n \times \mathbb{R})$ satisfying the following conditions:*

- (i) $\|\mathcal{K}h\|_{L_p(\mathbb{R}^n \times \mathbb{R})} \leq c_1 \|h\|_{L_p(\mathbb{R}^n \times \mathbb{R})}$ for all $h \in L_p(\mathbb{R}^n \times \mathbb{R})$,
- (ii) $\int_{|t-t_0|>2\delta} \|(\mathcal{K}h)(\cdot, t)\|_{L_p(\mathbb{R}^n)} dt \leq c_2 \int_{\mathbb{R}} \|h(\cdot, t)\|_{L_p(\mathbb{R}^n)} dt$ for all $\delta > 0$ and for all functions h with support in the layer $|t - t_0| < \delta$ such that $\int_{\mathbb{R}} h(x, t) dt \equiv 0$.

Then the inequality

$$\|\mathcal{K}h\|_{L_{p,q}(\mathbb{R}^n \times \mathbb{R})} \leq c \|h\|_{L_{p,q}(\mathbb{R}^n \times \mathbb{R})}$$

holds for arbitrary q , $1 < q < p$. Here the constant c depends only on c_1 , c_2 , p and q .

The condition (ii) of the last lemma can be verified in some cases by means of the following lemma (cf. [8, Lemma 10]).

Lemma 2.5 Suppose that the kernel of the integral operator (39) satisfies the estimate

$$\begin{aligned} & |K(x, y, t, \tau)| \\ & \leq c \frac{\delta}{(t - \tau)^{(n+4-r)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t - \tau}} \right)^{a+r} \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^b \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon_1} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon_2} \\ & \quad \times \frac{|x'|^{\mu-r}}{|y'|^\mu} \exp\left(\frac{-\kappa|x - y|^2}{t - \tau}\right) \end{aligned}$$

for $t > t_0 + 2\delta$, $|\tau - t_0| \leq \delta$, where $\kappa > 0$, $0 \leq r \leq 2$, $a + b > -m$, $-\frac{m}{p} - a < \mu < m - \frac{m}{p} + b$, $0 \leq \varepsilon_1 < 1/p$, $0 \leq \varepsilon_2 < 1 - 1/p$. Then

$$\int_{t_0+2\delta}^{\infty} \|(\mathcal{K}h)(\cdot, t)\|_{L_p(\mathcal{D})} dt \leq c \|h\|_{L_{p,1}(\mathcal{D} \times \mathbb{R})}$$

for all $h \in L_{p,1}(\mathcal{D} \times \mathbb{R})$ with support in the layer $|t - t_0| \leq \delta$. Here, the constant c is independent of t_0 and δ .

It is more easy to estimate the remainder $v = u - \Sigma$, where Σ is defined by (32). For this reason, we estimate the difference $\Sigma - \Sigma'$ first.

Lemma 2.6 Let Σ and Σ' be the functions (32) and (34), respectively. If $f \in L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})$, then $\partial_t^k \partial_x^\alpha (\Sigma - \Sigma') \in L_{p,q;\beta-2+2k+|\alpha|}(\mathcal{D} \times \mathbb{R})$ and

$$\|\partial_t^k \partial_x^\alpha (\Sigma - \Sigma')\|_{L_{p,q;\beta-2+2k+|\alpha|}(\mathcal{D} \times \mathbb{R})} \leq c_{k,\alpha} \|f\|_{L_{p,\beta}(\mathcal{D} \times \mathbb{R})}$$

for all k and α , $|\alpha| \leq 2$. Here, the constants $c_{k,\alpha}$ are independent of f . In particular, $\Sigma - \Sigma' \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$.

Proof We have

$$\Sigma - \Sigma' = - \sum_{\lambda_j^+ < \sigma} \int_{-\infty}^t \int_{\mathcal{D}} K_j(x, y, t, \tau) f(y, \tau) dy d\tau,$$

where K_j is given by (38). Let $\mathcal{K}_{j,k,\alpha}$ be the integral operator with the kernel

$$K_{j,k,\alpha}(x, y, t, \tau) = |x'|^{\beta-2+2k+|\alpha|} |y'|^{-\beta} \partial_x^\alpha \partial_t^k K_j(x, y, t, \tau),$$

where $|\alpha| \leq 2$. As was shown in the proof of Lemma 2.1, this operator is bounded in $L_p(\mathcal{D} \times \mathbb{R})$. Now let h be a function in $L_{p,1}(\mathcal{D} \times \mathbb{R})$ with support in the layer $|t - t_0| \leq \delta$ satisfying the condition $\int_{\mathbb{R}} h(x, t) dt \equiv 0$. Then

$$(\mathcal{K}_{j,k,\alpha} h)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \left(\int_{t_0}^{\tau} \frac{\partial}{\partial s} K_{j,k,\alpha}(x, y, t, s) ds \right) h(y, \tau) dy d\tau.$$

