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Global existence and nonexistence of solutions for quasilinear parabolic equation

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Abstract

This work is concerned with the global existence and nonexistence of solutions for a quasilinear parabolic equation with null Dirichlet boundary condition. Based on the Galerkin approximation technique and the theory of a family of potential wells, we obtain the invariant sets and vacuum isolating of global solutions including critical case, and we also give global nonexistence.

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1 Introduction

Our main interest lies in the following quasilinear p-Laplacian parabolic equation:

$$|u_t|^{m-2}u_t = \text{div}(|\nabla u|^{p-2}\nabla u) + bu^{1+q}, \quad x \in \Omega, t > 0, \tag{1.1}$$

subject to homogeneous Dirichlet boundary and initial conditions

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $b > 0$, $\Omega \subset R^N$ ($N \geq 1$) is a bounded domain with smooth boundary, $p < 2 + q < \infty$ if $N \leq p$; $p < 2 + q < \frac{Np}{N-p}$ if $N > p$ and $2 < m < 2 + q$. For simplicity, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ and $(u,v) = \int_\Omega uv\,dx$.

Many natural phenomena have been formulated as the nonlinear diffusive equation (1.1) such as the model of non-Newton flux in the mechanics of a fluid, the model of a population, biological species and filtration; we refer to [1, 2] and the references therein. In the non-Newtonian theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids, while media with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

There have been a lot of results on the global existence and nonexistence of solutions for nonlinear evolution equations and many effective methods have been developed such as the compactness method, the semi-group method, continuation of local solutions, the upper-lower solution method, and the concavity method; see [3–6]. Since the potential well was introduced by Sattinger [7] in order to prove the global existence of solutions for

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nonlinear hyperbolic equations which do not necessarily have positive definite energy, it has become an important method to study the global existence and nonexistence of solutions for various nonlinear evolution equations. It is an effective method to get a positive definite energy through one term, controlled by the other term in the principal part of the equations [7–19]. For instance, Levine [8] investigated the initial boundary value problem of the linear heat conduction equation

$$u_t = \Delta u, \quad x \in \Omega, t \in (0, T),$$

with nonlinear boundary condition, and he obtained the global existence and nonexistence of weak solutions. Payne and Sattinger [9] investigated the initial boundary value problem of the semilinear hyperbolic equation with fully nonlinear term

$$u_{tt} = \Delta u + f(u), \quad x \in \Omega, t > 0,$$

and proved the weak solution blows up in finite time. They also extended the results to the initial boundary value problem of the corresponding parabolic equation. Tsutsumi [10] studied the homogeneous Dirichlet initial boundary value problem of the nonlinear parabolic equation

$$u_t = \text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^{1+q}, \quad x \in \Omega, t > 0, \tag{1.4}$$

and he obtained the sufficient conditions of the existence of global weak solutions and the solutions blow up in finite time for the case $p < 2 + q$. Later, Liu [11] proved the global existence of solutions of the homogeneous Dirichlet initial boundary value problem for (1.4) with critical initial conditions. Pang and Zhang [12] investigated the initial boundary value problem of the quasilinear parabolic equation

$$|u_t|^{m-2}u_t = \Delta u + |u|^{q-1}u, \quad x \in \Omega, t > 0,$$

and they obtained the sufficient conditions as regards global existence and nonexistence of solution by using a potential well method.

However, the potential wells used in these works were defined by the same method as Sattinger [7] and their results were similar. Until Liu [20] firstly introduced the theory of a family of potential wells, described the structure of potential wells and the estimates of the depth of potential wells. And he firstly found the phenomenon of vacuum isolating of solutions for nonlinear evolution equations. The study of applications about a family of potential wells has attracted more and more attention [20–25]. For instance, Liu and Zhao [21] not only proved the global existence and nonexistence of solutions, but they also obtained the vacuum isolating of solutions of the initial boundary value problem for semilinear hyperbolic equations and parabolic equations.

As far as we know, there are fewer papers on the global existence and nonexistence of weak solutions for nonlinear parabolic equations by using the theory of a family of potential wells. In particular, for our problem (1.1)-(1.3), the analysis of the structure and depth of the potential well, the invariant sets, the vacuum isolating of global solutions, and the question of the global existence of solutions with critical initial conditions are still open.

It is difficult to obtain an *a priori* estimate of the approximate solution for the study of the existence of global solutions by using the general Galerkin approximation method, but the theory of the potential well often makes up for the defect. The combination of the two methods can be used to solve the existence of solutions effectively. Moreover, the study of the phenomenon of vacuum isolating will be helpful for us in studying the distribution of solutions in Sobolev space. But the depth of potential well d for the problem (1.1)-(1.3) is usually very small *etc.* Our goal is to improve the theory of a family of potential wells for studying the global existence and nonexistence of solutions for our problem (1.1)-(1.3), including the critical case, and we further generalize the results in [10–12, 20].

