# Weak solutions for the singular potential wave system 

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#### Abstract

We investigate the existence of weak solutions for a class of the system of wave equations with singular potential nonlinearity. We obtain a theorem which shows the existence of nontrivial weak solution for a class of the wave system with singular potential nonlinearity and the Dirichlet boundary condition. We obtain this result by using the variational method and critical point theory for indefinite functional. MSC: 35L51; 35L70 Keywords: class of wave system; singular potential nonlinearity; Dirichlet boundary condition; variational method; critical point theorem for indefinite functional; (PS)c condition


## 1 Introduction

Let $D$ be an open subset in $R^{n}$ with compact complement $C=R^{n} \backslash D, n \geq 2$. In this paper we investigate the multiplicity of the solutions for a class of the system of nonlinear wave equations with the Dirichlet boundary condition and periodic condition:

$$
\begin{align*}
& \left(u_{1}\right)_{t t}-\left(u_{1}\right)_{x x}=\frac{\partial}{\partial u_{1}} G\left(x, t,\left(u_{1}(x, t), \ldots, u_{n}(x, t)\right)\right) \quad \text { in }\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R \\
& \left(u_{2}\right)_{t t}-\left(u_{2}\right)_{x x}=\frac{\partial}{\partial u_{2}} G\left(x, t,\left(u_{1}(x, t), \ldots, u_{n}(x, t)\right)\right) \quad \text { in }\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R \\
& \vdots  \tag{1.1}\\
& \left(u_{n}\right)_{t t}-\left(u_{n}\right)_{x x}=\frac{\partial}{\partial u_{n}} G\left(x, t,\left(u_{1}(x, t), \ldots, u_{n}(x, t)\right)\right) \quad \text { in }\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\
& u_{i}\left( \pm \frac{\pi}{2}, t\right)=0 \\
& u_{i}(x, t)=u_{i}(-x, t)=u_{i}(x,-t)=u_{i}(x, t+\pi), \quad i=1, \ldots, n
\end{align*}
$$

where $G \in C^{2}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times R^{1} \times D, R^{1}\right)$. Let $U=\left(u_{1}, \ldots, u_{n}\right)$. We assume that $G$ satisfies the following conditions:
(G1) There exists $R_{0}>0$ such that

$$
\sup \left\{|G(x, t, U)|+\left\|\operatorname{grad}_{U} G(x, t, U)\right\|_{R^{n}} \left\lvert\,(x, t, U) \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times R^{1} \times\left(R^{n} \backslash B_{R_{0}}\right)\right.\right\}<+\infty
$$

[^0](G2) There is a neighborhood $Z$ of $C$ in $R^{n}$ such that
$$
G(x, t, U) \geq \frac{A}{d^{2}(U, C)} \quad \text { for }(x, t, U) \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R \times Z
$$
where $d(U, C)$ is the distance function from $U$ to $C$ and $A>0$ is a constant. The system (1.1) can be rewritten as
\[

\left\{$$
\begin{array}{l}
U_{t t}-U_{x x}=\operatorname{grad}_{U} G(x, t, U(x, t))  \tag{1.2}\\
U\left( \pm \frac{\pi}{2}, t\right)=(0, \ldots, 0) \\
U(x, t)=U(x, t+\pi)=U(-x, t)=U(x,-t)
\end{array}
$$\right.
\]

where $U_{t t}-U_{x x}=\left(\left(u_{1}\right)_{t t}-\left(u_{1}\right)_{x x}, \ldots,\left(u_{n}\right)_{t t}-\left(u_{n}\right)_{x x}\right)$.

Remark We have a simple example satisfying the above conditions (G1)-(G2):

$$
G(x, t, U)=\frac{\cos 2 t}{\|U\|_{R^{2}}^{2}}, \quad U=\left(u_{1}, u_{2}\right), u_{1}=x^{2}, u_{2}=x^{2}+t^{2} .
$$

Our main result is the following.

Theorem 1.1 Assume that the nonlinear term G satisfies conditions (G1)-(G2). Then system (1.1) has at least one nontrivial weak solution.

