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Infinitely many solutions for a class of quasilinear elliptic equations with p-Laplacian in $\mathbb{R}^N$

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Abstract

In this paper, we study the multiplicity of solutions for a class of quasilinear elliptic equations with p-Laplacian in $\mathbb{R}^N$. In this case, the functional J is not differentiable. Hence, it is difficult to work under the classical framework of the critical point theory. To overcome this difficulty, we use a nonsmooth critical point theory, which provides the existence of critical points for nondifferentiable functionals.

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1 Introduction and main results

Recently, the multiplicity of solutions for the quasilinear elliptic equations has been studied extensively, and many fruitful results have been obtained. For example, in [1], Shibo Liu considered the existence of multiple nonzero solutions of the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

Moreover, Aouaoui studied the following quasilinear elliptic equation in [2]:

$$-\text{div}\big(A(x,u)\nabla u\big) + \frac{1}{2}A'_s(x,u)|\nabla u|^2 + \big(b(x) - \lambda\big)u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

and proved the multiplicity of solutions of the problem (1.2) by using the nonsmooth critical point theory. One can refer to [3, 4] and [5] for more results.

In this paper, we shall investigate the existence of infinitely many solutions of the following problem

$$-\text{div}\big(A(x,u)|\nabla u|^{p-2}\nabla u\big) + \frac{1}{p}A_s(x,u)|\nabla u|^p + \big(b(x) - \lambda\big)|u|^{p-2}u = f(x,u),$$

$$\text{in } \mathbb{R}^N, \tag{1.3}$$

where $2 < p < N$, $\lambda \in \mathbb{R}$ and $A_s(x, u) \equiv \frac{\partial A}{\partial s}(x, s)|_{s=u}$, $b(x)$ is a given continuous function satisfying

$$b(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N \quad \text{and} \quad \lim_{|x| \to +\infty} b(x) = +\infty.$$

In order to determine weak solutions of (1.3) in a suitable functional space E, we look for critical points of the functional $J : E \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{p} \int_{\mathbb{R}^N} A(x, v)|\nabla v|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} \big(b(x) - \lambda\big)|v|^p \, dx - \int_{\mathbb{R}^N} F(x, v) \, dx,$$
$$\forall v \in E, \tag{1.4}$$

where $F(x, \xi) = \int_0^\xi f(x, t) \, dt$. Under reasonable assumptions, the functional J is continuous, but not even locally Lipschitz. However, one can see from [4, 6] and [7] that the Gâteaux-derivative of J exists in the smooth directions, *i.e.*, it is possible to evaluate

$$
\begin{aligned}
\big\langle J'(u), v \big\rangle &= \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} \\
&= \int_{\mathbb{R}^N} A(x, u)|\nabla u|^{p-2} \nabla u \nabla v \, dx + \frac{1}{p} \int_{\mathbb{R}^N} A_s(x, u)|\nabla u|^p v \, dx \\
&\quad + \int_{\mathbb{R}^N} \big(b(x) - \lambda\big)|u|^{p-2} uv \, dx - \int_{\mathbb{R}^N} f(x, u) v \, dx
\end{aligned}
$$

for all $u \in E$ and $v \in E \cap L^\infty(\mathbb{R}^N)$.

Definition 1.1 A critical point u of the functional J is defined as a function $u \in E$ such that $\langle J'(u), v \rangle = 0$, $\forall v \in E \cap L^\infty(\mathbb{R}^N)$, *i.e.*,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} A(x, u)|\nabla u|^{p-2} \nabla u \nabla v \, dx + \frac{1}{p} \int_{\mathbb{R}^N} A_s(x, u)|\nabla u|^p v \, dx \\
&\quad + \int_{\mathbb{R}^N} \big(b(x) - \lambda\big)|u|^{p-2} uv \, dx - \int_{\mathbb{R}^N} f(x, u) v \, dx = 0, \quad \forall v \in E \cap L^\infty\big(\mathbb{R}^N\big). \tag{1.5}
\end{aligned}
$$

Our approach to study (1.3) is based on the nonsmooth critical point theory developed in [8] and [9]. Dealing with this class of problems, the main difficulty is that the associated functional is not differentiable in all directions.

The main goal here is to establish multiplicity of results for (1.3), when $f(x, s)$ is odd and $A(x, s)$ is even in s. Such solutions for (1.3) will follow from a version of the symmetric mountain pass theorem due to Ambrosetti and Rabinowitz [10, 11]. Compared with problem (1.2) in [2], problem (1.3) is much more difficult, since the discreteness of the spectrum is not guaranteed. Therefore, we only consider the first eigenvalue λ_1.