The outline of the paper is as follows. In Section 2, we firstly give the definition of the weak solution for problem (1.1)-(1.3), and the definition and properties of a family of potential wells. Then we prove the global existence of solutions for problem (1.1)-(1.3) by using the Galerkin approximation technique and the theory of a family of potential wells in Section 3. The invariant sets of global solutions and vacuum isolating are obtained in Section 4. Then the sufficient condition of global nonexistence of solutions is given in Section 5. Finally, we give the result of global existence with critical initial conditions.

2 Preliminaries

Due to the degeneracy of (1.1), problem (1.1)-(1.3) has no classical solutions in general. We need to give the definition of the weak solution firstly.

Definition 1 A function $u = u(x, t)$ is called a weak solution of problem (1.1)-(1.3) on $\Omega \times [0, T)$ if it satisfies the following conditions:

(1) $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$, $u_t \in L^m(0, T; L^m(\Omega))$;

(2) $\int_0^t ((|u_t|^{m-2} u_t, v) + (|\nabla u|^{p-2} \nabla u, \nabla v)) \, d\tau = \int_0^t (bu^{q+1}, v) \, d\tau$ for $\forall v \in W_0^{1,p}(\Omega)$, $t \in [0, T)$;

(3) $u(x, 0) = u_0(x)$,

where T is either infinity or the limit of the existence interval of solution.

In order to study the problem (1.1)-(1.3), we also consider the auxiliary equation

$$|u_t|^{m-2} u_t = \text{div}(|\nabla u|^{p-2} \nabla u) + b\varphi(u), \quad x \in \Omega, t > 0, \tag{1.1a}$$

where $\varphi(u) = \{u^{1+q}, \text{if } u \geq 0; 0, \text{if } u < 0\}$.

Next, we define the functionals $J(u)$, $I(u)$, and the potential well W as follows:

$$J(u) = \frac{1}{p} \|\nabla u\|_p^p - \frac{b}{q+2} \|u^+\|_{q+2}^{q+2},$$

$$I(u) = \|\nabla u\|_p^p - b \|u^+\|_{q+2}^{q+2},$$

$$W = \{u \in W_0^{1,p}(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\},$$

where $u^+ = \max\{u, 0\}$, $d = \inf_{u \in W_0^{1,p}, u \neq 0} (\sup_{\lambda \geq 0} J(\lambda u))$.

Furthermore, for problem (1.1)-(1.3) and $0 < \delta < 1$, we define the auxiliary functional $J_\delta(u)$ and function $d(\delta)$

$$J_\delta(u) = \frac{\delta}{p}\|\nabla u\|_p^p - \frac{b}{q+2}\|u^+\|_{q+2}^{q+2},$$

$$d(\delta) = \frac{1-\delta}{p}\left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{p}{q-p+2}},$$

where $C_* = \sup \frac{\|u\|_{q+2}}{\|\nabla u\|_p}$.

In the paper, we always assume that p and q satisfy (H):

$$p < 2 + q < \infty \quad \text{if } N \le p; \qquad p < 2 + q < \frac{Np}{N-p} \quad \text{if } N > p.$$

Before giving our main results, we show some preliminary lemmas which are very important in the following proofs. As for the proofs of these several lemmas, we will not repeat them again (see [11, 20]).

Lemma 1 ([11], Lemma 2.2) *For any given $u \in W_0^{1,p}(\Omega)$, $\|u^+\|_{q+2} \ne 0$, $g(\lambda) = J(\lambda u)$ possesses the following properties*:

(1) $\lim_{\lambda \to 0} g(\lambda) = 0$, $\lim_{\lambda \to +\infty} g(\lambda) = -\infty$;

(2) *There exists a unique $\overline{\lambda} = \overline{\lambda}(u) > 0$ such that $g'(\overline{\lambda}) = 0$*;

(3) $g'(\lambda) > 0$, *i.e. $g(\lambda)$ is increasing for $0 < \lambda < \overline{\lambda}$; $g'(\lambda) < 0$, i.e. $g(\lambda)$ is decreasing for $\overline{\lambda} < \lambda < +\infty$*;

(4) $g''(\lambda) < 0$.

Lemma 2 ([20], Lemmas 2.1-2.3) *The following sufficient and necessary conditions always hold*:

(1) *Let $J(u) \le d(\delta)$, then $J_\delta(u) > 0$ if and only if $0 < \|\nabla u\|_p < (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$.*

(2) *Let $J(u) \le d(\delta)$, then $J_\delta(u) < 0$ if and only if $\|\nabla u\|_p > (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$.*

(3) *Let $J(u) = d(\delta)$, then $J_\delta(u) = 0$ if and only if $\|\nabla u\|_p = (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$.*

Lemma 3 ([20], Lemma 2.4) *The function $d(\delta)$ possesses the following properties on the interval $0 \le \delta \le 1$*:

(1) $d(0) = d(1) = 0$;

