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Positive solution for boundary value problems with *p*-Laplacian in Banach spaces

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Abstract

In this article, by using the fixed point theorem of strict-set-contractions operator, we discuss the existence of positive solution for boundary value problems with p-Laplacian

$$\begin{cases} \left(\phi_p\left(u\prime\left(t\right)\right)\right)\prime + f\left(u\left(t\right)\right) = \theta, & 0 < t < 1, \\ u\prime\left(0\right) = \theta, & u\left(1\right) = \theta, \end{cases}$$

in Banach spaces E, where θ is the zero element of E. Although the fixed point theorem of strict-set-contractions operator is used extensively in yielding positive solutions for boundary value problems in Banach spaces, this method has not been used to study those boundary value problems with p-Laplacian in Banach spaces. So this article may be regarded as an illustration of fixed point theorem of strict-set-contractions operator in a new area.

MSC: 34B18.

Keywords: boundary value problems, p-Laplacian, positive solution, strict-set-contractions

1 Introduction

In the last ten years, the theory of ordinary differential equations in Banach spaces has become an important new branch, so boundary value problems in Banach Space has been studied by some researchers, we refer the readers to [1-9] and the references therein.

For abstract space, it is here worth mentioning that Guo and Lakshmikantham [10] discussed the multiple solutions of the following two-point boundary value problems (BVP for short) of ordinary differential equations in Banach space

$$\begin{cases} u\prime\prime\left(t\right) + f\left(u\left(t\right)\right) = \theta, & 0 < t < 1, \\ u\prime\left(0\right) = \theta, & u\left(1\right) = \theta. \end{cases}$$

Very recently, by using the fixed-point principle in cone and the fixed-point index theory for strict-set-contraction operator, Zhang et al. [11] investigated the existence, nonexistence, and multiplicity of positive solutions for the following nonlinear three-point boundary value problems of nth-order differential equations in ordered Banach spaces

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \cdots, x^{(n-2)}(t)) = \theta, t \in (0,1), \\ x^{(i)}(0) = \theta, i = 0, 1, 2, \cdots, n-2, \\ x^{(n-2)}(1) = \rho x^{(n-2)}(\eta). \end{cases}$$

On the other hand, boundary value problems with p-Laplacian have received a lot of attention in recent years. They often occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory, and the turbulent flow of gas in porous medium [12-19]. Many studies have been carried out to discuss the existence of solutions or positive solutions and multiple solutions for the local or nonlocal boundary value problems.

However, to the authors' knowledge, this is the first article can be found in the literature on the existence of positive solutions for boundary value problems with p-Laplacian in Banach spaces. As is well known, the main difficulty that appears when passing from $p = 2$ to $p \neq 2$ is that, when $p = 2$, we can change the differential equation into a equivalent integral equation easily and therefore a Green's function exists, so we can easily prove the equivalent integral operator is a strict-set-contractions operator, which is a very important result for discussing positive solution for boundary value problems in Banach space. However, for $p \neq 2$, it is impossible for us to find a Green's function in the equivalent integral operator since the differential operator $(\varphi_p(u'))'$ is nonlinear. To authors' knowledge, this is the first article to use the fixed point theorem of strict-set-contractions to deal with boundary value problems with p-Laplacian in Banach spaces. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

Basic facts about an ordered Banach space E can be found in [1,4]. Here we just recall a few of them. Let the real Banach spaces E with norm $||\cdot||$ be partially ordered by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$, and P^* denotes the dual cone of P. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $||x|| \leq N||y||$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P (it is clear, $N \geq 1$). Set $I = 0$ [1], $(C[I, E], ||\cdot||_C)$ is a Banach space with $||x||_C = \max_{t \in I} ||x(t)||$. Clearly, $Q = \{x \in C[I, E] | x(t) \geq \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$.

For a bounded set S in a Banach space, we denote by $\alpha(S)$ the Kuratowski measure of noncompactness. In this article, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness of a bounded set in E and in $C[I, E]$.