For the proof of Theorem 1.1, we approach the variational method and use the critical point theory for indefinite functional. In Section 2, we introduce a Banach space and the associated functional $I$ of (1.1), and recall the critical point theory for indefinite functional. In Section 3, we prove that $I$ satisfies the geometric assumptions of the critical point theorem for indefinite functional and prove Theorem 1.1.

## 2 Variational approach

The eigenvalue problem

$$
\begin{align*}
& v_{t t}-v_{x x}=\lambda v \quad \text { in }\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\
& v\left( \pm \frac{\pi}{2}, t\right)=0,  \tag{2.1}\\
& v(x, t)=v(-x, t)=v(x,-t)=v(x, t+\pi)
\end{align*}
$$

has infinitely many eigenvalues

$$
\lambda_{m n}=(2 n+1)^{2}-4 m^{2} \quad(m, n=0,1,2, \ldots)
$$

and corresponding normalized eigenfunctions $\phi_{m n}(x, t), m, n>0$, given by

$$
\begin{aligned}
& \phi_{0 n}=\frac{\sqrt{2}}{\pi} \cos (2 n+1) x \text { for } n \geq 0, \\
& \phi_{m n}=\frac{2}{\pi} \cos 2 m t \cos (2 n+1) x \text { for } m>0, n \geq 0 .
\end{aligned}
$$

Let $\Omega$ be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $E^{\prime}$ be the Hilbert space defined by

$$
E^{\prime}=\left\{v \in L^{2}(\Omega) \mid v \text { is even in } x \text { and } t, \int_{\partial \Omega} v=0\right\} .
$$

The set of functions $\left\{\phi_{m n}\right\}$ is an orthonormal basis in $E^{\prime}$. Let us denote an element $v$, in $E^{\prime}$, as

$$
v=\sum h_{m n} \phi_{m n},
$$

and we define a subspace $E$ of $E^{\prime}$ as

$$
E=\left\{v \in E^{\prime}\left|\sum\right| \lambda_{m n} \mid h_{m n}^{2}<\infty\right\} .
$$

This is a complete normed space with a norm

$$
\|v\|=\left[\sum\left|\lambda_{m n}\right| h_{m n}^{2}\right]^{\frac{1}{2}} .
$$

Since $\left\{\lambda_{m n} \mid m, n=0,1,2, \ldots\right\}$ is unbounded from above and from below and has no finite accumulation point, it is convenient for the following to rearrange the eigenvalues $\lambda_{m n}$ by increasing magnitude: from now on we denote by $\left(\rho_{i}^{-}\right)_{i \geq 1}$ the sequence of negative eigenvalues of (2.1), by $\left(\rho_{i}^{+}\right)_{i \geq 1}$ the sequence of positive ones, so that

$$
\cdots \leq \rho_{i}^{-} \leq \cdots \leq \rho_{2}^{-} \leq \rho_{1}^{-} \leq \rho_{1}^{+} \leq \rho_{2}^{+} \leq \cdots \leq \rho_{i}^{+} \leq \cdots
$$

We will denote by the sequence $\left(\rho_{i}\right)$ all the sequences ( $\rho_{i}^{+}$) and ( $\rho_{i}^{-}$). Let ( $\left.e_{i}^{-}, e_{i}^{+}, i \geq 1\right)$ be an orthonormal system of the eigenfunctions associated with the eigenvalues $\left\{\rho_{i}^{-}, \rho_{i}^{+}, i \geq 1\right\}$. We will denote by the sequence $\left(e_{i}\right)$ the sequences $\left(e_{i}^{+}\right),\left(e_{i}^{-}\right)$. Let $E^{+}$be the span of closure of eigenfunctions associated with positive eigenvalues and $E^{-}$be the span of closure of eigenfunctions associated with negative eigenvalues. Let $H$ be the $n$ Cartesian product space of $E$, i.e.,

$$
H=E \times E \times \cdots \times E .
$$

Let $H^{+}$and $H^{-}$be the subspaces on which the functional

$$
U \mapsto Q(U)=\int_{\Omega}\left[-\left|U_{t}\right|^{2}+\left|U_{x}\right|^{2}\right] d x d t, \quad U=\left(u_{1}, \ldots, u_{n}\right)
$$