To state and prove our main result, we consider the following assumptions.

Suppose that $N \geq 3$ and $p^* = \frac{Np}{N-p}$.

(H$_1$) Let $A(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function such that

- for each $s \in \mathbb{R}$, $A(x, s)$ is measurable with respect to x;
- for a.e. $x \in \mathbb{R}^N$, $A(x, s)$ is a function of class C^1 with respect to s;

- there exist $0 < \alpha < \beta < +\infty$ such that

$$\alpha \le A(x,s) \le \beta, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}, \tag{1.6}$$

$$\left| A_s(x,s) \right| \le \beta, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{1.7}$$

(H_2) There exist $\theta > p$, $1 < \gamma < \frac{\theta}{p}$ and $\alpha_1 > 0$ such that

$$0 \le \frac{\gamma}{p} A_s(x,s)s \le \left(\frac{\theta}{p} - \gamma \right) A(x,s) - \alpha_1, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{1.8}$$

(H_3) Let a Carathéodory function $f(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy $f(x,0) = 0$, a.e. $x \in \mathbb{R}^N$ and

$$0 < \theta F(x,s) \le f(x,s)s, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \neq 0 \text{ in } \mathbb{R}, \tag{1.9}$$

where θ is the same as that in (H_2).

(H_4) There exists $p - 1 \le q < p^* - 1$ such that

$$\left| f(x,s) \right| \le c_0 |s|^q, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}, \tag{1.10}$$

where c_0 is a positive constant.

Example 1.1 Let $p = 3$. The following function satisfies hypotheses (H_1) and (H_2)

$$A(x,s) = 2\left(\sin\left(|x|^2 \right) + 2 \right) \arctan\left(s^2 \right) + 20,$$

and the corresponding constants are

$$\alpha = 20, \qquad \beta = 20 + 3\pi, \qquad \theta = 34, \qquad \gamma = 10, \qquad \alpha_1 = 1.$$

Example 1.2 The following function satisfies hypotheses (H_3) and (H_4)

$$f(x,s) = |s|^{q-1}s, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}.$$

On the other hand, we define the operator $Lu = -\Delta_p u + b(\cdot)u$. It follows from [12] that the discreteness of the spectrum is not guaranteed. Hence, we only consider the first eigenvalue λ_1, where

$$\lambda_1 = \inf\left\{ \|u\|^p; u \in E, \|u\|_{L^p(\mathbb{R}^N)} = 1 \right\}.$$

Next, we can state the main theorem of the paper.

Theorem 1.1 *Assume that $A(x,s)$ and $f(x,s)$ satisfy (H_1)-(H_4). Moreover, let $A(x,-s) = A(x,s)$ and $f(x,-s) = -f(x,s)$, a.e. $x \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$. If there exists a positive number μ such that $\lambda \in (-\infty, \mu\lambda_1)$, then problem (1.3) has infinitely many distinct solutions in $E \cap L^\infty(\mathbb{R}^N)$, i.e., there exists a sequence $\{u_n\} \subset E \cap L^\infty(\mathbb{R}^N)$, satisfying (1.3) and $J(u_n) \to +\infty$, as $n \to \infty$.*

To explain our result, we introduce some functional spaces. We define the reflexive Banach space E of all functions $u : \mathbb{R}^N \to \mathbb{R}$ with the norm $\|u\|^p = \int_{\mathbb{R}^N} (|\nabla u|^p + b(x)|u|^p) \, dx < \infty$.

Such a weighted Sobolev space has been used in many previous papers, see [13] and [14]. Now, we give an important property of the space E, which will play an essential role in proving our main results.

Remark 1.1 One can easily deduce $E \hookrightarrow L^z(\mathbb{R}^N)$ and $E \hookrightarrow\hookrightarrow L^z(\mathbb{R}^N)$ for $p \leq z < p^*$. More details can be found in [2].

Throughout this paper, let $\| \cdot \|$ denote the norm of E and $u_n \to u$ ($u_n \rightharpoonup u$) means that u_n converges strongly (weakly) in corresponding spaces. $\hookrightarrow$ stands for a continuous map, and $\hookrightarrow\hookrightarrow$ means a compact embedding map. C denotes any universal positive constant unless specified.