(2) $d(\delta)$ *takes the maximum $d(\delta_0) = \frac{1}{\alpha b^\beta C_*^\alpha}$ at $\delta_0 = \frac{p}{q+2}$, where $\alpha = \frac{p(q+2)}{q-p+2}$, $\beta = \frac{p}{q-p+2}$;*

(3) $d(\delta)$ *is increasing on $[0, \delta_0]$ and decreasing on $[\delta_0, 1]$;*

(4) *For any given $e \in (0, d(\delta_0))$, the equation $d(\delta) = e$ has exactly two roots $\delta_1 \in (0, \delta_0)$ and $\delta_2 \in (\delta_0, 1)$.*

Lemma 4 ([20], Lemma 2.5) $d(\delta) = \inf J(u)$, *where $u \in W_0^{1,p}(\Omega)$, $\|\nabla u\|_p \ne 0$, $J_\delta(u) = 0$.*

Proposition 1 $d = d(\delta_0) = \inf(J(u))$, *where $u \in W_0^{1,p}(\Omega)$, $\|\nabla u\|_p \ne 0$, $I(u) = 0$.*

Proof The result can easily be obtained by Lemma 4 and the fact that $J_{\delta_0}(u) = 0$ is equivalent to $I(u) = 0$. □

Now we can define a family of potential wells as follows:

$$W_\delta = \left\{u \in W_0^{1,p}(\Omega) \big| J_\delta(u) > 0, J(u) < d(\delta)\right\} \cup \{0\}, \quad 0 < \delta < 1,$$

$$\overline{W}_\delta = W_\delta \cup \partial W_\delta = \left\{u \in W_0^{1,p}(\Omega) \big| J_\delta(u) \geq 0, J(u) \leq d(\delta)\right\}.$$

Obviously, we have $W_{\delta_0} = W$.

Remark 1 From $J(u) = \frac{1-\delta}{p}\|\nabla u\|_p^p + J_\delta(u)$, we see that $J_\delta(u) > 0$ implies that $J(u) > 0$.

In the following, we define

$$V_\delta = \left\{u \in W_0^{1,p}(\Omega) \big| J_\delta(u) < 0, J(u) < d(\delta)\right\}, \quad 0 < \delta < 1,$$

$$\overline{V}_\delta = V_\delta \cup \partial V_\delta = \left\{u \in W_0^{1,p}(\Omega) \big| J_\delta(u) \leq 0, J(u) \leq d(\delta)\right\},$$

$$V = \left\{u \in W_0^{1,p}(\Omega) \big| I(u) < 0, J(u) < d\right\},$$

$$B_\delta = \left\{u \in W_0^{1,p} \Big| \|\nabla u\|_p < \left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{1}{q-p+2}}\right\},$$

$$\overline{B}_\delta = B_\delta \cup \partial B_\delta = \left\{u \in W_0^{1,p} \Big| \|\nabla u\|_p \leq \left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{1}{q-p+2}}\right\},$$

$$B_\delta^c = \left\{u \in W_0^{1,p} \Big| \|\nabla u\|_p > \left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{1}{q-p+2}}\right\}.$$

Obviously, we have $V_{\delta_0} = V$.

Note that $J(u) \leq \frac{1}{p}\|\nabla u\|_p^p$, hence for any given $\delta \in (0,1)$, when $0 < \|\nabla u\|_p < (1-\delta)^{\frac{1}{2}} \times (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$, we have $J(u) < d(\delta)$ and $J_\delta(u) > 0$. This implies that $B_{\overline{\delta}} \subset W_\delta$, where $\overline{\delta}$ satisfies $(\frac{q+2}{pbC_*^{q+2}}\overline{\delta})^{\frac{1}{q-p+2}} = (1-\delta)^{\frac{1}{2}}(\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$.

Lemma 5 ([20], Theorem 2.7) *Suppose that W_δ, V_δ, B_δ, B_δ^c, and $\overline{\delta}$ are defined as the above, then*

$$B_{\overline{\delta}} \subset W_\delta \subset B_\delta, \qquad V_\delta \subset B_\delta^c.$$

Lemma 6 ([20], Lemma 2.10) *Assume that $0 < J(u) < d$ for some given $u \in W_0^{1,p}(\Omega)$, $\delta_1 < \delta_2$ are the two roots of the equation $d(\delta) = J(u)$, then the sign of $J_\delta(u)$ is not changed for $\delta \in (\delta_1, \delta_2)$.*

Lemma 7 ([11], Lemma 2.8) *Let p and q satisfy* (H), *then the solutions given in Theorem* 1 *satisfy*

$$\int_0^t \|u_t\|_m^m \, d\tau + J(u) \leq J(u_0), \quad \forall t \geq 0.$$

3 Existence of global weak solutions

In this section, we obtain the global existence of solutions for problem (1.1)-(1.3) by combining the Galerkin approximation technique and the theory of a family of potential wells.