is positive definite and negative definite, respectively. Then

$$
H=H^{+} \oplus H^{-} .
$$

Let $P^{+}$be the projection from $H$ onto $H^{+}$and $P^{-}$be the projection from $H$ onto $H^{-}$. The norm in $H$ is given by

$$
\|U\|^{2}=\left\|P^{+} U\right\|^{2}+\left\|P^{-} U\right\|^{2}, \quad U=\left(u_{1}, \ldots, u_{n}\right)
$$

where $\left\|P^{+} U\right\|^{2}=\sum_{i=1}^{n}\left\|P^{+} u_{i}\right\|^{2},\left\|P^{-} U\right\|^{2}=\sum_{i=1}^{n}\left\|P^{-} u_{i}\right\|^{2}, U=\left(u_{1}, \ldots, u_{n}\right)$.

Let $\left(H_{n}\right)_{n}$ be a sequence of closed finite dimensional subspace of $H$ with the following assumptions: $H_{n}=H_{n}^{-} \oplus H_{n}^{+}$, where $H_{n}^{+} \subset H^{+}, H_{n}^{-} \subset H^{-}$for all $n$ ( $H_{n}^{+}$and $H_{n}^{-}$are subspaces of $H$ ), $\operatorname{dim} H_{n}<+\infty, H_{n} \subset H_{n+1}, \bigcup_{n \in N} H_{n}$ is dense in $H$.

In this paper we are trying to find the weak solutions $U \in C^{2}(\Omega, D) \cap H$ of system (1.1), that is, $U=\left(u_{1}, \ldots, u_{n}\right) \in C^{2}(\Omega, D) \cap H$ such that

$$
\begin{aligned}
\int_{\Omega} & {\left[-\left(u_{1}\right)_{t} \cdot\left(\phi_{1}\right)_{t}-\left(u_{2}\right)_{t} \cdot\left(\phi_{2}\right)_{t}-\cdots-\left(u_{n}\right)_{t} \cdot\left(\phi_{n}\right)_{t}\right.} \\
& \left.+\left(u_{1}\right)_{x} \cdot\left(\phi_{1}\right)_{x}+\cdots+\left(u_{n}\right)_{x} \cdot\left(\phi_{n}\right)_{x}\right] d x d t \\
& -\int_{\Omega} \frac{\partial}{\partial u_{1}} G(x, t, U(x, t)) \cdot \phi_{1}-\int_{\Omega} \frac{\partial}{\partial u_{2}} G(x, t, U(x, t)) \cdot \phi_{2}-\cdots \\
& -\int_{\Omega} \frac{\partial}{\partial u_{n}} G(x, t, U(x, t)) \cdot \phi_{n}=0
\end{aligned}
$$

for all $\phi=\left(\phi_{1}, \ldots, \phi_{n}\right) \in C^{2}(\Omega, D) \cap H$, i.e.,

$$
\int_{\Omega}\left[-U_{t} \cdot \phi_{t}+U_{x} \cdot \phi_{x}\right] d x d t-\int_{\Omega} \operatorname{grad}_{U} G(x, t, U(x, t)) \cdot \phi=0 \quad \text { for all } \phi \in C^{2}(\Omega, D) \cap H .
$$

Let us introduce an open set of the Hilbert space $H$ as follows:

$$
X=\left\{U \in H \mid U(x, t) \in D \subset R^{n},(x, t) \in \Omega\right\} .
$$

Let us consider the functional on $X$

$$
\begin{align*}
I(U) & =\frac{1}{2} \int_{\Omega}\left[-\left|U_{t}\right|^{2}+\left|U_{x}\right|^{2}\right] d x d t-\int_{\Omega} G(x, t, U) d x d t \\
& =Q(U)-\int_{\Omega} G(x, t, U) d x d t \\
& =\frac{1}{2}\left\|P^{+} U\right\|^{2}-\frac{1}{2}\left\|P^{-} U\right\|^{2}-\int_{\Omega} G(x, t, U) d x d t, \tag{2.2}
\end{align*}
$$

where $Q(U)=\frac{1}{2} \int_{\Omega}\left[-\left|U_{t}\right|^{2}+\left|U_{x}\right|^{2}\right] d x d t$ and $\|U\|^{2}=\sum_{i=1}^{n}\left\|u_{i}\right\|^{2}$. The Euler equation for (2.1) is (1.1). By the following Lemma 2.1, $I \in C^{1}(X, R)$, and so the weak solutions of system (1.1) coincide with the critical points of the associated functional $I(U)$.

