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# Global existence and blow-up of solutions for $p$ -Laplacian evolution equation with nonlinear memory term and nonlocal boundary condition

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## Abstract

In this paper, we deal with an initial boundary value problem for a  $p$ -Laplacian evolution equation with nonlinear memory term and inner absorption term subject to a weighted linear nonlocal boundary condition. We find the effects of a weighted function as regards determining blow-up of nonnegative solutions or not and establish the precise blow-up estimate for the linear diffusion case under some suitable conditions.

**Keywords:**  $p$ -Laplacian evolution equation; nonlinear memory; global existence; blow-up; weight function

## 1 Introduction

In the past decades, there have been many works dealing with global existence and blow-up properties of solutions for nonlinear parabolic equations, especially the initial boundary value problems with nonlocal terms in equations or boundary conditions, we refer to [1–10] and references therein. For the study of an initial boundary value problem for local parabolic equations with nonlocal boundary condition, we refer to [1–4]. For example, Friedman [1] studied the linear parabolic equation

$$u_t - Au = 0, \quad (x, t) \in \Omega \times (0, T),$$

subject to the weighted linear nonlocal Dirichlet boundary condition

$$u(x, t) = \int_{\Omega} f(x, y) u(y, t) dy, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.1)$$

where  $A$  is an elliptic operator,

$$A = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x) \quad \text{and} \quad c(x) \leq 0,$$

and the nonnegative continuous function  $f(x, y)$  satisfies suitable conditions. He proved that when  $\int_{\Omega} f(x, y) dy \leq \rho < 1$ , the solution approaches to 0 monotonously and exponen-

tially as  $t \rightarrow \infty$ . As regards more general discussions on an initial boundary value problem for a linear parabolic equation with a weighted linear nonlocal Neumann boundary condition, one can refer to [2] by Pao, where the following problem:

$$\begin{aligned} u_t - Lu &= g(x, u), \quad x \in \Omega, t > 0, \\ Bu &= \int_{\Omega} f(x, y)u(y, t) dy, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \end{aligned}$$

where

$$Lu = \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{j=1}^n b_j(x)u_{x_j}, \quad Bu = \alpha_0 \frac{\partial u}{\partial n} + u,$$

was considered. He studied the asymptotic behavior of solutions and found the influence of the weight function on the existence of global and blow-up solutions. Later, Akila [3] adopted the method of an upper-lower solution to consider the semilinear parabolic equation

$$u_t = \Delta u + g(x, t, u), \quad (x, t) \in \Omega \times (0, \infty),$$

under a similar weighted linear nonlocal boundary condition. Wang *et al.* [4] studied a porous medium equation with power form source term,

$$u_t = \Delta u^m + u^p, \quad (x, t) \in \Omega \times (0, +\infty),$$

under the weighted linear nonlocal Dirichlet boundary condition (1.1). By virtue of the method of an upper-lower solution, they obtained global existence, blow-up properties, and blow-up rate of solutions.

For the study of initial boundary value problem with nonlocal parabolic equation, especially the nonlocal problem with time-integral, we refer to [5–10]. Under a homogeneous Dirichlet boundary condition, Li and Xie [5] studied the nonlinear diffusion equation

$$u_t - \Delta u = u^q \int_0^t u^p ds, \quad (x, t) \in \Omega \times (0, T),$$

where  $p \geq 0$ ,  $q > 0$ . They obtained the sufficient conditions of global existence and blow-up of solutions under appropriate critical conditions. Furthermore, under the following assumptions:

$$\text{there exists } t_0 \in (0, T) \text{ such that } u_t(x, t_0) \geq 0 \text{ for all } x \in \bar{\Omega}, \quad (1.2)$$

and

$$\begin{aligned} \Omega &= \{x \in R^N : |x| < R\}, \quad u_0(x) = u_0(|x|) \equiv u_0(r), \\ u'_0(r) &< 0 \quad \text{and} \quad u''_0(r) < 0, \end{aligned} \quad (1.3)$$

they derived the following blow-up rate:

$$C_1(T-t)^{-\frac{2}{p-1}} \leq \max_{x \in \Omega} u(x, t) \leq C_2(T-t)^{-\frac{2}{p-1}}, \quad t \rightarrow T, \quad (1.4)$$

for the special case  $p > 1$  and  $q = 0$ . It is necessary to point out that assumption (1.2) seems to be reasonable, but unfortunately, the authors of [5] did not give a relationship between  $u_0(x)$  and equation (1.2). The characterization of the monotonicity condition (1.2) was given by Souplet in [6], who proved the existence of monotone in time solutions for the above problem and obtained the blow-up rate (1.4) without the assumption of condition (1.3).

Zhou *et al.* [7] considered the following singular diffusion equation with memory term:

$$(u^k)_t - \Delta u = u^q \int_0^t u^p ds, \quad (x, t) \in \Omega \times (0, T),$$

where  $p \geq 0$ ,  $q \geq 0$ ,  $0 < k < 1$ . They got similar results by the method of upper-lower solution. We should notice that this kind of equation can be turned into a degenerate porous medium equation by suitable transformation. In addition, for the system of porous medium equations with nonlinear memory terms and a homogeneous Dirichlet boundary condition, one can refer for example to [8, 9].

