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Multiplicity result for a critical elliptic system with concave-convex nonlinearities

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Abstract

We study the existence of multiple solutions of a strongly indefinite elliptic system involving the critical Sobolev exponent and concave-convex nonlinearities. By using a suitable version of the dual fountain theorem established in this paper, we prove the existence of infinitely many small energy solutions.

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1 Introduction

In this paper, we study the existence and multiplicity of solutions for the following elliptic system of noncooperative type:

$$
\begin{cases}
-\Delta u = \lambda |u|^{p-2}u - \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\
\Delta v = \mu |v|^{q-2}v - \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\
u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{cases}
\tag{$S_{\lambda,\mu}$}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$, $\lambda > 0$, and $\mu > 0$, $1 < p, q < 2$ and $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^{\star}$, where $2^{\star} := 2N/N - 2$ denotes the critical Sobolev exponent. The solutions of $(S_{\lambda,\mu})$ are steady states of reaction-diffusion systems, which serve as a class of models with applications in physics, chemistry and biology (see, for instance, [1, 2]).

It is well known that weak solutions of $(S_{\lambda,\mu})$ are critical points of the following functional defined on the Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$:

$$
J(u,v) := \int_{\Omega}\left[\frac{1}{2}\left(|\nabla u|^2 - |\nabla v|^2\right) - \frac{\lambda}{p}|u|^p - \frac{\mu}{q}|v|^q + \frac{1}{\alpha+\beta}|u|^{\alpha}|v|^{\beta}\right]\mathrm{d}x.
\tag{1}
$$

We call (u,v) a weak solution of $(S_{\lambda,\mu})$ if $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and

$$
\int_{\Omega}\bigg[\nabla u\nabla\phi - \nabla v\nabla\varphi - \lambda|u|^{p-2}u\phi - \mu|v|^{q-2}v\varphi
$$
$$
+ \frac{1}{\alpha+\beta}\big(\alpha|u|^{\alpha-2}|v|^{\beta}u\phi + \beta|u|^{\alpha}|v|^{\beta-2}v\varphi\big)\bigg]\mathrm{d}x = 0
$$

for every $(\phi, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

The celebrated papers by Brezis and Nirenberg [3] and by Ambrosetti *et al.* [4] have inspired research on differential equations and systems with critical terms, and with concave-convex nonlinearities, respectively, see, for example, [5–12] and the many references therein. In recent years, more and more attention have been paid to the existence and multiplicity of solutions of the following elliptic system of cooperative type:

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v = \mu |v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \qquad (P_{\lambda,\mu})$$

In [9], by extracting Palais-Smale sequences on the Nehari manifold, Hsu and Lin proved that if λ and μ are small, then $(P_{\lambda,\mu})$ has at least two solutions. Subsequently, their result was extended by Chen and Wu [13], who considered $(P_{\lambda,\mu})$ with sign-changing weight functions. See also [12, 14] and the references therein for some related results.

However, what can be said about the noncooperative case? To the best of our knowledge, there is no result on problem $(S_{\lambda,\mu})$, even in the subcritical case $\alpha + \beta < 2^\star$. In addition to the combined effects of the concave and convex terms, two main difficulties arise when studying $(S_{\lambda,\mu})$. The first one is that the energy functional J above is strongly indefinite in the sense that it is neither bounded from above nor from below, even on subspaces of finite codimension. Therefore, the usual critical point theorems such as the mountain pass theorem cannot be used. Moreover, the method of the Nehari manifold, which is extensively used in the cooperative case does not apply in this situation. We refer to [15] for a unified approach on the method of the Nehari manifold. The second difficulty arises with the critical Sobolev exponent. Indeed, it is well known that the embedding $H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega)$ is not compact because of the action of dilatations. Therefore, the energy functional is not expected to satisfy the Palais-Smale condition (see Definition 5 below). Usually, the best we can expect is to find a bounded from above subset $I \subset \mathbb{R}$ such that the functional satisfies the Palais-Smale condition at every level $c \in I$, see, for instance, [3, 9, 16, 17]. But surprisingly enough, we will show that in our case the functional J satisfies the Palais-Smale condition.