Lemma 2.1 Assume that G satisfies conditions (G1)-(G2). Then $I(U)$ is continuous and Fréchet differentiable in $X$ with Fréchet derivative

$$
D I(U) V=\int_{\Omega}\left[-U_{t} \cdot V_{t}+U_{x} \cdot V_{x}-\operatorname{grad}_{U} G(x, t, U(x, t)) \cdot V(x, t)\right] d x d t \quad \forall V \in X
$$

Moreover, $D I \in C$. That is, $I \in C^{1}$.

Proof First we prove that $I(U)$ is continuous. For $U, V \in X$,

$$
\begin{aligned}
& |I(U+V)-I(U)| \\
& \quad=\left\lvert\, \frac{1}{2} \int_{\Omega}\left((U+V)_{t t}-(U+V)_{x x}\right) \cdot(U+V) d x d t\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{\Omega} G(x, t, U+V) d x d t \\
& \left.-\frac{1}{2} \int_{\Omega}\left(U_{t t}-U_{x x}\right) \cdot U d x d t+\int_{\Omega} G(x, t, U) d x d t \right\rvert\, \\
= & \left\lvert\, \frac{1}{2} \int_{\Omega}\left(\left(U_{t t}-U_{x x}\right) \cdot V+\left(V_{t t}-V_{x x}\right) \cdot U+\left(V_{t t}-V_{x x}\right) \cdot V\right) d x d t\right. \\
& -\int_{\Omega}(G(x, t, U+V)-G(x, t, U)) d x d t \mid .
\end{aligned}
$$

We have

$$
\begin{align*}
& \left|\int_{\Omega}[G(x, t, U+V)-G(x, t, U)] d x d t\right| \\
& \quad \leq\left|\int_{\Omega}\left[\operatorname{grad}_{U} G(x, t, U(x, t)) \cdot V+O\left(\|V\|_{R^{n}}\right)\right] d x d t\right|=O\left(\|V\|_{R^{n}}\right) . \tag{2.3}
\end{align*}
$$

Thus we have

$$
|I(U+V)-I(U)|=O\left(\|V\|_{R^{n}}\right)
$$

Next we shall prove that $I(U)$ is Fréchet differentiable in $X$. For $U, V \in X$,

$$
\begin{aligned}
\mid I(U+ & +V)-I(U)-\nabla I(U) V \mid \\
= & \left\lvert\, \frac{1}{2} \int_{\Omega}\left((U+V)_{t t}-(U+V)_{x x}\right) \cdot(U+V) d x d t-\int_{\Omega} G(x, t, U+V) d x d t\right. \\
& -\frac{1}{2} \int_{\Omega}\left(U_{t t}-U_{x x}\right) \cdot U d x d t+\int_{\Omega} G(x, t, U) d x d t \\
& -\int_{\Omega}\left(U_{t t}-U_{x x}-\operatorname{grad}_{U} G(x, t, U(x, t))\right) \cdot V d x d t \mid \\
= & \left\lvert\, \frac{1}{2} \int_{\Omega}\left[\left(V_{t t}-V_{x x}\right) \cdot U+\left(V_{t t}-V_{x x}\right) \cdot V\right] d x\right. \\
& -\int_{\Omega}[G(x, t, U+V)-G(x, t, U)] d x d t+\int_{\Omega} \operatorname{grad}_{U} G(x, t, U(x, t)) \cdot V d x d t \mid
\end{aligned}
$$

Thus by (2.3), we have

$$
\begin{equation*}
|I(U+V)-I(U)-D I(U) V|=O\left(\|V\|_{R^{n}}\right) \tag{2.4}
\end{equation*}
$$

Similarly, it is easily checked that $I \in C^{1}$.