The paper is organized as follows. In Section 2, we introduce the nonsmooth critical framework and preliminaries to our work. In Section 3, we give some lemmas to prove the main result. Finally, the proof of Theorem 1.1 is presented in Section 4.

2 Nonsmooth critical framework and preliminaries

Our results are based on the techniques of nonsmooth critical point theory. In this section, we recall some basic tools from [8] and [9].

Definition 2.1 Let (X, d) be a metric space, let $I : X \to \mathbb{R}$ be a continuous functional and $u \in X$. We denote by $|dI|(u)$ the supremum of the σ's in $[0, +\infty)$ such that there exist $\delta > 0$ and a continuous map $H : B(u, \delta) \times [0, \delta] \to X$, satisfying

$$d\big(H(v, t), v\big) \leq t \quad \text{and} \quad I\big(H(v, t)\big) \leq I(v) - \sigma t, \quad (v, t) \in B(u, \delta) \times [0, \delta].$$

The extended real number $|dI|(u)$ is called the weak slope of I at u.

Note that the notion above was independently introduced in [15], as well.

Definition 2.2 Let (X, d) be a metric space, let $I : X \to \mathbb{R}$ be a continuous functional and $c \in \mathbb{R}$. We say that I satisfies $(P - S)_c$, *i.e.,* the Palais-Smale condition at level c, if every sequence $\{u_n\}$ in X with $|dI|(u_n) \to 0$ and $I(u_n) \to c$ admits a strongly convergent subsequence.

In order to treat the Palais-Smale condition, we need to introduce an auxiliary notion.

Definition 2.3 Let c be a real number. We say that functional I satisfies the concrete Palais-Smale condition at level c ($(C - P - S)_c$ for short) if every sequence $\{u_n\} \subset E$ satisfying

$$\lim_{n \to +\infty} I(u_n) = c \quad \text{and} \quad \big|\langle I'(u_n), v \rangle\big| \leq \epsilon_n \|v\|, \quad \forall v \in E \cap L^\infty(\mathbb{R}^N)$$

possesses a strongly convergent subsequence in E, where $\{\epsilon_n\}$ is some real number converging to zero.

Remark 2.1 Under assumptions (H_1)-(H_4), if the functional J satisfies (1.4), then J is continuous, and for every $u \in E$ we have

$$|dJ|(u) \geq \sup\{\langle J'(u), v\rangle; v \in E \cap L^\infty(\mathbb{R}^N), \|v\| \leq 1\},$$

where $|dJ|(u)$ denotes the weak slope of J at u.

Remark 2.2 Let c be a real number. If J satisfies $(C - P - S)_c$, then J satisfies $(P - S)_c$.

Proof Let $\{u_n\} \subset E$ be a sequence such that

$$\lim_{n \to +\infty} |dJ|(u_n) = 0 \quad \text{and} \quad \lim_{n \to +\infty} J(u_n) = c.$$

Note that for $v \subset E \cap L^\infty(\mathbb{R}^N)$,

$$\left|\langle J'(u_n), v\rangle\right| \leq \left\|J'(u_n)\right\| \|v\| = \sup\{\langle J'(u_n), v\rangle, \|v\| \leq 1\}\|v\|.$$

By Remark 2.1, we have $|dJ|(u_n) \geq \sup\{\langle J'(u_n), v\rangle, \|v\| \leq 1\}$. Taking $\epsilon_n = \sup\{\langle J'(u_n), v\rangle, \|v\| \leq 1\}$, the conclusion follows. □

3 Basic lemmas

To derive our main theorem, we need the following lemmas. The first lemma is the version of the Ambrosetti-Rabinowitz mountain pass lemma [10, 11] and [16].

Lemma 3.1 *Let X be an infinite-dimensional Banach space, and let $I : X \to \mathbb{R}$ be a continuous even functional satisfying $(P - S)_c$ for every $c \in \mathbb{R}$. Assume that*
  (i) *there exist $\varrho > 0$, $\alpha > I(0)$ and a subspace $V \subset X$ of finite codimension such that*

$$\forall u \in \{V : \|u\| = \varrho\} \quad \Rightarrow \quad I(u) \geq \alpha,$$

  (ii) *for every finite-dimensional subspace $W \subset X$, there exists $\mathbf{R} > 0$ such that*

$$\forall u \in \{W : \|u\| = \mathbf{R}\} \quad \Rightarrow \quad I(u) \leq I(0).$$

Then there exists a sequence $\{c_h\}$ of critical values of I with $c_h \to \infty$.