The operator $T : D \to E (D \subset E)$ is said to be a k-set contraction if $T : D \to E$ is continuous and bounded and there is a constant $k \geq 0$ such that $\alpha(T(S)) \leq k\alpha(S)$ for any bounded $S \subset D$; a k-set contraction with $k < 1$ is called a strict set contraction.

In this article, we will consider the boundary value problems with p-Laplacian

$$(\phi_p(u'(t)))' + f(u(t)) = \theta, \quad 0 < t < 1, \tag{1}$$

$$u'(0) = \theta, \quad u(1) = \theta, \tag{2}$$

in Banach spaces E, where $\varphi_p(s) = s^{p-1}$, $p > 1$, $(\varphi_p)^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, θ is the zero element of E, $f \in C(P, P)$.

A function u is called a positive solution of BVP (1) and (2) if it satisfies (1) and (2) and $u \in Q$, $u(t)$ ⊠ Q.

The main tool of this article is the following fixed point Theorems.

Theorem 1. [5] Let K be a cone in a Banach space E and $K_{r,R} = \{x \in K, r \le ||x|| \le R\}$, $R > r > 0$. Suppose that $A : K_{r,R} \to K$ is a strict-set contraction such that one of the following two conditions is satisfied:

$$(a) \quad \|Ax\| \ge \|x\|, \forall x \in K, \|x\| = r; \ \|Ax\| \le \|x\|, \forall x \in K, \|x\| = R.$$

$$(b) \quad \|Ax\| \le \|x\|, \forall x \in K, \|x\| = r; \ \|Ax\| \ge \|x\|, \forall x \in K, \quad \|x\| = R.$$

Then, A has a fixed point $x \in K_{r,R}$ such that $r \le ||x|| \le R$.

2 Preliminaries

Lemma 2.1. *If $y \in C[I, E]$, then the unique solution of*

$$(\phi_p(u'(t)))' + y(t) = \theta, \quad 0 < t < 1, \tag{3}$$

$$u'(0) = \theta, \quad u(1) = \theta, \tag{4}$$

is

$$u(t) = \int_t^1 \phi_q \left(\int_0^s y(\tau) d\tau \right) ds.$$

Lemma 2.2. *If $y \in Q$, then the unique solution u of the problem* (3) *and* (4) *satisfies $u(t) \ge \theta$, $t \in I$, that is $u \in Q$.*

Lemma 2.3. *Let $\delta \in (0, \frac{1}{2})$, $J_\delta = [\delta, 1\text{-}\delta]$, then for any $y \in Q$, the unique solution u of the problem* (3) *and* (4) *satisfies $u(t) \ge \delta u(s)$, $t \in J_\delta$, $s \in I$.*

Lemma 2.4. *We define an operator T by*

$$(Tu)(t) = \int_t^1 \phi_q \left(\int_0^s f(u(\tau)) d\tau \right) ds. \tag{5}$$

Then u is a solution of problem (1) *and* (2) *if and only if u is a fixed point of T.*

In the following, the closed balls in spaces E and $C[I, E]$ are denoted by $T_r = \{x \in E| ||x|| \le r\}$ $(r > 0)$ and $B_r = \{x \in C[I, E]| ||x||_c \le r\}$, $M = \sup \{||f(u)||: u \in Q \cap B_r\}$.

Lemma 2.5. *Suppose that, for any $r > 0$, f is uniformly continuous and bounded on $P \cap T_r$ and there exists a constant L_r with*

$$(q-1)M^{q-2}L_r < 1, \tag{6}$$

such that

$$\alpha(f(D)) \le L_r \alpha(D), \quad \forall D \subset P \cap T_r. \tag{7}$$

Then, for any $r > 0$, operator T is a strict-set-contraction on $D \subset P \cap T_r$.