Let

$$
X^{+}=X \cap H^{+}, \quad X^{-}=X \cap H^{-} .
$$

Lemma 2.2 Assume that G satisfies conditions (G1)-(G2). Let $\left\{U_{k}\right\} \subset X^{-}$and $U_{k} \rightharpoonup U$ weakly in $X$ with $U \in \partial X$. Then $I\left(U_{k}\right) \rightarrow-\infty$.

Proof To prove the conclusion, it suffices to prove that

$$
\int_{\Omega} G\left(x, t, U_{k}(x, t)\right) d x d t \rightarrow+\infty
$$

Since $G(x, t, U(x, t))$ is bounded from below, it suffices to prove that there is a subset $\tilde{\Omega}$ of $\Omega$ such that

$$
\int_{\tilde{\Omega}} G\left(x, t, U_{k}(x, t)\right) d x d t \rightarrow+\infty
$$

$U \in \partial X$ means that there exists $\left(x^{*}, t^{*}\right) \in \Omega$ such that $U\left(x^{*}, t^{*}\right) \in \partial D$. Let us set

$$
\Omega_{\delta}\left(x^{*}, t^{*}\right)=\left\{(x, t) \in \Omega \mid \sqrt{\left\|x-x^{*}\right\|_{R^{n}}^{2}+\left\|t-t^{*}\right\|_{R^{n}}^{2}}<\delta\right\} .
$$

By (G1) and (G2), there exists a constant $B$ such that

$$
G(x, t, U) \geq \frac{A}{d^{2}(U, C)}-B
$$

Thus we have

$$
\int_{\Omega_{\delta}\left(x^{*}, t^{*}\right)} G(x, t, U(x, t)) d x d t \geq \int_{\Omega_{\delta}\left(x^{*}, t^{*}\right)}\left(\frac{A}{\left\|U(x, t)-U\left(x^{*}, t^{*}\right)\right\|_{R^{n}}^{2}}-B\right) d x d t
$$

for all $\delta>0$. By Schwarz's inequality, we have

$$
\begin{aligned}
\left\|U(x, t)-U\left(x^{*}, t^{*}\right)\right\|_{R^{n}} & \leq \sqrt{\left\|x-x^{*}\right\|_{R^{n}}^{\frac{1}{2}}+\left\|t-t^{*}\right\|_{R^{n}}^{\frac{1}{2}}}\left(\int_{\Omega}\left(\left\|U_{x}\right\|_{R^{n}}^{2}+\left\|U_{t}\right\|_{R^{n}}^{2}\right)\right)^{\frac{1}{2}} \\
& \leq \delta^{\frac{1}{2}}\left(\int_{\Omega}\left(\left\|U_{x}\right\|_{R^{n}}^{2}+\left\|U_{t}\right\|_{R^{n}}^{2}\right)\right)^{\frac{1}{2}}
\end{aligned}
$$

Thus we have

$$
\int_{\Omega_{\delta}\left(x^{*}, t^{*}\right)} G(x, t, U(x, t)) d x d t \geq \int_{\Omega_{\delta}\left(x^{*}, t^{*}\right)}\left(\frac{A}{\delta\left(\int_{\Omega}\left(\left\|U_{x}\right\|_{R^{n}}^{2}+\left\|U_{t}\right\|_{R^{n}}^{2}\right)\right)}-B\right) d x d t \rightarrow \infty .
$$

Hence

$$
\int_{\Omega_{\delta}\left(x^{*}, t^{*}\right)} G(x, t, U(x, t)) d x d t=\infty .
$$

Since the embedding $X \hookrightarrow C\left(\Omega, R^{n}\right)$ is compact, we have

$$
\max \left\{\left\|U(x, t)-U_{k}(x, t)\right\|_{R^{n}}^{2} \mid(x, t) \in \Omega\right\} \rightarrow 0 \quad \text { as } k \rightarrow \infty
$$

Thus by Fatou's lemma, we have

$$
\begin{aligned}
\liminf \int_{G_{\delta}\left(x^{*}, t^{*}\right)} G\left(x, t, U_{k}(x, t)\right) & \geq \int_{G_{\delta}\left(x^{*}, t^{*}\right)} \liminf G\left(x, t, U_{k}(x, t)\right) \\
& =\int_{G_{\delta}\left(x^{*}, t^{*}\right)} G(x, t, U(x, t))=+\infty
\end{aligned}
$$