Lemma 3.2 *If $u \in E$ is a critical point of J, then $u \in L^\infty(\mathbb{R}^N)$.*

Proof For $r > 1$, $M > 0$, consider the real functions T_r, U_M and W_M defined in $\mathbb{R}$ by

$$T_r(s) = \begin{cases} s - r, & \text{if } s > r, \\ 0, & \text{if } -r \leq s \leq r, \\ s + r, & \text{if } s < -r, \end{cases} \tag{3.1}$$

$U_M = \min(T_r(s), M)$ and $W_M = \max(T_r(s), -M)$. Denoting $s^+ = \max(s, 0)$ and $s^- = \min(s, 0)$, we can take $v = U_M(u^+) \in E \cap L^\infty(\mathbb{R}^N)$ as a test function in (1.5). Therefore,

$$\int_{\mathbb{R}^N} A(x, u)|\nabla u|^{p-2}\nabla u\nabla(U_M(u^+))\, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_s(x, u)|\nabla u|^p U_M(u^+)\, dx$$

$$+ \int_{\mathbb{R}^N}(b(x) - \lambda)|u|^{p-2}u U_M(u^+)\, dx = \int_{\mathbb{R}^N} f(x, u)U_M(u^+)\, dx.$$

Noting that $u^+ \cdot U_M(u^+) \geq 0$ and $A_s(x, u^+)U_M(u^+) \geq 0$, we get

$$\int_{\mathbb{R}^N} A(x, u^+)|\nabla u^+|^{p-2}\nabla u^+ \nabla(U_M(u^+))\, dx$$

$$\leq |\lambda|\int_{\mathbb{R}^N}|u^+|^{p-2}u^+|U_M(u^+)|\, dx + \int_{\mathbb{R}^N}|f(x, u^+)||U_M(u^+)|\, dx.$$

From (1.10) and the fact $|U_M(u^+)| \leq u^+$ we deduce

$$\int_{\{u^+>r\}} A(x, u^+)|\nabla u^+|^{p-2}\nabla u^+ \nabla(U_M(u^+))\, dx \leq C\int_{\{u^+>r\}}(u^+)^{q+1}\, dx.$$

Since $U_M(u^+) \to T_r(u^+)$ a.e. in $\mathbb{R}^N$ and $U_M(u^+) \rightharpoonup T_r(u^+)$ in E as $M \to +\infty$. It follows from $M \to +\infty$ that

$$\int_{\{u^+>r\}} A(x, u^+)|\nabla u^+|^p\, dx \leq C\int_{\{u^+>r\}}(u^+)^{q+1}\, dx.$$

Denote $\Omega_r^+ = \{x \in \mathbb{R}^N, u^+(x) > r\}$. If $\mathrm{mes}(\Omega_r^+) = 0$, then the result is true. In the following discussion, $\mathrm{mes}(\Omega_r^+) \neq 0$ is assumed. By (1.6), we obtain

$$\int_{\Omega_r^+}|\nabla u^+|^p\, dx \leq C\int_{\Omega_r^+}(u^+ - r)^{q+1}\, dx + Cr^{q+1}\,\mathrm{mes}(\Omega_r^+). \tag{3.2}$$

Note that $(\int_{\Omega_r^+}(u^+ - r)^{q+1}\, dx)^{1-\frac{p}{q+1}} \leq C$, then we can get

$$\int_{\Omega_r^+}(u^+ - r)^{q+1}\, dx \leq C\left(\int_{\Omega_r^+}(u^+ - r)^{q+1}\, dx\right)^{\frac{p}{q+1}}. \tag{3.3}$$

On the other hand, we have

$$\int_{\Omega_r^+} r^{p^*}\, dx \leq \int_{\Omega_r^+}|u|^{p^*}\, dx = C,$$

which implies that

$$r^{p^*} \leq \frac{C}{\mathrm{mes}(\Omega_r^+)}. \tag{3.4}$$

Eventually, one can deduce from (3.2)-(3.4) that

$$\int_{\Omega_r^+}|\nabla u^+|^p\, dx \leq C\left(\int_{\Omega_r^+}(u^+ - r)^{q+1}\, dx\right)^{\frac{p}{q+1}} + Cr^p\,\mathrm{mes}(\Omega_r^+)^{1-\frac{q+1}{p^*}+\frac{p}{p^*}}, \quad \forall r > 1. \tag{3.5}$$