Theorem 1 *Assume that p and q satisfy* (H), $u_0(x) \in W_0^{1,p}(\Omega)$. *If* $0 < J(u_0) < d$, $\delta_1 < \delta_2$ *are the two roots of the equation* $d(\delta) = J(u_0)$ *and* $J_{\delta_2}(u_0) > 0$ *or* $\|\nabla u_0\|_p = 0$, *then problem* (1.1a)-(1.3) *admits a global weak solution* $u(x,t)$ *such that* $u \in L^\infty(0, \infty; W_0^{1,p}(\Omega))$, $u_t \in L^m(0, \infty; L^m(\Omega))$ *and* $u \in W_\delta$ *for* $\delta \in (\delta_1, \delta_2)$ *and* $0 \le t < \infty$. *Furthermore, we have*

(1) $\|u(x,t)\|_m \le \|u(x,s)\|_m$ *for* $t \ge s \ge 0$;

(2) *If* $N < p$, *then the solution is uniquely determined by the initial function*;

(3) *If* $u_0(x) \ge 0$ *a.e. in* Ω, *the solution* $u(x,t) \ge 0$ *a.e. in* Ω *for any fixed* $t > 0$, *hence* $u(x,t)$ *is a solution of the problem* (1.1)-(1.3).

Proof Let $\{\omega_j(x)\}$ be a system of base functions of $W_0^{1,p}(\Omega)$. Construct approximate solutions $u_n(x,t)$ in the form

$$u_n(t) = \sum_{j=1}^n g_{jn}(t)\omega_j(x), \quad n = 1, 2, \ldots,$$

satisfying

$$\left(|u_{nt}|^{m-2}u_{nt}, \omega_s\right) + \left(|\nabla u_n|^{p-2}\nabla u_n, \nabla \omega_s\right) = \left(bu_n^{q+1}, \omega_s\right), \quad s = 1, 2, \ldots, n,$$

$$u_n(x, 0) = \sum_{j=1}^n a_{jn}\omega_j(x) \to u_0(x), \quad \text{in } W_0^{1,p}(\Omega).$$

Multiplying (1.1a) by $g'_{sn}(t)$, summing over s and integrating with respect to t, we obtain

$$J_n(t) = \frac{1}{p}\|\nabla u_n\|_p^p - \frac{b}{q+2}\|u_n^+\|_{q+2}^{q+2} + \int_0^t \|u_{nt}\|_m^m \, d\tau$$

$$= \frac{1}{p}\|\nabla u_n(0)\|_p^p - \frac{b}{q+2}\|u_n^+(0)\|_{q+2}^{q+2} = J_n(u_0). \tag{3.1}$$

Note that $J_{\delta_2}(u_0) > 0$ implies $\|\nabla u_0\|_p \ne 0$. By Lemma 6, we have $J_\delta(u_0) > 0$ for $\delta \in (\delta_1, \delta_2)$. From this and $J(u_0) = d(\delta_1) = d(\delta_2) < d(\delta)$, we obtain $u_0(x) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. If $\|\nabla u_0\|_p = 0$, then $u_0(x) \in W_\delta$ for $\delta \in (0, 1)$. For any fixed $\delta \in (\delta_1, \delta_2)$, we have $J_\delta(u_n(0)) > 0$ and $J_n(u_0) < d(\delta)$ (if $J_{\delta_2}(u_0) > 0$) or $u_n(0) \in B_{\overline{\delta}}$ (if $\|\nabla u_0\|_p = 0$ and $\overline{\delta}$ is defined in Lemma 5), thereby $u_n(0) \in W_\delta$ for sufficiently large n.

Next, we prove that $u_n(t) \in W_\delta$ for sufficiently large n and $t > 0$. Otherwise, there must be a $t_0 > 0$ such that $u_n(t_0) \in \partial W_\delta$, *i.e.* $J_\delta(u_n(t_0)) = 0$ and $\|\nabla u_n(t_0)\|_p \ne 0$ or $J(u_n(t_0)) = d(\delta)$. From (3.1), we have

$$J(u_n(t)) \le J_n(u_0) < d(\delta), \quad t > 0,$$

hence $J(u_n(t_0)) = d(\delta)$ is impossible. If $J_\delta(u_n(t_0)) = 0$ and $\|\nabla u_n(t_0)\|_p \ne 0$, then by Lemma 4, we have $J(u_n(t_0)) \ge d(\delta)$, which is also impossible. Thus from (3.1) and Lemma 2, we obtain

$$\|\nabla u_n(t)\|_p < \left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{1}{q-p+2}},$$

$$\|u_n(t)\|_{q+2} < C_*\|\nabla u_n(t)\|_p < C_*\left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{1}{q-p+2}},$$

and

$$\int_0^t \left\| u_{nt}(\tau) \right\|_m^m \, d\tau < d(\delta),$$

for $t > 0$ and sufficiently large n. From these and the compactness method, we can prove that problem (1.1a)-(1.3) admits a global weak solution $u(x,t)$ such that $u \in L^\infty(0,\infty; W_0^{1,p}(\Omega))$, $u_t \in L^m(0,\infty; L^m(\Omega))$ and $u \in W_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $0 \le t < \infty$.