Thus

$$
\liminf \int_{G_{\delta}\left(x^{*}, t^{*}\right)} G\left(x, t, U_{k}(x, t)\right)=+\infty
$$

Thus

$$
\begin{aligned}
I\left(U_{k}\right) & =\int_{\Omega}\left[\frac{1}{2}\left(-\left|\left(U_{k}\right)_{t}\right|^{2}+\left|\left(U_{k}\right)_{x}\right|^{2}\right)-G\left(x, t, U_{k}(x)\right)\right] d x d t \\
& =\frac{1}{2}\left\|P^{+} U_{k}\right\|^{2}-\frac{1}{2}\left\|P^{-} U_{k}\right\|^{2}-\int_{\Omega} G\left(x, t, U_{k}(x)\right) d x d t \\
& =-\frac{1}{2}\left\|P^{-} U_{k}\right\|^{2}-\int_{\Omega} G\left(x, t, U_{k}(x)\right) d x d t \rightarrow-\infty
\end{aligned}
$$

so we prove the lemma.

We recall the critical point theorem for the indefinite functional (cf. [1]).
Let

$$
\begin{aligned}
& B_{r}=\{u \in X \mid\|u\| \leq r\}, \\
& S_{r}=\{u \in X \mid\|u\|=r\} .
\end{aligned}
$$

Theorem 2.1 (Critical point theorem for the indefinite functional) Let $X$ be a real Hilbert space with $X=X_{1} \oplus X_{2}$ and $X_{2}=X_{1}^{\perp}$. Suppose that $I \in C^{1}(X, R)$ satisfies $(P S)$, and
(I1) $I(u)=\frac{1}{2}(L u, u)+b u$, where $L u=L_{1} P_{1} u+L_{2} P_{2} u$ and $L_{i}: X_{i} \rightarrow X_{i}$ is bounded and self-adjoint, $i=1,2$,
(I2) $b^{\prime}$ is compact, and
(I3) there exists a subspace $\tilde{X} \subset X$ and sets $S \subset X, Q \subset \tilde{X}$ and constants $\alpha>\omega$ such that
(i) $S \subset X_{1}$ and $\left.I\right|_{S} \geq \alpha$,
(ii) $Q$ is bounded and $\left.I\right|_{\partial Q} \leq \omega$,
(iii) $S$ and $\partial Q$ link.

Then I possesses a critical value $c \geq \alpha$.

## 3 Proof of Theorem 1.1

We shall show that the functional $I(U)$ satisfies the geometric assumptions of the critical point theorem for indefinite functional.

Lemma 3.1 (Palais-Smale condition) Assume that G satisfies conditions (G1) and (G2). Then $I(u)$ satisfies the $(P S)$ condition in $X$.

Proof We shall prove the lemma by contradiction. We suppose that there exists a sequence $\left\{U_{k}\right\} \subset X$ satisfying $I\left(U_{k}\right) \rightarrow \gamma$ and

$$
\begin{equation*}
D I\left(U_{k}\right)=\left(U_{k}\right)_{t t}-\left(U_{k}\right)_{x x}-\operatorname{grad}_{U} G\left(x, t, U_{k}(x, t)\right) \rightarrow \theta \quad \text { in } X, \tag{3.1}
\end{equation*}
$$

or equivalently

$$
U_{k}-\left(D_{t t}-D_{x x}\right)^{-1}\left(\operatorname{grad}_{U} G\left(x, t, U_{k}(x, t)\right)\right) \rightarrow \theta,
$$

where $\theta=(0, \ldots, 0)$ and $\left(D_{t t}-D_{x x}\right)^{-1}$ is a compact operator. We claim that the sequence $\left\{U_{k}\right\}$, up to a subsequence, converges. It suffices to prove that the sequence $\left\{U_{k}\right\}$ is bounded in $X$. By contradiction, we suppose that $\left\|U_{k}\right\|_{R^{n}} \rightarrow \infty$. Then, for large $k$, we have

$$
\begin{equation*}
\left\|U_{k}\right\|_{R^{n}} \geq R_{0} \tag{3.2}
\end{equation*}
$$