By Theorem 5.2 of [17], we get that $u^+ \in L^\infty(\mathbb{R}^N)$. Replacing $U_M(u^+)$ by $W(u^-)$, we can similarly prove that $u^- \in L^\infty(\mathbb{R}^N)$. We conclude that $u \in L^\infty(\mathbb{R}^N)$, and the proof of Lemma 3.2 is completed. □

Lemma 3.3 *Let $\{u_n\}$ be a bounded sequence in E with*

$$\langle J'(u_n), v \rangle \le \epsilon_n \|v\|, \quad \forall v \in E \cap L^\infty(\mathbb{R}^N), \tag{3.6}$$

where $\{\epsilon_n\}$ is a sequence of real numbers converging to zero. Then there exists $u \in E$ such that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ and, up to a subsequence, $\{u_n\}$ is weakly convergent to u in E. Moreover, we have

$$\langle J'(u), v \rangle = 0, \quad \forall v \in E \cap L^\infty(\mathbb{R}^N), \tag{3.7}$$

i.e., u is a critical point of J.

Proof Since $\{u_n\}$ is bounded in E, and there is a $u \in E$ (see [18]) such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } E, \qquad u_n \to u \quad \text{in } L^p(\mathbb{R}^N), \qquad u_n \to u \quad \text{a.e. in } \mathbb{R}^N.$$

Moreover, since $\{u_n\}$ satisfies (3.6), by Theorem 2.1 of [19], we have, up to a further subsequence, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$.

We will use the device of [20]. We consider the test functions

$$v_n = \varphi \exp\{-Mu_n^+\}, \tag{3.8}$$

where $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, $\varphi \ge 0$ and $M > 0$. According to (1.6) and (1.7), we have

$$MA(x, u_n) \ge \frac{1}{p}|A_s(x, u_n)|.$$

Since (3.6) holds by density for every $v \in E \cap L^\infty(\mathbb{R}^N)$, we can put $v = v_n$ in (3.6) and obtain that

$$\int_{\mathbb{R}^N} A(x, u_n)|\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \exp\{-Mu_n^+\}\,dx$$

$$+ \int_{\mathbb{R}^N} \left(\frac{1}{p}A_s(x, u_n)|\nabla u_n|^p - MA(x, u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla u_n^+\right)\varphi \exp\{-Mu_n^+\}\,dx$$

$$+ \int_{\mathbb{R}^N} \left(b(x) - \lambda\right)|u_n|^{p-2}u_n \varphi \exp\{-Mu_n^+\}\,dx$$

$$- \int_{\mathbb{R}^N} f(x, u_n)\varphi \exp\{-Mu_n^+\}\,dx \ge -\epsilon_n \|v\|. \tag{3.9}$$

On the other hand, note that

$$\int_{\mathbb{R}^N} \left(\frac{1}{p}A_s(x, u_n)|\nabla u_n|^p - MA(x, u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla u_n^+\right)\varphi \exp\{-Mu_n^+\}\,dx \le 0. \tag{3.10}$$

One can deduce from (3.10) and Fatou's lemma that

$$\int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla \varphi \exp\{-Mu^+\} \, dx$$

$$+ \int_{\mathbb{R}^N} \left(\frac{1}{p}A_s(x,u)|\nabla u|^p - MA(x,u)|\nabla u|^{p-2}\nabla u\nabla u^+\right)\varphi \exp\{-Mu^+\} \, dx$$

$$+ \int_{\mathbb{R}^N} \left(b(x)-\lambda\right)|u|^{p-2}u\varphi \exp\{-Mu^+\} \, dx - \int_{\mathbb{R}^N} f(x,u)\varphi \exp\{-Mu^+\} \, dx$$

$$\geq 0. \tag{3.11}$$

We consider the test functions $\varphi_n = \varphi g(\frac{u}{n})\exp\{Mu^+\}$ with $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, $\varphi \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$, $g \in C^1(\mathbb{R})$, $0 \leq g \leq 1$,

$$g = 1 \quad \text{on} \left[-\frac{1}{p}, \frac{1}{p}\right], \qquad g = 0 \quad \text{on } [-\infty, -1] \cup [1, \infty].$$

This together with (3.11) can prove that

$$\int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_s(x,u)|\nabla u|^p \varphi \, dx$$

$$+ \int_{\mathbb{R}^N} \left(b(x)-\lambda\right)|u|^{p-2}u\varphi \, dx - \int_{\mathbb{R}^N} f(x,u)\varphi \, dx \geq 0,$$

$$\forall \varphi \in E \cap L^\infty(\mathbb{R}^N), \text{as } n \to \infty. \tag{3.12}$$