The main result of the paper is the following.

Theorem 1 *Let $\lambda, \mu > 0$. Then problem $(S_{\lambda,\mu})$ has a sequence of solutions (u_k, v_k) such that $J(u_k, v_k) < 0$ and $J(u_k, v_k) \to 0$ as $k \to \infty$.*

Remark 2 Theorem 1 remains true in the subcritical case, *i.e.*, if we assume that $\alpha + \beta > 2$ if $N = 1, 2$ or $2 < \alpha + \beta < 2^\star$ if $N = 3$. In this case, the proof of Theorem 1 is considerably simplified, since the embedding $H_0^1(\Omega) \hookrightarrow L^{\alpha+\beta}(\Omega)$ is now compact.

The paper is organized as follows. In order to prove Theorem 1, we need a new critical point theorem for strongly indefinite functionals, which will be provided in Section 2. This critical point theorem generalizes the dual fountain theorem of Bartsch and Willem, and is not based on any reduction method. It should be noted that problem $(S_{\lambda,\mu})$ does not fit into the framework of the theorem proved in [18], which is only suitable for finding large energy solutions. Finally the proof of our main result is presented in Section 3.

2 Critical point theory

Let Y be a closed subspace of a separable Hilbert space X endowed with the inner product $(\cdot)$ and the associated norm $\|\cdot\|$. We denote by $P : X \to Y$ and $Q : X \to Z := Y^\perp$ the orthogonal projections.

We fix an orthonormal basis $(e_j)_{j \geq 0}$ of Y, and we consider on $X = Y \oplus Z$ the τ-topology introduced by Kryszewski and Szulkin in [19]; that is, the topology associated to the following norm

$$\|\|u\|\| := \max\left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |(Pu, e_j)|, \|Qu\| \right), \quad u \in X.$$

Clearly $\|Qu\| \leq \|\|u\|\| \leq \|u\|$. Moreover, if (u_n) is a bounded sequence in X, then

$$u_n \xrightarrow{\tau} u \Longleftrightarrow Pu_n \rightharpoonup Pu \quad \text{and} \quad Qu_n \to Qu.$$

Definition 3 Let $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ and $a, b \in \mathbb{R}$, $a < b$. We say that Φ' is τ-weak sequentially compact in $\Phi^{-1}([a,b])$ if for every sequence u_n which τ-converges to u in $\Phi^{-1}([a,b])$, there is a subsequence (u_{n_k}) such that $\Phi'(u_{n_k}) \rightharpoonup \Phi'(u)$.

Lemma 4 (Deformation lemma) *Let $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional which is τ-upper semicontinuous, and such that $\nabla\varphi$ is τ-weak sequentially compact in $\varphi^{-1}([a,b])$ for any $a, b \in \mathbb{R}$. Let $S \subset X$ with $-S = S$, and $c \in \mathbb{R}$, $\varepsilon, \theta > 0$ such that*

$$\forall u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\theta}, \quad \|\varphi'(u)\| \geq \frac{8\varepsilon}{\theta}.$$

Then there exists $\eta \in \mathcal{C}([0,1] \times \varphi^{c+2\varepsilon}, X)$ such that
   (i) *$\eta(t, u) = u$ if $t = 0$ or if $u \notin \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\theta}$,*
   (ii) *$\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$,*
   (iii) *$\|\eta(t, u) - u\| \leq \frac{\theta}{2} \ \forall u \in \varphi^{c+2\varepsilon}, \forall t \in [0, 1]$,*
   (iv) *$\varphi(\eta(\cdot, u))$ is non increasing, $\forall u \in \varphi^{c+2\varepsilon}$,*
   (v) *Each point $(t, u) \in [0, 1] \times \varphi^{c+2\varepsilon}$ has a τ-neighborhood $N_{(t,u)}$ such that*
      $\{v - \eta(s, v) | (s, v) \in N_{(t,u)} \cap ([0, 1] \times \varphi^{c+2\varepsilon})\}$ is contained in a finite-dimensional subspace of X,
   (vi) *η is τ-continuous,*
   (vii) *$\eta(t, \cdot)$ is odd $\forall t \in [0, 1]$,*
where $S_\alpha := \{u \in X | \mathrm{dist}(u, S) \leq \alpha\} \ \forall \alpha > 0$ and $\varphi^a := \{u \in X | \varphi(u) \leq a\} \ \forall a \in \mathbb{R}$.