Proof. Since f is uniformly continuous and bounded on $P \cap T_r$, we see from Lemma 2.4 that T is continuous and bounded on $Q \cap B_r$. Now, let $S \subset Q \cap B_r$ be given arbitrary, there exists a partition $S = \cup_{i=1}^m S_i$. We set $\alpha\{y : y \in S\} = \alpha(S)\cdot$

By virtue of Lemma 2.4, it is easy to show that the functions $\{Ty|y \in S\}$ are uniformly bounded and equicontinuous, and so by [11],

$$\alpha(T(S)) = \sup_{t\in I} \alpha(T(S(t))), \tag{8}$$

where $T(S(t)) = \{Tu(t)|u \in S, t \text{ is fixed}\} \subset P \cap T_r$ for any $t \in I$.

Let $u_1, u_2 \in S_i$,

$$\begin{aligned}
|(Tu_1 - Tu_2)(t)| &= \left| \int_t^1 \left[\phi_q\left(\int_0^s f(u_1(\tau))d\tau\right) - \phi_q\left(\int_0^s f(u_2(\tau))d\tau\right)\right] ds \right| \\
&\leq \int_t^1 \left| \phi_q\left(\int_0^s f(u_1(\tau))d\tau\right) - \phi_q\left(\int_0^s f(u_2(\tau))d\tau\right)\right| ds \\
&\leq (q-1)M^{q-2} \int_t^1 \left(\int_0^s \left| f(u_1(\tau)) - f(u_2(\tau)) \right| d\tau \right) ds \\
&\leq (q-1)M^{q-2} \int_t^1 \int_0^s d\tau ds \max_{0\leq t\leq 1} \left| f(u_1(t)) - f(u_2(t)) \right| \\
&\leq \tfrac{1}{2}(q-1)M^{q-2} \max_{0\leq t\leq 1} \left| f(u_1(t)) - f(u_2(t)) \right|
\end{aligned}$$

So, we have

$$\alpha(Tu) \leq \frac{1}{2}(q-1)M^{q-2}\alpha(f(S)) \leq \frac{1}{2}(q-1)M^{q-2}L_r\alpha(B),$$

where $B = \{y(s)| s \in I, y \in S\} \subset P \cap T_r$. Similarly, to the proof of [10], we have $\alpha(B) \leq 2\alpha(S)$. It follows from (6), (7), and (8), that

$$\alpha(T(S)) < (q-1)M^{q-2}L_r\alpha(S) < \alpha(S), \quad \forall S \subset Q \cap B_r,$$

and consequently T is a strict-set-contraction on $S \subset Q \cap B_r$ because of $(q-1)M^{q-2} L_r < 1$. □

3 Existence of positive solution to BVP (1) and (2)

In the following, for convenience, we set

$$f^\beta = \lim \sup_{\|u\|\to\beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \qquad f_\beta \lim \inf_{\|u\|\to\beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \quad (\psi f)_\beta = \lim \inf_{\|u\|\to\beta} \frac{\psi(f(u))}{\phi_p(\|u\|)},$$

where $\beta = 0$ or ∞, $\psi \in P^*$ and $||\psi|| = 1$.

Furthermore, we list some condition:

(H_1): For any $r > 0$, f is uniformly continuous and bounded on $P \cap T_r$ and there exists a constant L_r with $(q-1)M^{q-2}L_r < 1$ such that

$$\alpha(f(D)) \leq L_r\alpha(D), \quad \forall D \subset P \cap T_r.$$

Theorem 3.1. *Let* (H_1) *hold, cone* P *be normal. If* $\phi_q(f^0) < 1 < \frac{1}{2}\delta\phi_q\left[\left(\frac{1}{2}-\delta\right)(\psi f)_\infty\right]$, *then BVP* (1) *and* (2) *has at least one positive solution.*