Analogously to the proof of Lemma 2.1, we obtain

$$\left| \frac{\partial}{\partial s} K_{j,k,\alpha}(x, y, t, s) \right| \leq c(t - s)^{-1-n/2} \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-\sigma-2}}{|y'|^{\beta-\sigma}} \exp\left(-\frac{|x - y|^2}{8(t - s)}\right) \quad (42)$$

for $|\alpha| \leq 2$. Since $|x'| \leq |x'| + \sqrt{t-s} \leq 2|x'|$ and $|y'| \leq |y'| + \sqrt{t-s} \leq 3|y'|$ on the support of $K_{j,k,\alpha}(x, y, t, s)$, we can append the factors

$$\left(\frac{|x'|}{|x'| + \sqrt{t-s}} \right)^a \quad \text{and} \quad \left(\frac{|y'|}{|y'| + \sqrt{t-s}} \right)^b$$

with arbitrary exponents a and b on the right-hand side of (42). For $t > t_0 + 2\delta$ and $|\tau - s| < |\tau - t_0| < \delta$, we obviously have $(t - \tau)/2 < t - s < 2(t - \tau)$. Consequently,

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} K_{j,k,\alpha}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t - \tau)^{1+n/2}} \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-\sigma-2}}{|y'|^{\beta-\sigma}} \\ \times \left(\frac{|x'|}{|x'| + \sqrt{t-\tau}} \right)^a \left(\frac{|y'|}{|y'| + \sqrt{t-\tau}} \right)^b \exp\left(-\frac{|x-y|^2}{8(t-s)}\right)$$

for $t > t_0 + 2\delta$ and $|\tau - t_0| < \delta$, where a and b are arbitrary real numbers and ε is an arbitrarily small positive real number. Hence, by Lemmas 2.4 and 2.5, the operator $\mathcal{K}_{j,k,\alpha}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for $1 < q \leq p$.

We consider the operator $\tilde{\mathcal{K}}_{j,k,\alpha}$ with the kernel

$$\tilde{K}_{j,k,\alpha}(x, y, t, \tau) = K_{j,k,\alpha}(y, x, -\tau, -t) = (-1)^k \frac{|y'|^{\beta-2+2k+|\alpha|}}{|x'|^\beta} \partial_\tau^k \partial_y^\alpha K_j(y, x, -\tau, -t).$$

It follows from the boundedness of the operator $\mathcal{K}_{j,k,\alpha}$ in L_p that $\tilde{\mathcal{K}}_{j,k,\alpha}$ is bounded in $L_{p'}(\mathcal{D} \times \mathbb{R})$, $p' = p/(p-1)$. Furthermore, one can check that

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{K}_{j,k,\alpha}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t - \tau)^{1+n/2}} \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon} \frac{|x'|^{\sigma-\beta}}{|y'|^{\sigma-\beta+2}} \\ \times \left(\frac{|x'|}{|x'| + \sqrt{t-\tau}} \right)^a \left(\frac{|y'|}{|y'| + \sqrt{t-\tau}} \right)^b \exp\left(-\frac{|x-y|^2}{8(t-s)}\right)$$

with arbitrary a and b . Thus, as in the first part of the proof, we conclude that $\tilde{\mathcal{K}}_{j,k,\alpha}$ (and therefore also the adjoint operator of $\mathcal{K}_{j,k,\alpha}$) is bounded in $L_{p',q'}(\mathcal{D} \times \mathbb{R})$ for $1 < q' < p'$. This means that $\mathcal{K}_{j,k,\alpha}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for all $p, q > 1$. The lemma is proved. \square

By means of Lemma 2.5, it is also possible to prove the assertion of [1, Theorem 3.7] under the weaker assumption on Ω of the present paper.

Theorem 2.2 *Let $f \in L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})$, where p and β satisfy the condition (28) and q is an arbitrary real number, $1 < q < \infty$. Then there exists a solution of the problem (1), (2) which has the form*

$$u = \sum_{\lambda_j^+ < 2-\beta-m/p} u_j^{(m_j)}(x', \partial_t - \Delta_{x''}) H_j(x, t) + w,$$

where $u_j^{(m_j)}$, H_j are given by (12) and (35), respectively, and $w \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$. The functions H_j are extensions of the functions (36) depending only on $|x'|$, x'' and t and satisfy the

estimate

$$\left\| \partial_t^k \partial_x^\alpha \partial_{x'}^\gamma H_j \right\|_{L_{p,q;\beta+\lambda_j^++2k+|\alpha|+|\gamma|-2}(\mathcal{D} \times \mathbb{R})} \leq c_{k,\alpha,\gamma} \|f\|_{L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})} \quad (43)$$

for all k, α, γ such that $|\alpha| \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$.

Proof We have to show that the integral operator $\mathcal{K}^{(k,\alpha)}$ with the kernel

$$K^{(k,\alpha)}(x, y, t, \tau) = \frac{|x'|^{\beta-2+2k+|\alpha|}}{|y'|^\beta} \partial_t^k \partial_x^\alpha (G - \chi_1 \chi_2 G_\sigma)(x', y', t - \tau) \Phi(x'', y'', t - \tau)$$

is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for $2k + |\alpha| \leq 2$. For $p = q$ this is true by Theorem 2.1. Let Ψ_1 , Ψ_2 , and Ψ_3 be the same functions as in the proof of Lemma 2.3 and let

$$K_j^{(k,\alpha)}(x, y, t, \tau) = |x'|^{\beta-2+2k+|\alpha|} |y'|^{-\beta} \partial_x^\alpha \partial_t^k \Psi_j(x, y, t, \tau).$$