Proof We define

$$w(v) := 2\|\nabla\varphi(v)\|^{-2} \nabla\varphi(v), \quad \forall v \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]).$$

We claim that for every $v \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon])$, there exists a τ-open neighborhood N_v of v such that $(\nabla\varphi(u), w(v)) > 1 \ \forall u \in N_v$ (N_v can be chosen to be symmetric, that is, $-N_v = N_v$). In fact, if this is not true, then we can find a sequence $(v_n) \subset \varphi^{-1}([c-2\varepsilon, c+2\varepsilon])$ which τ-converges to an element $v \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon])$ and such that $(\nabla\varphi(v_n), w(v)) \leq 1$. Since $\nabla\varphi$ is τ-weak sequentially compact, there is a subsequence (v_{n_k}) such that $\nabla\varphi(v_{n_k}) \rightharpoonup \nabla\varphi(v)$, and this implies that $(\nabla\varphi(v), w(v)) \leq 1$, which is in contradiction with the definition of w.

The rest of the proof follows the same lines as the proof of Lemma 8 in [18] with $G = \mathbb{Z}_2$.

$\square$

We introduce the following notations:

$$Y_k := \overline{\bigoplus_{j=k}^{\infty} \mathbb{R}e_j}, \qquad Z_k := \left(\bigoplus_{j=0}^{k} \mathbb{R}e_j\right) \oplus Z,$$

$$B_k := \{u \in Y_k | \|u\| \leq \sigma_k\}, \qquad N_k := \{u \in Z_k | \|u\| = s_k\}, \quad \text{with } 0 < s_k < \sigma_k, k \geq 2.$$

Before we state and prove our critical point theorem, we recall the following definition.

Definition 5 A functional $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ is said to satisfy the Palais-Smale condition (resp. $(PS)_c$ condition) if every sequence $(u_n) \subset X$ such that $(\Phi(u_n))$ is bounded (resp. $\Phi(u_n) \to c$) and $\Phi'(u_n) \to 0$, has a convergent subsequence.

Theorem 6 (Generalized dual fountain theorem) *Let $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional which is τ-lower semicontinuous and such that $\nabla \Phi$ is τ-weak sequentially compact in $\Phi^{-1}([a, b])$ for any $a, b \in \mathbb{R}$. If, for every $k \geq k_0$, there exist $\sigma_k > s_k > 0$ such that*

(B$_1$) $a_k := \inf_{\substack{u \in Y_k \\ \|u\| = \sigma_k}} \Phi(u) \geq 0,$
(B$_2$) $b_k := \sup_{\substack{u \in Z_k \\ \|u\| = s_k}} \Phi(u) < 0,$
(B$_3$) $d_k := \inf_{\substack{u \in Y_k \\ \|u\| \leq \sigma_k}} \Phi(u) \to 0, k \to \infty,$
(B$_4$) Φ *satisfies the* $(PS)_c$ *condition* $\forall c \in [d_{k_0}, 0[.$

Then Φ has a sequence of critical points (u_k) such that $\Phi(u_k) < 0$ and $\Phi(u_k) \to 0$ as $k \to \infty$.