It follows from (3.2) that

$$
\begin{equation*}
\left|\int_{\Omega} G\left(x, t, U_{k}\right) d x d t\right| \leq|\Omega| \sup \left\{\left|G\left(x, t, U_{k}\right)\right| \mid\left(x, t, U_{k}\right) \in \Omega \times\left(R^{n} \backslash B_{R_{0}}\right)\right\} . \tag{3.3}
\end{equation*}
$$

Let us set $W_{k}=\frac{u_{k}}{\left\|u_{k}\right\|}$. Then $\left\|W_{k}\right\|=1$, and hence the subsequence $\left\{W_{k}\right\}$, up to a subsequence, converges weakly to $W$ with $\|W\|=1$. By (3.1), we have

$$
\begin{align*}
0 & \leftarrow \frac{D I\left(U_{k}\right) U_{k}}{\left\|U_{k}\right\|_{H}}=\int_{\Omega}\left(\left(W_{k}\right)_{t t}-\left(W_{k}\right)_{x x}\right) \cdot W_{k} d x d t-\int_{\Omega} \frac{G\left(x, t, U_{k}\right)}{\left\|U_{k}\right\|^{2}} \\
& =\left\|P^{+} W_{k}\right\|^{2}-\left\|P^{-} W_{k}\right\|^{2}-\int_{\Omega} \frac{G\left(x, t, U_{k}\right)}{\left\|U_{k}\right\|^{2}} \tag{3.4}
\end{align*}
$$

Letting $k \rightarrow \infty$ in (3.4), by (3.3), we have

$$
\begin{align*}
0 & =\lim _{k \rightarrow \infty}\left\|P^{+} W_{k}\right\|^{2}-\lim _{k \rightarrow \infty}\left\|P^{-} W_{k}\right\|^{2} \\
& =\int_{\Omega}\left(W_{t t}-W_{x x}\right) \cdot W d x d t \\
& =\left\|P^{+} W\right\|^{2}-\left\|P^{-} W\right\|^{2} . \tag{3.5}
\end{align*}
$$

Thus we have

$$
\lim _{k \rightarrow \infty}\left\|P^{+} W_{k}\right\|^{2}=\left\|P^{+} W\right\|^{2}, \quad \lim _{k \rightarrow \infty}\left\|P^{-} W_{k}\right\|^{2}=\left\|P^{-} W\right\|^{2} .
$$

Thus

$$
\lim _{k \rightarrow \infty}\left\|W_{k}\right\|=\|W\|
$$

and by (3.5), $W$ is the weak solution of the equation

$$
W_{t t}-W_{x x}=0 \quad \text { in } X .
$$

We claim that $N(A) \cap X=\{(0, \ldots, 0)\}$, where $A U=U_{t t}-U_{x x}$ and $N(A)$ is the kernel of $A$. In fact, let $W \in N(A) \cap X, W=\left(w_{1}, \ldots, w_{n}\right)$. Then

$$
w_{i}=\sum_{m, n}\left(w_{i}\right)_{m n} \cos 2 m t \cos (2 n+1) x, \quad i=1, \ldots, n
$$

We note that

$$
(2 n+1)^{2}-4 m^{2} \neq 0 \quad \Longrightarrow \quad\left(w_{i}\right)_{m n}=0, \quad m, n=0,1,2, \ldots
$$

Thus $N(A) \cap X=\{(0, \ldots, 0)\}$. Thus $W=(0, \ldots, 0)$, which is absurd to the fact that $\|W\|=1$. Thus $\left\{U_{k}\right\}$ is bounded. Thus the subsequence, up to a subsequence, $U_{k}$ converges weakly to $U$ in $X$. By Lemma 2.2, $U \in X$ and that $\left\|\operatorname{grad}_{U} G\left(\cdot, U_{k}\right)\right\|$ is bounded. Since $\left(D_{t t}-D_{x x}\right)^{-1}$ is compact and (3.1) holds, $\left\{U_{k}\right\}$ converges strongly to $U$. Thus we prove the lemma.