Then $K^{(k,\alpha)} = K_1^{(k,\alpha)} + K_2^{(k,\alpha)} + K_3^{(k,\alpha)}$. We show that the operators $K_j^{(k,\alpha)}$ satisfy the condition (ii) of Lemma 2.4. Let h be a function in $L_{p,1}(\mathcal{D} \times \mathbb{R})$ with support in the layer $|t - t_0| \leq \delta$ satisfying the condition $\int_{\mathbb{R}} h(x, t) dt = 0$ for all x . Then

$$(\mathcal{K}_j^{(k,\alpha)} h)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \left(\int_{t_0}^\tau \frac{\partial}{\partial s} K_j^{(k,\alpha)}(x, y, t, s) ds \right) h(y, \tau) dy d\tau.$$

Using Theorem 1.1, we get

$$\begin{aligned} & \left| \partial_s K_1^{(k,\alpha)}(x, y, t, s) \right| \\ & \leq c(t-s)^{-k-1-(n+|\alpha|)/2} \left(\frac{|x'|}{|x'| + \sqrt{t-s}} \right)^{\sigma-|\alpha|} \left(\frac{|y'|}{|y'| + \sqrt{t-s}} \right)^{\lambda_1^+-\varepsilon} \\ & \quad \times \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-2+2k+|\alpha|}}{|y'|^\beta} \exp\left(-\frac{\kappa|x-y|^2}{t-s}\right). \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{t_0}^\tau \frac{\partial}{\partial s} K_1^{(k,\alpha)}(x, y, t, s) ds \right| \\ & \leq c \frac{\delta}{(t-\tau)^{(n+2k+|\alpha|+2)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t-\tau}} \right)^{\sigma-|\alpha|} \left(\frac{|y'|}{|y'| + \sqrt{t-\tau}} \right)^{\lambda_1^+-\varepsilon} \\ & \quad \times \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-2+2k+|\alpha|}}{|y'|^\beta} \exp\left(-\frac{\kappa|x-y|^2}{(t-\tau)}\right) \end{aligned}$$

for $t > t_0 + 2\delta$ and $|\tau - t_0| < \delta$. Applying Lemma 2.5 with $r = 2 - 2k - |\alpha|$, $a = \sigma + 2k - 2$ and $b = \lambda_1^+ - \varepsilon$, we conclude that

$$\int_{t_0+2\delta}^\infty \left\| (\mathcal{K}_j^{(k,\alpha)} h)(\cdot, t) \right\|_{L_p(\mathcal{D})} dt \leq c \|h\|_{L_{p,1}(\mathcal{D} \times \mathbb{R})} \quad (44)$$

for $j = 1$ and $2k + |\alpha| \leq 2$. Analogously, the estimate (6) yields

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} K_2^{(k,\alpha)}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t - \tau)^{(n+2k+|\alpha|+2)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t - \tau}} \right)^a \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^{\lambda_1^+ - \varepsilon} \\ \times \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-2+2k+|\alpha|}}{|y'|^\beta} \exp\left(-\frac{\kappa|x-y|^2}{(t - \tau)}\right)$$

for $t > t_0 + 2\delta$ and $|\tau - t_0| < \delta$, where a is an arbitrary real number. Here, we used the fact that $|x'| \leq |x'| + \sqrt{t - \tau} \leq 2|x'|$ on the support of $K_2^{(k,\alpha)}$. Thus, by Lemma 2.5, the inequality (44) holds for $j = 2$ and $2k + |\alpha| \leq 2$.

Analogously to the estimation of the kernel $K_3^{(\alpha)}$ in the proof of Lemma 2.3, we obtain the estimate

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} K_3^{(k,\alpha)}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t - \tau)^{(n+2+2k+|\alpha|)/2}} \left(\frac{|x'|}{\sqrt{t - \tau}} \right)^{\sigma - |\alpha|} \left(\frac{|y'|}{\sqrt{t - \tau}} \right)^{2\lambda_1^+ - \sigma} \\ \times \left(\frac{d(x')}{|x'|} \right)^{-\varepsilon} \frac{|x'|^{\beta-2+2k+|\alpha|}}{|y'|^\beta} \exp\left(-\frac{\kappa|x-y|^2}{t - \tau}\right)$$

by means of (37). We may assume, without loss of generality, that $\sigma < 2\lambda_1^+ + m - \beta - m/p$ in addition to (30) and (31). Then we conclude from Lemma 2.5 that (44) is valid for $j = 3$ and $2k + |\alpha| \leq 2$. Hence, by Lemma 2.4, the operator $\mathcal{K}^{(k,\alpha)}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for $1 < q \leq p$ if $2k + |\alpha| \leq 2$.