Proof Let Γ_k be the set of maps $\gamma : B_k \to X$ such that

(a) γ is odd, τ-continuous and $\gamma|_{\partial B_k} = \text{id}$,
(b) each $u \in \text{int}(B_k)$ has a τ-neighborhood $\mathcal{N}_u$ in Y_k such that $(\text{id} - \gamma)(\mathcal{N}_u \cap \text{int}(B_k))$ is contained in a finite dimensional subspace of X,
(c) $\Phi(\gamma(u)) \geq \Phi(u) \ \forall u \in B_k.$

Define

$$c_k := \sup_{\gamma \in \Gamma_k} \inf_{u \in B_k} \Phi(\gamma(u)).$$

It follows from the definitions that $d_k \leq c_k$. Lemma 10 in [18] implies that $\gamma(B_k) \cap N_k \neq \emptyset$ $\forall \gamma \in \Gamma_k$, which in turn implies that $c_k \leq b_k$.

Let $\varepsilon \in]0, \frac{a_k - c_k}{2}[$, $\theta > 0$, and let $\gamma \in \Gamma_k$ such that

$$c_k - \varepsilon \leq \inf_{u \in B_k} \Phi(\gamma(u)). \tag{2}$$

We claim that

$$\exists u \in \Phi^{-1}([c_k - 2\varepsilon, c_k + 2\varepsilon]) \cap (\gamma(B_k))_{2\theta} \quad \text{such that } \|\Phi'(u)\| \leq \frac{8\varepsilon}{\theta}. \tag{3}$$

Hence, there exists a sequence $(u_n) \subset X$ such that

$$\Phi(u_n) \to c_k \quad \text{and} \quad \Phi'(u_n) \to 0, \quad n \to \infty.$$

The conclusion of Theorem 6 then follows from (B_4) and (B_3).

To complete the proof of Theorem 6, it remains to show that (3) holds.

We proceed by contradiction by assuming that (3) does not hold, and then, we apply Lemma 4 with $\varphi = -\Phi$, $c = -c_k$ and $S = \gamma(B_k)$. We may assume that $c_k + 2\varepsilon < a_k$. Next, by following the proof of Theorem 11 in [18], one can easily verify that the map β defined on B_k by $\beta(u) := \eta(1, \gamma(u))$ belongs to Γ_k. Now, (2) and (ii) of Lemma 4 imply that

$$\inf_{u \in B_k} \Phi(\beta(u)) \geq c_k + \varepsilon,$$

which contradicts the definition of c_k. □

3 Proof of the main results

In this section, we denote by $|u|_p$ the usual $L^p(\Omega)$ norm.

We consider the Sobolev space $H_0^1(\Omega)$ endowed with the norm $\|u\| = |\nabla u|_2$ and $X := H_0^1(\Omega) \times H_0^1(\Omega)$ with the product norm $\|(u, v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$.

The functional J defined by (1) then reads as follows:

$$J(u, v) := \frac{1}{2}\|u\|^2 - \frac{1}{2}\|v\|^2 - \frac{\lambda}{p}|u|_p^p - \frac{\mu}{q}|v|_q^q + \frac{1}{\alpha + \beta}\int_\Omega |u|^\alpha |v|^\beta \, dx. \tag{4}$$

A standard argument shows that J is of class $\mathcal{C}^1$ on X and

$$\langle J'(u, v), (\phi, \varphi)\rangle = \int_\Omega \Bigg[\nabla u \nabla \phi - \nabla v \nabla \varphi - \lambda |u|^{p-2} u\phi - \mu |v|^{q-2} v\varphi$$
$$+ \frac{1}{\alpha + \beta}\big(\alpha |u|^{\alpha-2}|v|^\beta u\phi + \beta |u|^\alpha |v|^{\beta-2} v\varphi\big)\Bigg] dx. \tag{5}$$

Lemma 7 *Let $(u_n, v_n) \subset X$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in X. Then there exists a subsequence (u_{n_k}, v_{n_k}) such that $J'(u_{n_k}, v_{n_k}) \rightharpoonup J'(u, v)$.*