In a similar way, by considering the test functions $v_n = \varphi \exp\{Mu_n^-\}$, it is possible to prove that

$$\int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_s(x,u)|\nabla u|^p \varphi \, dx$$

$$+ \int_{\mathbb{R}^N} \left(b(x)-\lambda\right)|u|^{p-2}u\varphi \, dx - \int_{\mathbb{R}^N} f(x,u)\varphi \, dx \leq 0,$$

$$\forall \varphi \in E \cap L^\infty(\mathbb{R}^N), \text{as } n \to \infty. \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$\int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_s(x,u)|\nabla u|^p \varphi \, dx$$

$$+ \int_{\mathbb{R}^N} \left(b(x)-\lambda\right)|u|^{p-2}u\varphi \, dx - \int_{\mathbb{R}^N} f(x,u)\varphi \, dx = 0, \quad \forall \varphi \in E \cap L^\infty(\mathbb{R}^N). \tag{3.14}$$

Finally, we can deduce (3.7) from (3.14). □

Remark 3.1 (see [21]) Let $\{u_n\}$ be a sequence in E satisfying (3.6). Then $-\lambda|u_n|^p - f(x,u_n)u_n \in L^1(\mathbb{R}^N)$ and

$$\left| \int_{\mathbb{R}^N} \left(A(x,u_n)|\nabla u_n|^p + \frac{1}{p}A_s(x,u_n)u_n|\nabla u_n|^p + \left(b(x)-\lambda\right)|u_n|^p - f(x,u_n)u_n \right) dx \right|$$

$$\leq \epsilon_n \|u_n\|, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

In the following lemma, we will prove the boundedness of a $(C - P - S)_c$ sequence $\{u_n\} \subset E$ under (1.6), (1.8) and (1.9).

Lemma 3.4 *Let $c \in \mathbb{R}$ and $\{u_n\}$ be a sequence in E satisfying (3.6) and*

$$\lim_{n \to +\infty} J(u_n) = c. \tag{3.16}$$

Then $\{u_n\}$ is bounded in E.

Proof Calculating $\theta J(u_n) - \gamma \langle J'(u_n), u_n \rangle$, from (3.15) and (3.16), we obtain

$$\left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} A(x, u_n) |\nabla u_n|^p \, dx - \frac{\gamma}{p} \int_{\mathbb{R}^N} A_s(x, u_n) |\nabla u_n|^p u_n \, dx$$
$$+ \left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} \left(b(x) - \lambda \right) |u_n|^p \, dx + \int_{\mathbb{R}^N} \left(\gamma f(x, u_n) u_n - \theta F(x, u_n) \right) dx$$
$$\leq C \left(1 + \|u_n\| \right), \quad \forall n \in \mathbb{N}.$$

From (1.8) and (1.9), it follows that

$$\alpha_1 \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} \left(b(x) - \lambda \right) |u_n|^p \, dx + \theta(\gamma - 1) \int_{\mathbb{R}^N} F(x, u_n) \, dx$$
$$\leq C \left(1 + \|u_n\| \right), \quad \forall n \in \mathbb{N}. \tag{3.17}$$

Moreover, there exist $M > 0$ and $C_1(\lambda) > 0$ such that

$$\left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} \left(b(x) - \lambda \right) |u_n|^p \, dx \geq \left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} \frac{b(x)}{2} |u_n|^p \, dx - C_1 \int_{\{|x| < M\}} |u_n|^p \, dx.$$

Therefore, denoting $D_M = \{x \in \mathbb{R}^N, |x| < M\}$, we obtain from (3.17) that

$$\alpha_1 \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \left(\frac{\theta}{p} - \gamma \right) \int_{\mathbb{R}^N} \frac{b(x)}{2} |u_n|^p \, dx + \theta(\gamma - 1) \int_{\mathbb{R}^N} F(x, u_n) \, dx$$
$$\leq C \left(1 + \|u_n\| \right) + C_1 \|u_n\|_{L^p(D_M)}^p, \quad \forall n \in N. \tag{3.18}$$

By virtue of hypothesis (H$_3$), we know that there exist $a_0 > 0$ and $b_0 > 0$ such that

$$F(x, s) \geq a_0 |s|^\theta - b_0, \quad \text{a.e. } x \in D_M \text{ and } \forall s \in \mathbb{R}. \tag{3.19}$$