Furthermore, by Theorem 1 in [10] we can easily get the results (1)-(3), here we omit the proofs. □

Similarly, we can get the following conclusions directly.

Corollary 1 *Under the conditions of Theorem* 1, *we have* $u \in \overline{W}_{\delta_1}$ *for* $0 \le t < \infty$.

Corollary 2 *If the assumption* $I(u_0) > 0$ *or* $\|\nabla u_0\|_p = 0$ *is replaced by* $J_{\delta_2}(u_0) > 0$ *or* $\|\nabla u_0\|_p = 0$, *i.e.* $u_0(x) \in W$, *then the conclusion of Theorem* 1 *also holds.*

Corollary 3 *If the assumption* $u_0(x) \in B_{\delta_2}$ *is replaced by* $J_{\delta_2}(u_0) > 0$ *or* $\|\nabla u_0\|_p = 0$, *then problem* (1.1)-(1.3) *admits a global weak solution* $u(x,t)$ *such that* $u \in L^\infty(0,\infty; W_0^{1,p}(\Omega))$, $u_t \in L^m(0,\infty; L^m(\Omega))$ *and* $u \in \overline{B}_{\delta_1}$ *for* $0 \le t < \infty$.

4 Invariant property and vacuum isolating of global solutions

In this section, we discuss the invariance of some sets under the flow of (1.1)-(1.3) and vacuum isolating behavior of solutions for problem (1.1)-(1.3).

4.1 Invariant property of global solutions

Theorem 2 *Assume that* p *and* q *satisfy* (H), $0 \le u_0(x) \in W_0^{1,p}(\Omega)$. *If* $0 < e < d$, $\delta_1 < \delta_2$ *are the two roots of the equation* $d(\delta) = e$, *then the following hold.*

(1) *All solutions of problem* (1.1)-(1.3) *with initial energy* $0 < J(u_0) \le e$ *belong to* $\overline{W}_\delta$ *for* $\delta \in (\delta_1, \delta_2)$, *provided that* $I(u_0) > 0$ *or* $\|\nabla u_0\|_p = 0$.

(2) *All solutions of problem* (1.1)-(1.3) *with initial energy* $0 < J(u_0) \le e$ *belong to* $\overline{V}_\delta$ *for* $\delta \in (\delta_1, \delta_2)$, *provided that* $I(u_0) < 0$.

Proof Firstly, we consider the case of $J(u_0) = e$. Here we denote $u(x,t) \triangleq u(t)$.

(1) Let $u(t)$ be any solution of problem (1.1)-(1.3) with initial energy $J(u_0) = e$ and $I(u_0) > 0$ or $\|\nabla u_0\|_p = 0$, T be the existence time of $u(t)$. Since

$$J_{\delta_0}(u_0) = \frac{\delta_0}{p} \|\nabla u_0\|_p^p - \frac{b}{q+2} \|u_0\|_{q+2}^{q+2}$$

$$= \frac{1}{q+2} \|\nabla u_0\|_p^p - \frac{b}{q+2} \|u_0\|_{q+2}^{q+2} = \frac{1}{q+2} I(u_0) > 0,$$

by Theorem 1, we have $u_0(x) \in W_\delta$.

Next we prove $u(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 < t < T$. Otherwise, there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1, \delta_2)$, *i.e.* $J_\delta(u(t_0)) = 0$, $\|\nabla u(t_0)\|_p \ne 0$ or $J(u(t_0)) = d(\delta)$. From Lemma 7

$$J(u) + \int_0^t \|u_t\|_m^m \, d\tau \le J(u_0) < d(\delta), \tag{4.1}$$

we see that $J(u(t_0)) = d(\delta)$ is impossible. On the other hand, if $J_\delta(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_p \neq 0$, then by Lemma 4, we have $J(u(t_0)) \geq d(\delta)$, which contradicts (4.1).

(2) Let $u(t)$ be any solution of problem (1.1)-(1.3) with initial energy $J(u_0) = e$ and $I(u_0) < 0$, T be the existence time of $u(t)$. Since the sign of $J_\delta(u_0)$ is not changed for $\delta \in (\delta_1, \delta_2)$, we have $J_\delta(u_0) < 0$ for $\delta \in (\delta_1, \delta_2)$. From this and $J(u_0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$ we obtain $u_0(x) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$.