In order to prove this for $q > p$, we consider the adjoint operator. Let $\tilde{\mathcal{K}}^{(k,\alpha)}$ and $\tilde{\mathcal{K}}_j^{(k,\alpha)}$ be the integral operators with the kernels

$$\tilde{K}^{(k,\alpha)}(x, y, t, \tau) = K^{(k,\alpha)}(y, x, -\tau, -t) \quad \text{and} \quad \tilde{K}_j^{(k,\alpha)}(x, y, t, \tau) = K_j^{(k,\alpha)}(y, x, -\tau, -t),$$

respectively. From the boundedness of $\mathcal{K}^{(k,\alpha)}$ in $L_p(\mathcal{D} \times \mathbb{R})$ it follows that $\tilde{\mathcal{K}}^{(k,\alpha)}$ is bounded in $L_{p'}(\mathcal{D} \times \mathbb{R})$, $p' = p/(p - 1)$. We show that

$$\int_{|t-t_0|>2\delta} \|(\tilde{\mathcal{K}}_j^{(k,\alpha)} h)(\cdot, t)\|_{L_p(\mathcal{D})} dt \leq c_2 \int_{\mathbb{R}} \|h(\cdot, t)\|_{L_p(\mathcal{D})} dt \quad (45)$$

for all $\delta > 0$, $j = 1, 2, 3$ and for all functions h with support in the layer $|t - t_0| < \delta$ such that $\int_{\mathbb{R}} h(\cdot, t) dt = 0$. Let h be such a function. Then

$$(\tilde{\mathcal{K}}_j^{(k,\alpha)} h)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \left(\int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{K}_j^{(k,\alpha)}(x, y, t, s) ds \right) h(y, \tau) dy d\tau.$$

By means of 1.1, we obtain

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{K}_1^{(k,\alpha)}(x, y, t, s) ds \right| \\ \leq c \frac{\delta}{(t - \tau)^{(n+2+2k+|\alpha|)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t - \tau}} \right)^{\lambda_1^+ - \varepsilon} \left(\frac{|y'|}{|y'| + \sqrt{t - \tau}} \right)^{\sigma - |\alpha|} \\ \times \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon} \frac{|x'|^{-\beta}}{|y'|^{-\beta+2-2k-|\alpha|}} \exp\left(-\frac{\kappa|x-y|^2}{t - \tau}\right).$$

Analogously, the estimate (6) implies

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{K}_2^{(k,\alpha)}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t-\tau)^{(n+2+2k+|\alpha|)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t-\tau}} \right)^{\lambda_1^+ - \varepsilon} \left(\frac{|y'|}{|y'| + \sqrt{t-\tau}} \right)^a \\ \times \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon} \frac{|x'|^{-\beta}}{|y'|^{-\beta+2-2k-|\alpha|}} \exp\left(-\frac{\kappa|x-y|^2}{t-\tau}\right),$$

where a is an arbitrary real number, since $|y'| \leq |y'| + \sqrt{t-\tau} \leq 2|y'|$ on the support of the function $\tilde{K}_2^{(k,\alpha)}(x, y, t, \tau)$. Applying Lemma 2.5, we obtain (45) for $2k + |\alpha| \leq 2$ and $j \leq 2$. Using the representation for G_σ , the estimate (37), and the fact that $|x'| \leq |y'| \leq 2\sqrt{t-\tau}$ on the support of $\tilde{K}_3^{(k,\alpha)}(x, y, t, \tau)$, we obtain

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{K}_3^{(k,\alpha)}(x, y, t, s) ds \right| \leq c \frac{\delta}{(t-\tau)^{(n+2+2k)/2}} \left(\frac{|x'|}{|x'| + \sqrt{t-\tau}} \right)^{2\lambda_1^+ - \sigma} \left(\frac{|y'|}{|y'| + \sqrt{t-\tau}} \right)^{\sigma} \\ \times \left(\frac{d(y')}{|y'|} \right)^{-\varepsilon} \frac{|x'|^{-\beta}}{|y'|^{-\beta+2-2k}} \exp\left(-\frac{\kappa|x-y|^2}{t-\tau}\right).$$

We may assume again that $\sigma < 2\lambda_1^+ + m - \beta - m/p$ in addition to (30) and (31). Then it follows from Lemma 2.5 that (45) is valid for $j = 3$ and $2k + |\alpha| \leq 2$. Therefore, by Lemma 2.4, the operator $\tilde{\mathcal{K}}^{(k,\alpha)}$ is bounded in $L_{p',q'}(\mathcal{D} \times \mathbb{R})$ for $1 < q' < p'$ if $2k + |\alpha| \leq 2$. This means that $\mathcal{K}^{(k,\alpha)}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for all q if $2k + |\alpha| \leq 2$. The proof of the theorem is complete. \square

3 Another representation for the coefficients

As was proved [1, Lemma 4.1], the functions H_j in Theorem 2.1 can be replaced by other extensions \tilde{H}_j of the functions $h_j(x'', t)$ provided these extensions also satisfy the conditions (40) and (41). Note that the proof of this assertion in [1] is also correct under our assumptions on the boundary of Ω . Moreover, it was proved in [1, Lemma 4.4], for the particular case $p = q$, that the extension

$$\tilde{H}_j(x, t) = (\mathcal{E}h_j)(x, t) = \int_0^\infty \int_{\mathbb{R}^{n-m}} T(\tau) R(z'') h_j(x'' - rz'', t - r^2\tau) dz'' d\tau$$

satisfies the conditions (40) and (41). Here $T(\tau)$ is a smooth function with support in $[0, \infty)$ satisfying the conditions

$$|\partial_\tau^k T(\tau)| \leq c_{k,M} \tau^{-M} \exp(-\kappa \tau^{-1}) \quad \text{for all } M > 0,$$

with certain positive constants $c_{k,M}$, κ and

$$\int T(\tau) d\tau = 1, \quad \int T(\tau) \tau^k d\tau = 0 \quad \text{for } k = 1, 2, \dots$$