From (3.18) and (3.19), it follows that

$$\min \left(\alpha_1, \frac{1}{2} \left(\frac{\theta}{p} - \gamma \right) \right) \|u_n\|^p + \theta(\gamma - 1) a_0 \|u_n\|_{L^\theta(D_M)}^\theta$$
$$\leq C \left(1 + \|u_n\| \right) + b_0 \theta(\gamma - 1) \operatorname{mes}(D_M) + C_1 \|u_n\|_{L^p(D_M)}^p. \tag{3.20}$$

On the other hand, by Hölder's inequality and Young's inequality, for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$C_1 \|u_n\|_{L^p(D_M)}^p \leq C \|u_n\|_{L^\theta(D_M)}^p \leq C_\epsilon + \epsilon \|u_n\|_{L^\theta(D_M)}^\theta. \tag{3.21}$$

Using (3.20) and (3.21), we get

$$\min\left(\alpha_1, \frac{1}{2}\left(\frac{\theta}{p} - \gamma\right)\right)\|u_n\|^p$$

$$\leq C\left(1 + \|u_n\|\right) + b_0\theta(\gamma - 1)\operatorname{mes}(D_M) + C_\epsilon + \left(\epsilon - \theta(\gamma - 1)a_0\right)\|u_n\|_{L^\theta(D_M)}^\theta. \tag{3.22}$$

Choosing $0 < \epsilon < \theta(\gamma - 1)a_0$ in (3.22), we find that $\{u_n\}$ is bounded in E. □

Lemma 3.5 *Let $\{u_n\}$ be the same as that in Lemma* 3.3. *Then $\{u_n\}$, up to a subsequence, converges strongly to u in E.*

Proof By Lemma 3.3, we know that u is a critical point of the functional J. Then, from Lemma 3.2, we get $u \in L^\infty(\mathbb{R}^N)$. Therefore, taking $v = u$ as a test function in (3.7), we get

$$\int_{\mathbb{R}^N} A(x, u)|\nabla u|^p \, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_s(x, u)u|\nabla u|^p \, dx + \int_{\mathbb{R}^N} \left(b(x) - \lambda\right)|u|^p \, dx$$

$$= \int_{\mathbb{R}^N} f(x, u)u \, dx. \tag{3.23}$$

By virtue of $\{u_n\}$ is bounded in E, we can assume that there exists $u \in E$ satisfying

$$u_n \rightharpoonup u \quad \text{in } E \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \quad \text{a.e. } x \in \mathbb{R}^N.$$

By Lemma 3.3, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Then by Fatou's lemma, we have

$$\int_{\mathbb{R}^N} A_s(x, u)u|\nabla u|^p \, dx \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^N} A_s(x, u_n)u_n|\nabla u_n|^p \, dx. \tag{3.24}$$

Moreover, by $E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N)$ and $E \hookrightarrow\hookrightarrow L^{q+1}(\mathbb{R}^N)$, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} f(x, u_n)u_n \, dx = \int_{\mathbb{R}^N} f(x, u)u \, dx, \tag{3.25}$$

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^p \, dx = \int_{\mathbb{R}^N} |u|^p \, dx. \tag{3.26}$$

By using (3.23)-(3.26) and passing to limit in (3.15), we obtain

$$\limsup_{n \to +\infty}\left(\int_{\mathbb{R}^N} A(x, u_n)|\nabla u_n|^p \, dx + \int_{\mathbb{R}^N} b(x)|u_n|^p \, dx\right)$$

$$\leq \int_{\mathbb{R}^N} A(x, u)|\nabla u|^p \, dx + \int_{\mathbb{R}^N} b(x)|u|^p \, dx. \tag{3.27}$$

On the other hand, by Lebesgue's dominated convergence theorem and the weak convergence of u_n to u in E, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} A(x, u_n)\nabla u_n|\nabla u|^{p-2}\nabla u \, dx = \int_{\mathbb{R}^N} A(x, u)|\nabla u|^p \, dx, \tag{3.28}$$

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} A(x, u_n)|\nabla u|^p \, dx = \int_{\mathbb{R}^N} A(x, u)|\nabla u|^p \, dx, \tag{3.29}$$

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} b(x)|u|^{p-2}uu_n \, dx = \int_{\mathbb{R}^N} b(x)|u|^p \, dx. \tag{3.30}$$

Moreover, since $A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n$ and $b(x)^{\frac{p-1}{p}}|u_n|^{p-2}u_n$ are bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$, then we have

$$A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup A(x, u)|\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N),$$

$$b(x)^{\frac{p-1}{p}}|u_n|^{p-2}u_n \rightharpoonup b(x)^{\frac{p-1}{p}}|u|^{p-2}u \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N).$$