Proof. Set

$$K = \{u \in Q|u(t) \geq \delta u(s), \forall t \in J_\delta, s \in I\}.$$

It is clear that K is a cone of the Banach space $C[I, E]$ and $K \subset Q$. By Lemma 2.4, we know $T(Q) \subset K$, and so

$$T(K) \subset K.$$

We first assume that $\varphi_q(f^0) < 1$ Then, there exists a constant $\bar{r}_1 > 0$ such that, for any $u \in K$, $\|u\| \le \bar{r}_1$, we have $\|f(u)\| \le (f^0+\varepsilon_1)\varphi_p(\|u\|)$, where $\varepsilon_1 > 0$ satisfies $\varphi_q(f^0 + \varepsilon_1) \le 1$. Let $r_1 \in (0, \bar{r}_1)$, then for any $t \in I$, $u \in K$, $\|u\|_C = r_1$, we have

$$\begin{aligned}\|(Tu)(t)\| &\le \int_0^1 \phi_q\left(\int_0^s \|f(u(\tau))\| d\tau\right) ds \\ &\le \phi_q \int_0^1 \left(\left(f^0+\varepsilon_1\right)\phi_p(\|u\|)\,ds\right) \\ &\le \phi_q\left(f^0+\varepsilon_1\right)\|u\|_C \le \|u\|_C,\end{aligned} \tag{9}$$

i.e., $u \in K$, $\|u\|_C = r_1$ implies $\|Tu\|_C \le \|u\|_C$.

On the other hand, since $1 < \frac{1}{2}\delta\phi_q\left[\left(\frac{1}{2}-\delta\right)(\psi f)_\infty\right]$, there exists $\bar{r}_2 > 0$ such that

$$\psi\left(f\left(u(t)\right)\right) \ge \left((\psi f)_\infty - \varepsilon_2\right)\phi_p(\|u\|), \forall t \in I, x \in K, \|u\| \ge \bar{r}_2,$$

where $\varepsilon_2 > 0$ satisfies $\frac{1}{2}\delta\phi_q\left[\left(\frac{1}{2}-\delta\right)\left((\psi f)_\infty - \varepsilon_2\right)\right] \ge 1$.

Choose $r_2 = \max\{2r_1, \frac{\bar{r}_2}{\delta}\}$, then, for any $t \in J_\delta$, $u \in K$, $\|u\|_C = r_2$, we have

$$\|u(t)\| \ge \delta\|u\|_C \ge \delta r_2 \ge \bar{r}_2,$$

then,

$$\begin{aligned}&\left\|(Tu)\left(\tfrac{1}{2}\right)\right\| \ge \psi\left((Tu)\left(\tfrac{1}{2}\right)\right) = \int_{\frac{1}{2}}^1 \phi_q\left[\int_0^s \psi(f(u(\tau)))d\tau\right] ds \\ &\ge \int_{\frac{1}{2}}^1 \phi_q\left[\int_\delta^{\frac{1}{2}} \psi(f(u(\tau)))d\tau\right] ds \\ &\ge \int_{\frac{1}{2}}^1 \phi_q\left[\int_\delta^{\frac{1}{2}} \left((\psi f)_\infty - \varepsilon_2\right)\phi_p(\|u\|)\,d\tau\right] ds \\ &\ge \int_{\frac{1}{2}}^1 \phi_q\left[\int_\delta^{\frac{1}{2}} \left((\psi f)_\infty - \varepsilon_2\right)\phi_p(\delta\|u\|_C)\,d\tau\right] ds \\ &= \tfrac{1}{2}\delta\phi_q\left[\left(\tfrac{1}{2}-\delta\right)\left((\psi f)_\infty - \varepsilon_2\right)\right]\|u\|_C \\ &\ge \|u\|_C,\end{aligned} \tag{10}$$

i.e., for any $u \in K$, $\|u\|_C = r_2$, we have

$$\|Tu\|_C \ge \|u\|_C.$$

On the other hand, by Lemma 2.5, T is a strict set contraction from $B_r^{(1)}$ into $B_r^{(1)}$. Consequently, Theorem 1 implies that T has a fixed point in $B_r^{(1)}$, and the proof is complete. □

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Authors' contributions

All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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