Next we prove $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$ and $0 < t < T$. Otherwise, there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial V_\delta$ for some $\delta \in (\delta_1, \delta_2)$, *i.e.* $J_\delta(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta)$. From (4.1) we see that $J(u(t_0)) = d(\delta)$ is impossible. On the other hand, let t_0 be the first time such that $J_\delta(u(t_0)) = 0$, then $J_\delta(u(t)) < 0$ for $0 \leq t < t_0$. From (4.1) and Lemma 2, we have $\|\nabla u(t)\|_p > (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$ for $0 \leq t < t_0$. Hence we have $\|\nabla u(t_0)\|_p > (\frac{q+2}{pbC_*^{q+2}}\delta)^{\frac{1}{q-p+2}}$, thus by Lemma 4, we get $J(u(t_0)) \geq d(\delta)$, which contradicts (4.1).

For the case of $0 < J(u_0) < e$, we can obtain the same results as the case $J(u_0) = e$ by Lemma 4, we omit it here. □

Remark 2 Assume that $0 < J(u_0) \leq e$, then W_δ and V_δ are invariant under the flow of (1.1)-(1.3) for any $\delta \in (\delta_1, \delta_2)$.

From the above Theorem 2 and Lemma 2, we can easily get the following conclusions.

Theorem 3 *Let p and q satisfy* (H), $0 \leq u_0(x) \in W_0^{1,p}(\Omega)$. *Assume that* $0 < e < d$, $\delta_1 < \delta_2$ *are the two roots of the equation $d(\delta) = e$, then the following hold.*

(1) *All solutions of problem (1.1)-(1.3) with initial energy $0 < J(u_0) \leq e$ and $u_0(x) \in B_{\delta_0}$ belong to $\overline{B}_{\delta_1}$ for $\delta \in (\delta_1, \delta_2)$.*

(2) *All solutions of problem (1.1)-(1.3) with initial energy $0 < J(u_0) \leq e$ and $u_0(x) \in B_{\delta_0}^c$ belong to $\overline{B}_{\delta_2}^c$ for $\delta \in (\delta_1, \delta_2)$.*

Remark 3 Let p and q satisfy (H), $0 \leq u_0(x) \in W_0^{1,p}(\Omega)$. Assume that $0 < J(u_0) \leq e$, then B_δ and B_δ^c are invariant under the flow of (1.1)-(1.3) for any $\delta \in (\delta_1, \delta_2)$.

4.2 Vacuum isolating of global solutions

The result of Theorem 3 shows that for any given $e \in (0, d)$, there exists a corresponding vacuum region of solutions

$$U_e = \left\{ u \in W_0^{1,p}(\Omega) \middle| \left(\frac{q+2}{pbC_*^{q+2}}\delta_1 \right)^{\frac{1}{q-p+2}} < \|\nabla u\|_p < \left(\frac{q+2}{pbC_*^{q+2}}\delta_2 \right)^{\frac{1}{q-p+2}} \right\},$$

for the set of all solutions of problem (1.1)-(1.3) with initial energy $J(u_0)$ satisfying $0 < J(u_0) \leq e$, there is no solution in U_e and all solutions are isolated by U_e. This phenomenon is called the phenomenon of vacuum isolating of solutions. Obviously, the vacuum region U_e of solutions becomes bigger and bigger with decreasing of e. As the limit case $e = 0$, we obtain the biggest vacuum region of solutions (for $J(u_0) \geq 0$)

$$U_0 = \left\{ u \in W_0^{1,p}(\Omega) \middle| 0 < \|\nabla u\|_p < \left(\frac{q+2}{pbC_*^{q+2}} \right)^{\frac{1}{q-p+2}} \right\}.$$

Theorem 4 *Let p and q satisfy* (H), $0 \le u_0(x) \in W_0^{1,p}(\Omega)$. *All nontrivial solutions of problem* (1.1)-(1.3) *with initial energy* $J(u_0) = 0$ *lie outside of the ball* B_1 *(maybe in* ∂B_1).

Theorem 5 *Let p and q satisfy* (H), $0 \le u_0(x) \in W_0^{1,p}(\Omega)$. *All nontrivial solutions of problem* (1.1)-(1.3) *with initial energy* $J(u_0) < 0$ *satisfy*

$$\|\nabla u\|_p > \left(\frac{q+2}{pbC_*^{q+2}} \right)^{\frac{1}{q-p+2}}$$

and

$$\|\nabla u\|_p \ge \left[\frac{(q+2)\sqrt{\frac{-4J(u_0)}{p}}}{bC_*^{q+2}} \right]^{\frac{2}{2q-p+4}}.$$

Remark 4 The proofs of Theorems 4-5 are similar to Theorems 4.7-4.8 in [20], we omit them.

5 Nonexistence of global solutions
In this section, we given the sufficient condition of global nonexistence of solutions.

Theorem 6 *Assume that* $2 < m < 2 + q$, $u_0(x) \in W_0^{1,p}(\Omega)$, $u(x,t)$ *is a local solution of problem* (1.1)-(1.3) *on* $[0, T]$, *then no solution of* (1.1)-(1.3) *can exist on* $[0, \infty)$ *when* $J(u_0) < 0$.