Furthermore, R is a smooth function with support on the cube $[0, (n-m)^{-1/2}]^{n-m}$ having the form

$$R(x'') = R(x_{m+1}, \dots, x_n) = \prod_{j=m+1}^n \psi(x_j),$$

where

$$\int_{\mathbb{R}} \psi(s) ds = 1, \quad \int_{\mathbb{R}} s^j \psi(s) ds = 0 \quad \text{for } j = 1, 2, \dots, N_0 \quad (46)$$

with a sufficiently large integer N_0 .

We extend the result of [1, Lemma 4.4] to the case $q \neq p$. First, note that $\mathcal{E}h_j = \mathcal{K}_j f$, where \mathcal{K}_j is the integral operator

$$(\mathcal{K}_j f)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} K_j(x, y, t - \tau) f(y, \tau) dy d\tau$$

with the kernel

$$K_j(x, y, t) = r^{m-n-2} \int_0^t \int_{\mathbb{R}^{n-m}} T\left(\frac{t-s}{r^2}\right) R\left(\frac{x'' - z''}{r}\right) c_j(y', s) \Phi(y'', z'', s) dz'' ds.$$

Our goal is to show that the operator

$$L_{p,q;\beta}(\mathcal{D} \times \mathbb{R}) \ni f \rightarrow \partial_t^k \partial_x^\alpha \partial_{x''}^\gamma \mathcal{K}_j f \in L_{p,q;\beta+\lambda_j^++2k+|\alpha|+|\gamma|-2}(\mathcal{D} \times \mathbb{R})$$

is bounded if $|\alpha| \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$. Since the function $(x, t) \rightarrow (\mathcal{K}_j f)(x, t)$ depends only on the variables $r = |x'|$, x'' , and t , it suffices to prove that the operator

$$L_{p,q;\beta}(\mathcal{D} \times \mathbb{R}) \ni f \rightarrow \partial_t^k \partial_r^l \partial_{x''}^\gamma \mathcal{K}_j f \in L_{p,q;\beta+\lambda_j^++2k+l+|\gamma|-2}(\mathcal{D} \times \mathbb{R})$$

is bounded if $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$.

We define the operator $\mathcal{K}_j^{k,l,\gamma}$ as

$$\mathcal{K}_j^{k,l,\gamma} h = r^{\beta+\lambda_j^++2k+l+|\gamma|-2} \partial_t^k \partial_r^l \partial_{x''}^\gamma \mathcal{K}_j(r^{-\beta} h).$$

This means that $\mathcal{K}_j^{k,l,\gamma}$ is the integral operator with the kernel

$$K_j^{k,l,\gamma}(x, y, t, \tau) = r^{\beta+\lambda_j^++2k+l+|\gamma|-2} \rho^{-\beta} \partial_t^k \partial_r^l \partial_{x''}^\gamma K_j(x, y, t - \tau),$$

where $r = |x'|$ and $\rho = |y'|$. As was shown in [1], the operator $\mathcal{K}_j^{k,l,\gamma}$ is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ if $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$. In order to prove the boundedness in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for $q \neq p$, we verify the condition (ii) of Lemma 2.4. For this, we apply the following lemma.

Lemma 3.1 Suppose that the kernel of the integral operator (39) satisfies the condition

$$|K(x, y, t, \tau)| \leq c \frac{\delta}{(t - \tau)^{M/2}} r^{\mu+M-n-4} \rho^{-\mu} \exp\left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t - \tau}\right)$$

for $t > t_0 + 2\delta$, $|\tau - t_0| \leq \delta$, where $r = |x'|$, $\rho = |y'|$, $\kappa > 0$, $M > 4 + n - m$ and $-\frac{m}{p} - M + n + 4 < \mu < m - \frac{m}{p}$. Then

$$\int_{t_0+2\delta}^{\infty} \|(\mathcal{K}h)(\cdot, t)\|_{L_p(\mathcal{D})} dt \leq c \|h\|_{L_{p,1}(\mathcal{D} \times \mathbb{R})}$$

for all $h \in L_{p,1}(\mathcal{D} \times \mathbb{R})$ with support in the layer $|t - t_0| \leq \delta$. Here, the constant c is independent of t_0 and δ .