Proof Let $(\phi, \varphi) \in X$. Using (5), we have

$$\langle J'(u_n, v_n) - J'(u, v), (\phi, \varphi)\rangle = \int_\Omega \Bigg[\nabla(u_n - u)\nabla\phi - \nabla(v_n - v)\nabla\varphi$$
$$- \lambda\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\phi - \mu\big(|v_n|^{q-2}v_n - |v|^{q-2}v\big)\varphi$$
$$+ \frac{\alpha}{\alpha + \beta}\big(|u_n|^{\alpha-2}u_n|v_n|^\beta - |u|^{\alpha-2}u|v|^\beta\big)\phi$$
$$+ \frac{\beta}{\alpha + \beta}\big(|u_n|^\alpha|v_n|^{\beta-2}v_n - |u|^\alpha|v|^{\beta-2}v\big)\varphi\Bigg] dx.$$

It is clear that

$$\int_\Omega \nabla(u_n - u)\nabla\phi \, dx \to 0 \quad \text{and} \quad \int_\Omega \nabla(v_n - v)\nabla\varphi \, dx \to 0.$$

Now up to a subsequence we have

$$u_n \to u \quad \text{and} \quad v_n \to v \quad \text{a.e. on } \Omega.$$

By the Sobolev embedding theorem (u_n) and (v_n) are bounded in $L^p(\Omega)$, $L^q(\Omega)$ and $L^{\alpha+\beta}(\Omega)$. So $(|u_n|^{p-2}u_n)$ (resp. $(|v_n|^{q-2}v_n)$) is bounded in $L^{\frac{p}{p-1}}(\Omega)$ (resp. in $L^{\frac{q}{q-1}}(\Omega)$), $(|u_n|^{\alpha-2}u_n|v_n|^\beta)$ and $(|u_n|^\alpha|v_n|^{\beta-2}v_n)$ are bounded in $L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega)$.

By Theorem 10.36 in [20], we have

$$|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u \quad \text{in } L^{\frac{p}{p-1}}(\Omega), \qquad |v_n|^{q-2}v_n \rightharpoonup |v|^{q-2}v \quad \text{in } L^{\frac{q}{q-1}}(\Omega),$$

$$|u_n|^{\alpha-2}u_n|v_n|^\beta \rightharpoonup |u|^{\alpha-2}u|v|^\beta \quad \text{and} \quad |u_n|^\alpha|v_n|^{\beta-2}v_n \rightharpoonup |u|^\alpha|v|^{\beta-2}v \quad \text{in } L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega).$$

Since $\phi, \varphi \in H_0^1(\Omega) \subset L^p(\Omega) \cap L^q(\Omega) \cap L^{\alpha+\beta}(\Omega)$, it follows that

$$\int_\Omega \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\phi \, dx \to 0, \qquad \int_\Omega \left(|v_n|^{q-2}v_n - |v|^{q-2}v\right)\varphi \, dx \to 0,$$

$$\int_\Omega \left(|u_n|^{\alpha-2}u_n|v_n|^\beta - |u|^{\alpha-2}u|v|^\beta\right)\phi \, dx \to 0,$$

$$\int_\Omega \left(|u_n|^\alpha|v_n|^{\beta-2}v_n - |u|^\alpha|v|^{\beta-2}v\right)\varphi \, dx \to 0,$$

and $\langle J'(u_n, v_n) - J'(u, v), (\phi, \varphi)\rangle \to 0$ for every $(\phi, \varphi) \in X$. Hence, $J'(u_n, v_n) \rightharpoonup J'(u, v)$. □

Lemma 8 *The functional J satisfies the Palais-Smale condition.*

Proof Let $(u_n, v_n) \subset X$ such that

$$d := \sup_n |J(u_n, v_n)| < \infty \quad \text{and} \quad J'(u_n, v_n) \to 0, \quad n \to \infty.$$

We deduce from (5) that

$$\langle J'(u_n, v_n), (u_n, 0)\rangle = \|u_n\|^2 - \lambda|u_n|_p^p + \frac{\alpha}{\alpha+\beta}\int_\Omega |u_n|^\alpha|v_n|^\beta \, dx$$

$$\geq \|u_n\|^2 - \lambda|u_n|_p^p.$$

Hence for n big enough, we have

$$\|u_n\|^2 - \lambda|u_n|_p^p \leq \|u_n\|,$$

which implies, since $p < 2$, that (u_n) is bounded.