Proof Assume for contradiction that there is a solution of (1.1)-(1.3) on $[0, \infty)$.
   Define

$$H(t) = \int_0^t \|u_t\|_m^m \, d\tau - J(u_0),$$

hence

$$\|u\|_{q+2}^{q+2} = \frac{q+2}{b} \left[\frac{1}{p} \|\nabla u\|_p^p + H(t) \right] \ge \frac{q+2}{b} H(t) \ge -\frac{q+2}{b} J(u_0) > 0. \tag{5.1}$$

Multiplying (1.1) by u and integrating over Ω, we have

$$0 = b\|u\|_{q+2}^{q+2} - \|\nabla u\|_p^p - \left(|u_t|^{m-2} u_t, u \right)$$
$$= \frac{b(q-p+2)}{q+2} \|u\|_{q+2}^{q+2} - pJ(u(t)) - \left(|u_t|^{m-2} u_t, u \right)$$
$$\ge \frac{b(q-p+2)}{q+2} \|u\|_{q+2}^{q+2} - pJ(u(t)) - \|u_t\|_{\frac{(m-1)(q+2)}{q+1}}^{m-1} \|u\|_{q+1}. \tag{5.2}$$

As $J(u(t)) < J(u_0) < 0$, we see from (5.1) and (5.2) that

$$\|u_t\|_{\frac{(m-1)(q+2)}{q+1}}^{m-1} \ge \frac{b(q-p+2)}{q+2} \|u\|_{q+2}^{q+1} \ge \frac{b(q-p+2)}{q+2} \left[\frac{q+2}{b} H(t) \right]^{\frac{q+1}{q+2}}. \tag{5.3}$$

Since $m < 2 + q$, we have $m \ge \frac{(m-1)(q+2)}{q}$. By the embedding theorem and (5.3), we have

$$H'(t) = \|u_t\|_m^m \ge C\|u_t\|_{\frac{(m-1)(q+2)}{q+1}}^{q+1} \ge C\left(H(t) \right)^{\frac{(q+1)m}{(m-1)(q+2)}}.$$

Since $m < 2 + q$, we have

$$\frac{(q+1)m}{(m-1)(q+2)} - 1 = \frac{q-m+2}{(m-1)(q+2)} > 0.$$

Hence we write $\frac{qm}{(m-1)(q+2)} = 1 + \theta$, $\theta > 0$, then

$$\frac{H'(t)}{H^{1+\theta}} > C.$$

By setting $H_0 = H(0) = -J(u_0) > 0$, we obtain

$$\frac{1}{\theta H_0^\theta} \geq \frac{1}{\theta (H(t))^\theta} + Ct.$$

This is impossible, since the left hand side is finite and the right hand side goes to ∞ as $t \to \infty$. □

6 Existence of global solution with critical initial conditions

In this section, we prove the result of global existence with critical initial conditions.

Theorem 7 *Assume that p and q satisfy* (H), $0 \leq u_0(x) \in W_0^{1,p}(\Omega)$. *If $J(u_0) = d(\delta)$, $J_{\delta_1}(u_0) > 0$ or $J_{\delta_1}(u_0) = 0$, $0 < J(u_0) \leq d(\delta)$, $\delta_1 < \delta_2$ are the two roots of the equation $J(u_0) = d(\delta)$, then problem* (1.1)-(1.3) *admits a global solution $u(x, t)$ such that $u \in L^\infty(0, \infty; W_0^{1,p}(\Omega))$, $u_t \in L^m(0, \infty; L^m(\Omega))$ and $u \in \overline{W}_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $0 \leq t < \infty$, where $\overline{W}_\delta = W_\delta \cup \partial W_\delta = \{u \in W_0^{1,p}(\Omega) | J_\delta(u) \geq 0, J(u) \leq d(\delta)\}$.*

Proof Let $\lambda_n = 1 - \frac{1}{n}$, $u_{0n}(x) = \lambda_n u_0(x)$, $n = 2, 3, \ldots$. Consider the initial condition

$$u(x, 0) = u_{0n}(x)$$

with the corresponding problem (1.1)-(1.3) and suppose that $\delta_1^n < \delta_2^n$ are two roots of the equation $J(u_{0n}) = d(\delta)$.