Proof Obviously,

$$\left(\frac{r}{\sqrt{t-\tau}}\right)^M \leq \left(\frac{r}{r+\sqrt{t-\tau}}\right)^M$$

for $M \leq 0$ and

$$\begin{aligned} \left(\frac{r}{\sqrt{t-\tau}}\right)^M &\leq c \min\left(1, \left(\frac{r}{\sqrt{t-\tau}}\right)^M\right) \exp\left(\frac{\kappa r^2}{2(t-\tau)}\right) \\ &\leq c \left(\frac{2r}{r+\sqrt{t-\tau}}\right)^M \exp\left(\frac{\kappa r^2}{2(t-\tau)}\right) \end{aligned}$$

for $M > 0$. Consequently, it follows from our assumption on K that

$$\begin{aligned} |K(x, y, t, \tau)| &\leq c \frac{\delta}{(t-\tau)^{(n+2)/2}} \left(\frac{r}{\sqrt{t-\tau}}\right)^{M-n-2} \frac{r^{\mu-2}}{\rho^\mu} \exp\left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-\tau}\right) \\ &\leq c \frac{\delta}{(t-\tau)^{(n+2)/2}} \left(\frac{r}{r+\sqrt{t-\tau}}\right)^{M-n-2} \frac{r^{\mu-2}}{\rho^\mu} \exp\left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{2(t-\tau)}\right). \end{aligned}$$

Thus, we can apply Lemma 2.5. \square

We will show that the operator $\mathcal{K}_j^{k,l,\gamma}$ satisfies the condition of the last lemma. This leads to the following assertion.

Lemma 3.2 Suppose that $p, q \in (1, \infty)$, $\lambda_j^+ < 2 - \beta - m/p$ and that at least one of the conditions $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$ is satisfied. Furthermore, we assume that the number N_0 in (46) is greater than $3 - \beta - \lambda_j^+ - m/p$. Then the operator $\mathcal{K}_j^{k,l,\gamma}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$.

Proof For the case $q = p$, we refer to [1, Lemma 4.4].

We consider the case $1 < q < p$. Let $h \in L_{p,1}(\mathcal{D} \times \mathbb{R})$ be an arbitrary function with support in the layer $|t - t_0| < \delta$ such that $\int h(x, t) dt = 0$ for all x . Then $(\mathcal{K}_j^{k,l,\gamma} h)(x, t) = 0$ for $t < t_0 + \delta$, while

$$(\mathcal{K}_j^{k,l,\gamma} h)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \left(\int_{t_0}^{\tau} \frac{\partial}{\partial s} K_j^{k,l,\gamma}(x, y, t-s) ds \right) h(y, \tau) dy d\tau \quad (47)$$

for $t > t_0 + \delta$. We verify the condition of Lemma 3.1 for the kernel of the last integral operator. To this end, we use the same decomposition

$$\partial_t^{k+1} \partial_{x''}^\gamma K_j(x, y, t-s) = \Gamma(x, y, t-s) + A(x, y, t-s) + \sum_{i=0}^k B_i(x, y, t-s)$$

for the t, x'' -derivatives of $K_j(x, y, t - s)$ as in the proof of [1, Lemma 4.4], where

$$\begin{aligned}\Gamma(x, y, t) &= \int_0^{t/2} \int_{\mathbb{R}^{n-m}} T^{(k+1)}\left(\frac{t-\xi}{r^2}\right) R^{(\gamma)}\left(\frac{x''-z''}{r}\right) c_j(y', \xi) \Phi(y'', z'', \xi) \frac{dz'' d\xi}{r^{n-m+4+2k+|\gamma|}}, \\ A(x, y, t) &= \int_0^{t/2} \int_{\mathbb{R}^{n-m}} T\left(\frac{\xi}{r^2}\right) R\left(\frac{x''-z''}{r}\right) \partial_t^{k+1} c_j(y', t-\xi) \partial_{z''}^\gamma \Phi(y'', z'', t-\xi) \frac{dz'' d\xi}{r^{n-m+2}}\end{aligned}$$

and

$$\begin{aligned}B_i(x, y, t) &= 2^i r^{m-n-2-2k+2i} T^{(k-i)}\left(\frac{t}{2r^2}\right) \int_{\mathbb{R}^{n-m}} R\left(\frac{x''-z''}{r}\right) \partial_t^i c_j(y', t/2) \partial_{z''}^\gamma \Phi(y'', z'', t/2) dz''.\end{aligned}$$

Here we used the notation $T^{(k)}(t) = \partial_t^k T(t)$ and $R^{(\gamma)}(x'') = \partial_{x''}^\gamma R(x'')$. Applying the estimates

$$\begin{aligned}\left| \partial_r^l r^{m-n-4-2k-l-|\gamma|} T^{(k+1)}\left(\frac{t-s-\xi}{r^2}\right) R^{(\gamma)}\left(\frac{x''-z''}{r}\right) \right| \\ \leq c r^{m-n-4-2k-l-|\gamma|} \left(\frac{r^2}{t-s}\right)^M \exp\left(-\frac{\kappa r^2}{t-s}\right)\end{aligned}$$

and

$$\frac{|y''-z''|^2}{4\xi} \geq \frac{|y''-z''|^2}{8\xi} + \frac{\kappa|x''-y''|^2}{4(t-s)} - \frac{\kappa r^2}{2(t-s)}$$

for $0 \leq \xi \leq (t-s)/2$, $|z''-x''| \leq r$ and $\kappa \leq 1/2$, we obtain