Therefore, from the definition of weak convergence, we obtain

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla u \, dx = \int_{\mathbb{R}^N} A(x, u)|\nabla u|^p \, dx, \tag{3.31}$$

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} b(x)|u_n|^{p-2}u_n u \, dx = \int_{\mathbb{R}^N} b(x)|u|^p \, dx. \tag{3.32}$$

Combining (3.27)-(3.32), it follows that

$$\limsup_{n \to +\infty}\left(\int_{\mathbb{R}^N} A(x, u_n)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u) \, dx \right.$$

$$\left. + \int_{\mathbb{R}^N} b(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u) \, dx \right) \le 0.$$

It is well known that the following inequality

$$\big(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta\big)(\xi - \eta) > 0 \tag{3.33}$$

holds for any $t > 1$, $\xi, \eta \in \mathbb{R}^N$ and $\xi \ne \eta$. Therefore,

$$\limsup_{n \to +\infty}\left(\int_{\mathbb{R}^N} A(x, u_n)|\nabla u_n - \nabla u|^p \, dx + \int_{\mathbb{R}^N} b(x)|u_n - u|^p \, dx \right) \le 0.$$

According to (1.6), we conclude that $\{u_n\}$ converges strongly to u in E. $\square$

Lemma 3.6 *For every real number c, the functional J satisfies* $(C - P - S)_c$.

Proof Let $\{u_n\}$ be a sequence in E satisfying (3.6) and (3.16). By Lemma 3.4, $\{u_n\}$ is bounded in E. Therefore, the conclusion can be deduced from Lemma 3.5. $\square$

4 Proof of Theorem 1.1

It is easy to check that the functional J is continuous and even. Moreover, by Remark 2.2 and Lemma 3.6, J satisfies $(P - S)_c$ for every $c \in \mathbb{R}$.

On the other hand, from (1.4), (1.6), (1.9) and (1.10), for $u \in E$, we have

$$J(u) \ge \frac{\min(1, \alpha)}{p}\|u\|^p - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p \, dx - C\|u\|^{q+1}. \tag{4.1}$$

We discuss (4.1) in the following two cases:

In case $\lambda \leq 0$, we get

$$J(u) \geq \frac{\min(1,\alpha)}{p}\|u\|^p - C\|u\|^{q+1}.$$

In case $\lambda > 0$, by the definition of λ_1, we get

$$J(u) \geq \frac{1}{p}\left(\min(1,\alpha) - \frac{\lambda}{\lambda_1}\right)\|u\|^p - C\|u\|^{q+1},$$

i.e., $\mu = \min(1,\alpha)$. Therefore, if λ satisfies $\lambda < \min(1,\alpha)\lambda_1$, there exist $\varrho > 0$ small enough and $\delta > 0$ such that

$$J(u) \geq \delta \quad \text{for } \|u\| = \varrho.$$

Hence, condition (i) of Lemma 3.1 holds with $V = E$.

Now we consider a finite-dimensional subspace W of E. Let $u \in W$ and $J(u) \geq 0$. From (1.6), we have

$$0 \leq J(u) \leq \max(1,\beta)\|u\|^p - \frac{\lambda}{p}\|u\|_{L^p(\mathbb{R}^N)}^p - \int_{\mathbb{R}^N} F(x,u)\,dx. \tag{4.2}$$

By virtue of (1.9) and (1.10), we know that there exist $z(x) \in L^\infty(\mathbb{R}^N)$, satisfying $z(x) > 0$ a.e. $x \in \mathbb{R}^N$ and a positive constant C_2 such that

$$F(x,s) \geq z(x)|s|^\theta - C_2|s|^p, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{4.3}$$

Combining (4.2)-(4.3), we have

$$\max(1,\beta)\|u\|^p \geq \int_{\mathbb{R}^N} z(x)|u|^\theta\,dx - C_2\int_{\mathbb{R}^N}|u|^p\,dx. \tag{4.4}$$

Since W is finite-dimensional, then all norms of W are equivalent. From (4.4), there exists $C_3 > 0$ such that

$$\|u\|^\theta \leq C_3\|u\|^p.$$

In view of $\theta > p$, we deduce that the set $\{u \in W, J(u) \geq 0\}$ is bounded in E and condition (ii) of Lemma 3.1 holds. By Lemma 3.1, the conclusion follows.

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