On the other hand, by using (4) and (5), we obtain

$$-J(u_n, v_n) + \frac{1}{\alpha+\beta}\langle J'(u_n, v_n), (u_n, v_n)\rangle = \left(\frac{1}{\alpha+\beta} - \frac{1}{2}\right)\|u_n\|^2 + \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|v_n\|^2$$

$$+ \lambda\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)|u_n|_p^p + \mu\left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)|v_n|_q^q.$$

Since (u_n) is bounded, $q < 2 < \alpha + \beta$ and

$$-J(u_n, v_n) + \frac{1}{\alpha + \beta} \langle J'(u_n, v_n), (u_n, v_n) \rangle \leq d + \|(u_n, v_n)\|$$

for n big enough, we easily deduce that (v_n) is bounded.

Consequently, we have, up to a subsequence,

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X, \tag{6}$$

$$u_n \to u \quad \text{in } L^p(\Omega), \qquad v_n \to v \quad \text{in } L^q(\Omega), \tag{7}$$

$$u_n \to u, \qquad v_n \to v \quad \text{a.e. on } \Omega,$$

$$\nabla J(u_n, v_n) \rightharpoonup \nabla J(u, v) \quad \text{(by Lemma 7)}.$$

On the other hand, $J'(u_n, v_n) \to 0$ and equation (6) imply that $J'(u, v) = 0$ and that

$$\langle J'(u_n, v_n), (u_n, 0) \rangle - \langle J'(u, v), (u, 0) \rangle \to 0, \qquad \langle J'(u_n, v_n), (0, v_n) \rangle - \langle J'(u, v), (0, v) \rangle \to 0.$$

Now, because of equation (7), it follows that

$$\|u_n\|^2 - \|u\|^2 + \frac{\alpha}{\alpha + \beta} \int_\Omega \left(|u_n|^\alpha |v_n|^\beta - |u|^\alpha |v|^\beta \right) dx \to 0,$$

$$-\|v_n\|^2 + \|v\|^2 + \frac{\beta}{\alpha + \beta} \int_\Omega \left(|u_n|^\alpha |v_n|^\beta - |u|^\alpha |v|^\beta \right) dx \to 0.$$

Let us combine the preceding equations in order to cancel the integrals. We then get

$$\beta \left(\|u_n\|^2 - \|u\|^2 \right) + \alpha \left(\|v_n\|^2 - \|v\|^2 \right) \to 0.$$

But since equation (6) leads to

$$\|u_n\|^2 = \|u_n - u\|^2 + \|u\|^2 + \circ(1), \qquad \|v_n\|^2 = \|v_n - v\|^2 + \|v\|^2 + \circ(1),$$

it is then easy to conclude that

$$\beta \|u_n - u\|^2 + \alpha \|v_n - v\|^2 \to 0.$$

Hence $(u_n, v_n) \to (u, v)$. □

Now, we select an orthonormal basis (e_j) of $H_0^1(\Omega)$, and we consider the τ-topology on $X = Y \oplus Z$, where Y and Z are defined by

$$Y := H_0^1(\Omega) \times \{0\} \quad \text{and} \quad Z := \{0\} \times H_0^1(\Omega).$$

Lemma 9 *J is τ-lower semicontinuous, and ∇J is τ-weak sequentially compact in $J^{-1}([a, b])$ for any $a, b \in \mathbb{R}$.*