$$\begin{aligned}\left| \partial_r^l \Gamma(x, y, t-s) \right| &\leq c r^{m-n-4-2k-l-|\gamma|} \rho^{\lambda_j^+} \left(\frac{r^2}{t-s}\right)^M \exp\left(-\kappa \frac{2r^2 + |x''-y''|^2}{4(t-s)}\right) \\ &\quad \times \int_0^{(t-s)/2} \int_{\mathbb{R}^{n-m}} \xi^{-\lambda_j^+ - n/2} \exp\left(-\frac{2\rho^2 + |y''-z''|^2}{8\xi}\right) dz'' d\xi \\ &\leq c \frac{r^{m-n-4-2k-l-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r^2}{t-s}\right)^M \exp\left(-\kappa' \frac{r^2 + \rho^2 + |x''-y''|^2}{t-s}\right)\end{aligned}$$

with arbitrary positive M and certain positive κ' . Furthermore, the estimates

$$\left| \partial_r^l r^{m-n-2-2k+2i} T^{(k-i)}\left(\frac{t-s}{2r^2}\right) R\left(\frac{x''-z''}{r}\right) \right| \leq c r^{m-n-2-2k-l+2i} \left(\frac{r^2}{t-s}\right)^M \exp\left(-\frac{\kappa r^2}{t-s}\right)$$

and

$$\begin{aligned}\left| \partial_t^i c_j\left(y', \frac{t-s}{2}\right) \partial_{z''}^\gamma \Phi\left(y'', z'', \frac{t-s}{2}\right) \right| \\ \leq c(t-s)^{-\lambda_j^+ - i - (n+|\gamma|)/2} \rho^{\lambda_j^+} \exp\left(-\kappa \frac{\rho^2 + |y''-z''|^2}{t-s}\right) \\ \leq c(t-s)^{-\lambda_j^+ - i - (n+|\gamma|)/2} \rho^{\lambda_j^+} \exp\left(-\kappa \frac{4\rho^2 + 2|y''-z''|^2 + |x''-y''|^2 - 2r^2}{4(t-s)}\right)\end{aligned}$$

for $|x'' - z''| \leq r$ with certain positive κ and arbitrary positive M yield

$$\begin{aligned} |\partial_r^l B_i(x, y, t-s)| &\leq c \frac{r^{m-n-4-2k-l-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r^2}{t-s} \right)^{M+i+1+|\gamma|/2} \left(\frac{\rho^2}{t-s} \right)^{\lambda_j^+ - 1 + m/2} \\ &\quad \times \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{4(t-s)} \right) \\ &\leq c \frac{r^{m-n-4-2k-l-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r^2}{t-s} \right)^{M+i+1+|\gamma|/2} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{8(t-s)} \right). \end{aligned}$$

Finally, (cf. formulas (4.7) and (4.8) in [1]), we get the estimates

$$\begin{aligned} |A(x, y, t-s)| &\leq c(t-s)^{-\lambda_j^+ - k - 1 - (n+|\gamma|)/2} \rho^{\lambda_j^+} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-s} \right) \\ &\leq c \frac{r^{m-n-4-2k-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r}{\sqrt{t-s}} \right)^{n-m+4+2k+|\gamma|} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{2(t-s)} \right) \end{aligned}$$

and

$$\begin{aligned} |\partial_r^l A(x, y, t-s)| &\leq c(t-s)^{-\lambda_j^+ - (n+N_0+1)/2} r^{N_0-1-2k-l-|\gamma|} \rho^{\lambda_j^+} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-s} \right) \\ &\leq c \frac{r^{m-n-4-2k-l-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r}{\sqrt{t-s}} \right)^{n-m+N_0+3} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{2(t-s)} \right) \end{aligned}$$

if $l \geq 1$. Thus,

$$\begin{aligned} |\partial_t^{k+1} \partial_r^l \partial_{x''}^\gamma K_j(x, y, t-s)| \\ \leq c \frac{r^{m-n-4-2k-l-|\gamma|}}{\rho^{\lambda_j^+ + m-2}} \left(\frac{r}{\sqrt{t-s}} \right)^M \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-s} \right), \end{aligned} \quad (48)$$

where

$$M = \begin{cases} n-m+4+2k+|\gamma|, & \text{if } l=0, \\ n-m+3+N_0, & \text{if } l \geq 1. \end{cases}$$

If $t > t_0 + 2\delta$, $|\tau - t_0| < \delta$, and s lies between t_0 and τ , we have $\frac{2}{3}(t-\tau) < t-s < 2(t-\tau)$. Consequently, it follows from (48) that

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} K_j^{k,l,\gamma}(x, y, t-s) ds \right| \leq c \frac{\delta}{(t-\tau)^{M/2}} \frac{r^{\beta+\lambda_j^+ + m-n-6+M}}{\rho^{\beta+\lambda_j^+ + m-2}} \exp \left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-\tau} \right)$$

for $t > t_0 + 2\delta$ and $|\tau - t_0| < \delta$. This means that the kernel of the integral operator (47) satisfies the condition of Lemma 3.1 if $M > n-m+6-\beta-\lambda_j^+ - m/p$. Hence, by Lemmas 2.4 and 3.1, the operator $\mathcal{K}_j^{k,l,\gamma}$ is bounded in $L_{p,q}(D \times \mathbb{R})$ if $l \geq 1$ or $2k+|\gamma| > 2-\beta-\lambda_j^+ - m/p$.