Let

$$
Q=\left(\bar{B}_{r} \cap X^{-}\right) \oplus\left\{r e \mid e \in B_{1} \cap X^{+}, 0<r<R\right\} .
$$

Lemma 3.2 Assume that $G$ satisfies conditions (G1) and (G2). Then there exist sets $S_{\rho} \subset$ $X^{+}$with radius $\rho>0, Q \subset X$ and constants $\alpha>0$ such that
(i) $S_{\rho} \subset X^{+}$and $\left.I\right|_{S_{\rho}} \geq \alpha$,
(ii) $Q$ is bounded and $\left.I\right|_{\partial Q} \leq 0$,
(iii) $S_{\rho}$ and $\partial Q$ link.

Proof (i) Let us choose $U \in X^{+} \subset X$. Then $U(x, t) \in D$. By (G1), $G(x, t, U)$ is bounded above and there exists a constant $C>0$

$$
I(U)=\frac{1}{2}\left\|P^{+} U\right\|^{2}-\frac{1}{2}\left\|P^{-} U\right\|^{2}-\int_{\Omega} G(x, t, U) d x d t \geq \frac{1}{2}\left\|P^{+} U\right\|^{2}-C
$$

for $C>0$. Then there exist constants $\rho>0$ and $\alpha>0$ such that if $U \in S_{\rho} \cap X^{+}$, then $I(U) \geq \alpha$.
(ii) Let us choose $e \in B_{1} \cap X^{+}$. Let $U \in \bar{B}_{r} \cap X^{-} \oplus\{r e \mid 0<r\}$. Then $U=V+W, V \in$ $\bar{B}_{r} \cap X^{-}, W=r e$. We note that:

$$
\text { If } V \in \bar{B}_{r} \cap X^{-}, \quad \text { then } \int_{\Omega}\left[-\left\|V_{t}\right\|_{R^{n}}^{2}+\left\|V_{x}\right\|_{R^{n}}^{2}\right] d x d t=-\left\|P^{-} U\right\|^{2} \leq 0 .
$$

By (G2), $G(x, t, V+r e)$ is bounded from below. Thus by Lemma 2.2, there exists a constant $A>0$ such that if $U=V+r e$, then we have

$$
\begin{aligned}
I(U) & =\frac{1}{2} r^{2}-\frac{1}{2}\left\|P^{-} V\right\|^{2}-\int_{\Omega} G(x, t, V+r e) d x d t \\
& \leq \frac{1}{2} r^{2}-\frac{1}{2}\left\|P^{-} V\right\|^{2}-\int_{\Omega} \frac{A}{d^{2}(V+r e, C)} d x d t
\end{aligned}
$$

We can choose a constant $R>r$ such that if $U=V+r e \in Q=\left(\bar{B}_{r} \cap X^{-}\right) \oplus\left\{r e \mid e \in B_{1} \cap\right.$ $\left.X^{+}, 0<r<R\right\}$, then $I(U)<0$. Thus we prove the lemma.

Proof of Theorem 1.1 By Lemma 2.1, $I(U)$ is continuous and Fréchet differentiable in $X$ and, moreover, $D I \in C$. By Lemma 2.2, if $\left\{U_{k}\right\} \subset X^{-}$and $U_{k} \rightharpoonup U$ weakly in $X$ with $U \in \partial X$, then $I\left(U_{k}\right) \rightarrow-\infty$. By Lemma 3.1, $I(u)$ satisfies the (PS) condition. By Lemma 3.2, there exist sets $S_{\rho} \subset X^{+}$with radius $\rho>0, Q \subset X$ and constant $\alpha>0$ such that $\left.I\right|_{S_{\rho}} \geq \alpha, Q$ is bounded and $\left.I\right|_{\partial Q} \leq 0$, and $S_{\rho}$ and $\partial Q$ link. By the critical point theorem, $I(U)$ possesses a critical value $c \geq \alpha$. Thus (1.1) has at least one nontrivial weak solution. Thus we prove Theorem 1.1

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read, checked and approved the final manuscript.

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## Acknowledgements

This work (Tacksun Jung) was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (KRF-2010-0023985).

Received: 13 February 2013 Accepted: 9 December 2013 Published: 07 Jan 2014

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[^1]:    10.1186/1687-2770-2014-7

    Cite this article as: Jung and Choi: Weak solutions for the singular potential wave system. Boundary Value Problems 2014, 2014:7