Since $J_{\delta_1}(u_0) \geq 0$, we obtain $\frac{\delta_1}{p} \|\nabla u_0\|_p^p - \frac{b}{q+2} \|u_0\|_{q+2}^{q+2} \geq 0$, i.e. $\left[\frac{\delta_1 (q+2) \|\nabla u_0\|_p^p}{bp \|u_0\|_{q+2}^{q+2}}\right]^{\frac{1}{q-p+2}} \geq 1$. Thus $\overline{\lambda} = \left(\frac{\|\nabla u_0\|_p^p}{bp \|u_0\|_{q+2}^{q+2}}\right)^{\frac{1}{q-p+2}} \geq \left(\frac{1}{\delta_1} \cdot \frac{p}{q+2}\right)^{\frac{1}{q-p+2}} \geq 1$. From $0 < \lambda_n < 1 \leq \overline{\lambda}$ and Lemma 1, we get

$$J(u_{0n}) = J(\lambda_n u_0) < J(u_0) \leq d(\delta) \leq d,$$

$$J_{\delta_1}(u_{0n}) = J_{\delta_1}(\lambda_n u_0) > 0.$$

Obviously, $J(u) = \frac{1-\delta}{p} \|\nabla u\|_p^p + J_\delta(u)$ implies that $J(u_{0n}) > 0$. As $\delta_2^n > \delta_0 > \delta_1$ and $J_\delta(u)$ is increasing with δ, it follows that $J_{\delta_2^n}(u_{0n}) > 0$.

Thus, by Theorem 1, the problem admits a global solution $u_n(x, t)$ such that $u_n \in L^\infty(0, \infty; W_0^{1,p}(\Omega))$, $u_{nt} \in L^m(0, \infty; L^m(\Omega))$ and $u_n \in W_\delta$ for any $\delta \in (\delta_1^n, \delta_2^n)$, satisfying

$$\left(|u_{nt}|^{m-2} u_{nt}, v\right) + \left(|\nabla u_n|^{p-2} \nabla u_n, \nabla v\right) = \left(bu_n^{q+1}, v\right), \quad \forall v \in W_0^{1,p}(\Omega), t \geq 0, \tag{6.1}$$

$$\int_0^t \|u_{nt}\|_m^m d\tau + J(u_n) \leq J(u_n(0)) = J(u_{0n}) < d(\delta), \quad t \geq 0, \tag{6.2}$$

i.e.

$$\int_0^t \|u_{nt}\|_m^m \, d\tau + \frac{1-\delta}{p} \|\nabla u_n\|_p^p + J_\delta(u_n) < d(\delta), \quad t \geq 0.$$

From $J_\delta(u_n) \geq 0$, we obtain

$$\int_0^t \|u_{nt}\|_m^m \, d\tau < d, \qquad \frac{1-\delta}{p} \|\nabla u_n\|_p^p < d, \quad t \geq 0. \tag{6.3}$$

Hence there exist u, ξ, and subsequence $\{u_\nu\}$ of $\{u_n\}$ such that

$$u_\nu \to u \quad \text{in } L^\infty\big(0, \infty; W_0^{1,p}(\Omega)\big) \text{ weak star,}$$

$$u_{\nu t} \to u \quad \text{in } L^m\big(0, \infty; L^m(\Omega)\big) \text{ weakly,}$$

$$bu_\nu^{q+1} \to \xi \quad \text{in } L^\infty\big(0, \infty; L^{(q+2)'}(\Omega)\big) \text{ weak star.}$$

By using the monotone operator method, we get $\xi = bu^{q+1}$.

In (6.1), letting $n = \nu \to \infty$ we obtain

$$\big(|u_t|^{m-2} u_t, \nu\big) + \big(|\nabla u|^{p-2}\nabla u, \nabla \nu\big) = \big(bu^{q+1}, \nu\big), \quad \forall \nu \in W_0^{1,p}(\Omega), t \geq 0.$$

On the other hand, letting $n = \nu \to \infty$ in $u_n(x, 0) = u_{0n}(x)$ we get $u(x, 0) = u_0(x)$ in $W_0^{1,p}(\Omega)$. Also $\delta_1^n \to \delta_1$, $\delta_2^n \to \delta_2$ as $n \to \infty$.

Furthermore, from (6.2) and Lemma 7, we have $J(u) \leq d(\delta)$ for $0 \leq t < \infty$. On the other hand, from (6.3) we have

$$\|\nabla u\|_p^p \leq \lim_{\nu \to \infty} \|\nabla u_\nu\|_p^p \leq \frac{p}{1-\delta} \cdot d(\delta) = \left(\frac{q+2}{pbC_*^{q+2}}\delta\right)^{\frac{p}{q-p+2}}.$$

By Lemma 2, we have $J_\delta(u) \geq 0$ and $u \in \overline{W}_\delta$ for any $\delta \in (\delta_1, \delta_2)$. $\qquad \square$

Remark 5 The invariant sets and vacuum of solutions for problem (1.1)-(1.3) with critical initial conditions also occur.

Remark 6 Taking $m = 2$ or $p = 2$, Theorem 7 is still satisfied and generalizes the results of [11]. Similarly, the invariant sets and vacuum isolating of solutions also occur.

Remark 7 In fact, all the results in our paper also hold for the homogeneous Dirichlet initial boundary value problem for the more general equation

$$|u_t|^{m-2} u_t = \text{div}\big(|\nabla u|^{p-2}\nabla u\big) + b|u|^q u.$$

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