In order to prove this for $q > p$, we consider the adjoint operator. Let $\tilde{\mathcal{K}}_j^{k,l,\gamma}$ be the integral operator with the kernel

$$\tilde{\mathcal{K}}_j^{k,l,\gamma}(x, y, t, \tau) = K_j^{k,l,\gamma}(y, x, -\tau, -t) = \rho^{\beta+\lambda_j^+ + 2k+l+|\gamma|-2} r^{-\beta} \partial_t^k \partial_\rho^l \partial_{y''}^\gamma K_j(y, x, t-\tau).$$

Since $\mathcal{K}_j^{k,l,\gamma}$ is bounded in $L_p(\mathcal{D} \times \mathbb{R})$ under the assumptions of the lemma, the operator $\tilde{\mathcal{K}}_j^{k,l,\gamma}$ is bounded in $L_{p'}(\mathcal{D} \times \mathbb{R})$, where $p' = p/(p-1)$. Suppose that $h \in L_{p',1}(\mathcal{D} \times \mathbb{R})$ is a function with support in the layer $|t - t_0| < \delta$ such that $\int h(x, t) dt = 0$ for all x . Then

$$(\tilde{\mathcal{K}}_j^{k,l,\gamma} h)(x, t) = \int_{-\infty}^t \int_{\mathcal{D}} \left(\int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{\mathcal{K}}_j^{k,l,\gamma}(x, y, t, s) ds \right) h(y, \tau) dy d\tau$$

for $t > t_0 + \delta$, where

$$\frac{\partial}{\partial s} \tilde{\mathcal{K}}_j^{k,l,\gamma}(x, y, t, s) = -\rho^{\beta+\lambda_j^++2k+l+|\gamma|-2} r^{-\beta} \partial_t^{k+1} \partial_\rho^l \partial_{y''}^\gamma \mathcal{K}_j(y, x, t - \tau).$$

As was shown above, the derivatives of \mathcal{K}_j satisfy the estimate

$$|\partial_t^{k+1} \partial_\rho^l \partial_{y''}^\gamma \mathcal{K}_j(y, x, t - \tau)| \leq c \frac{\rho^{m-n-4-2k-l-|\gamma|}}{r^{\lambda_j^++m-2}} \left(\frac{\rho}{\sqrt{t-s}} \right)^M \exp\left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t-s}\right)$$

with the same M as before. This implies

$$\left| \int_{t_0}^{\tau} \frac{\partial}{\partial s} \tilde{\mathcal{K}}_j^{k,l,\gamma}(x, y, t - s) ds \right| \leq c \frac{\delta}{(t - \tau)^{M/2}} \frac{r^{2-m-\beta-\lambda_j^+}}{\rho^{n-m+6-\beta-\lambda_j^+-M}} \exp\left(-\kappa \frac{r^2 + \rho^2 + |x'' - y''|^2}{t - \tau}\right).$$

Therefore, it follows from Lemma 3.1 that

$$\int_{t_0+2\delta}^{\infty} \|(\tilde{\mathcal{K}}_j^{k,l,\gamma} h)(\cdot, t)\|_{L_{p'}(\mathcal{D})} dt \leq c \|h\|_{L_{p',1}(\mathcal{D} \times \mathbb{R})}$$

for all $h \in L_{p',1}(\mathcal{D} \times \mathbb{R})$ with support in the layer $|t - t_0| \leq \delta$ if $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$. Applying Lemma 2.4, we conclude that $\tilde{\mathcal{K}}_j^{k,l,\gamma}$ is bounded in $L_{p',q'}(\mathcal{D} \times \mathbb{R})$ for $1 < q' < p'$ if $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$. Consequently, the operator $\mathcal{K}_j^{k,l,\gamma}$ is bounded in $L_{p,q}(\mathcal{D} \times \mathbb{R})$ for $p < q < \infty$ if $l \geq 1$ or $2k + |\gamma| > 2 - \beta - \lambda_j^+ - m/p$. The proof is complete. \square

Using the last lemma, we obtain the following result which generalizes [1, Corollary 4.5].

Theorem 3.1 *Let $f \in L_{p,q;\beta}(\mathcal{D} \times \mathbb{R})$, where p and β satisfy the condition (28) and q is an arbitrary real number, $1 < q < \infty$. Then there exists a solution of the problem (1), (2) which has the form*

$$u = \sum_{\lambda_j^+ < 2-\beta-m/p} u_j^{(m_j)}(x', \partial_t - \Delta_{x''}) \mathcal{E}h_j + w, \quad (49)$$

where $u_j^{(m_j)}$, h_j are given by (12) and (36), respectively, and $w \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$.

Proof By Lemma 3.2, the functions $\tilde{H}_j = \mathcal{E}h_j$ satisfy the same condition (43) as the functions H_j in Theorem 2.2. Thus, it follows from [1, Lemma 4.1] that

$$u_j^{(m_j)}(x', \partial_t - \Delta_{x''})(H_j - \tilde{H}_j) \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R}).$$

This together with Theorem 2.2 implies (49) with a remainder $w \in W_{p,q;\beta}^{2,1}(\mathcal{D} \times \mathbb{R})$. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors achieved the key results of the paper during a research stay of JR in Linköping in October 2012. Both authors read and approved the final manuscript.

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