Proof

1. Let $(u_n, v_n) \subset X$ and $C \in \mathbb{R}$ such that $(u_n, v_n) \xrightarrow{\tau} (u, v)$ in X and $J(u_n, v_n) \leq C$. By the definition of τ, (v_n) converges strongly to v in $H_0^1(\Omega)$. Clearly, we have

$$J(u_n, v_n) \geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{p}|u_n|_p^p - \frac{\mu}{q}|v_n|_q^q,$$

which implies, since (v_n) is bounded and $p < 2$, that (u_n) is bounded. Up to a subsequence, we may assume that $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$, $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ for almost every $x \in \Omega$. By using Fatou's lemma and the weak lower semicontinuity of the norm $\|\cdot\|$, we deduce from the inequality

$$C \geq J(u_n, v_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{p}|u_n|_p^p - \frac{\mu}{q}|v_n|_q^q + \frac{1}{\alpha+\beta}\int_\Omega |u_n|^\alpha |v_n|^\beta \, dx$$

that $C \geq J(u, v)$.

2. Assume that $(u_n, v_n) \xrightarrow{\tau} (u, v)$ in $J^{-1}([a, b])$, $a, b \in \mathbb{R}$. The same argument as above shows that (u_n, v_n) is bounded, and then $(u_n, v_n) \rightharpoonup (u, v)$. It follows from Lemma 7 that ∇J is τ-weak sequentially compact in $J^{-1}([a, b])$. $\qquad\square$

Proof of Theorem 1 We recall that

$$Y_k := \overline{\bigoplus_{j=k}^\infty \mathbb{R}e_j \times \{0\}} \quad \text{and} \quad Z_k := \left(\bigoplus_{j=0}^k \mathbb{R}e_j \times \{0\}\right) \oplus Z.$$

Let $(u, 0) \in Y_k$. Then, since $\lambda > 0$, we have

$$J(u, 0) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}|u|_p^p \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\theta_k^p \|u\|^p,$$

where

$$\theta_k = \sup_{\substack{w \in \bigoplus_{j=k}^\infty \mathbb{R}e_j \\ \|w\|=1}} |w|_p.$$

Therefore, for every $(u, 0) \in Y_k$ such that $\|u\| = \sigma_k := 2(\frac{2\lambda}{p}\theta_k^p)^{\frac{1}{2-p}}$, we have

$$J(u, 0) \geq \left(\frac{\lambda}{p}\theta_k^p\right)^{\frac{2}{2-p}} 2^{\frac{p}{2-p}} (2^2 - 2^p) > 0.$$

On the other hand, it is clear that

$$\frac{1}{2}\|u\|^2 \geq J(u, 0) \geq -\frac{\lambda}{p}\theta_k^p \|u\|^p.$$

Hence, for every $(u, 0) \in Y_k$ such that $\|u\| \leq \sigma_k$, we have

$$\frac{1}{2}\sigma_k^2 \geq J(u, 0) \geq -\frac{\lambda}{p}\theta_k^p \sigma_k^p.$$

We know by [21] that $\theta_k \to 0$ as $k \to \infty$, hence we deduce that $J(u, 0) \to 0$ as $k \to \infty$.

We have then proved that assumptions (B_1) and (B_3) of Theorem 6 are satisfied.

Let $(u, v) \in Z_k$. Since the norms $\| \cdot \|$ and $| \cdot |_p$ are equivalent on $\bigoplus_{j=0}^{k} \mathbb{R}e_j$, $\mu > 0$ and $H_0^1(\Omega)$ continuously embeds into $L^{\alpha+\beta}(\Omega)$, we have

$$J(u, v) \leq \frac{1}{2} \|u\|^2 - \frac{\lambda c_1}{p} \|u\|^p + \frac{c}{\alpha + \beta} \|u\|^{\alpha+\beta} - \frac{1}{2} \|v\|^2 + \frac{c}{\alpha + \beta} \|v\|^{\alpha+\beta},$$

where $c_1 > 0$ and $c > 0$ are constants. It is then easy to verify, since $\lambda > 0$, that assumption (B_2) of Theorem 6 is satisfied for s_k small enough.

By using Lemmas 8, 9, we can apply Theorem 6 and get the desired result. □

Authors' contributions
All authors contributed equally to the manuscript. In addition, both authors read and approved the final manuscript.

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