Recently, Liu and Mu [10] considered the following semilinear parabolic equation with memory term:

$$u_t = \Delta u + u^q \int_0^t u^p ds, \quad (x, t) \in \Omega \times (0, T),$$

subject to a weighted nonlinear nonlocal boundary,

$$u(x, t) = \int_{\Omega} f(x, y) u^l(y, t) dy, \quad (x, t) \in \partial\Omega \times (0, T),$$

where  $p, q \geq 0$ . They gave the conditions of global existence and blow-up of solutions and the blow-up rate of solutions for  $q = 0$ ,  $l = 1$  by establishing an auxiliary function.

In view of the works mentioned above, a nonlocal parabolic equation with time-integral term does not seem to be so much investigated as nonlocal equations with space-integral terms. Already at first glance, the problem with a memory term has some difficulties in proving the existence of non-global solutions. First if  $t$  is sufficiently small, the nonlinear memory term vanishes, and then it is not clear whether the comparison principle holds in proving the existence of global small solutions. As far as we know, there are a few papers about the blow-up phenomenon for the  $p$ -Laplacian evolution equation with nonlinear memory term. Motivated by it, we consider the global existence and blow-up properties of the following  $p$ -Laplacian evolution equation with nonlinear memory term and inner absorption term:

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = u^\alpha \int_0^t u^\beta ds - k u^m, \quad (x, t) \in \Omega \times (0, T), \quad (1.5)$$

subject to weighted linear nonlocal boundary and initial conditions

$$u(x, t) = \int_{\Omega} f(x, y) u(y, t) dy, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.6)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.7)$$

where  $p \geq 2$ ,  $\alpha \geq 0$ ,  $\beta > 0$ ,  $m > 0$ ,  $k > 0$ , and  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded domain with smooth boundary. The weight function  $f(x, y) \not\equiv 0$  in the boundary condition is continuous, nonnegative on  $\partial\Omega \times \Omega$ , and  $\int_{\Omega} f(x, y) dy > 0$  on  $\partial\Omega$ , while the nonnegative and nontrivial initial data  $u_0(x) \in L^{\infty}(\Omega) \cap W_0^{1,p}(\Omega)$  satisfies the compatibility conditions  $u_t(x, 0) = \operatorname{div}(|\nabla u_0(x)|^{p-2} \nabla u_0(x)) - k u_0^{\delta}(x)$  for  $x \in \Omega$  and  $u_0(x) = \int_{\Omega} f(x, y) u_0(y) dy$  for  $x \in \partial\Omega$ , which is the closed relationship for local solvability of our problem (1.5)-(1.7) (see Section 2).

The nonlocal diffusion model like equation (1.5) arises in many natural phenomena. In some sense, this kind of nonlocal problem is closer to the actual model than the local problem, such as the model of non-Newton flux through a porous medium, the model for compressible reactive gases, the model of population dynamics, and the model of biological species with human-controlled distribution (see [2, 11–14] and references therein). From a physics point of view, equation (1.5) with  $p = 2$ ,  $m, \alpha, \beta = 1$  and  $k = 0$  appears in the theory of nuclear reactor dynamics in which case the nonlocal term with time-integral is called the memory term [15]. In fact, there are some important phenomena formulated as parabolic equations which are coupled with weighted nonlocal boundary conditions in mathematical models, such as thermoelasticity theory. In this case, the solution  $u(x, t)$  describes the entropy per volume of the material (see [16, 17]).

Our main goal is to find the effects of weight function on global or non-global existence of solutions for problem (1.5)-(1.7), the suitable range of nonlinear exponent, and to give the blow-up rate estimate under some suitable conditions. In addition, we treat the nonlocal nonlinearity Hölder (non-Lipschitz) cases  $m$  or  $n \in (0, 1)$ , as well as the Lipschitz cases  $m, n \geq 1$  in this paper. We get our main results by establishing a modified comparison principle, constructing the suitable upper and lower solutions (including the self-similar lower solutions, the eigenfunction argument and the technique of ordinary differential equation and so on) and the auxiliary function. Moreover, our results extend part of or all results in [8–10]. The detailed results are stated as follows.

- For arbitrary  $f(x, y) > 0$ . If  $\alpha + \beta > \max\{p - 1, m\}$ , then the solution of problem (1.5)-(1.7) blows up in finite time for sufficiently large initial data.
- If  $\int_{\Omega} f(x, y) \geq 1$ , for  $x \in \partial\Omega$ . If  $\alpha + \beta > m \geq 1$ , then the solution of problem (1.5)-(1.7) blows up in finite time for all strictly positive initial data with  $T$  sufficiently large.
- If  $\int_{\Omega} f(x, y) < 1$ , for  $x \in \partial\Omega$ .  $\alpha + \beta > 1$ ,  $p = 2$ , the initial value  $u_0(x)$  satisfies conditions  $(H_1)$ -( $H_2$ ) (see Section 3) and  $u(x, t)$  is the blow-up solution of problem (1.2)-(1.4), then the blow-up rate is

$$c(T - t)^{-\frac{1}{\alpha+\beta-1}} \leq u(x, t) \leq C(T - t)^{-\frac{1}{\alpha+\beta-1}},$$

where  $c = [(\alpha + \beta - 1)T]^{-\frac{1}{\alpha+\beta-1}}$ ,  $C = [(\alpha + \beta - 1)\delta]^{-\frac{1}{\alpha+\beta-1}}$  and  $\delta > 0$  is a constant.

- If  $\int_{\Omega} f(x, y) < 1$ , for  $x \in \partial\Omega$ ,
  - (i) If  $\alpha + \beta \leq \max\{p - 1, m\}$ , then the solution of problem (1.5)-(1.7) exists globally for small initial data.
  - (ii) If  $p - 1 < \min\{\alpha, \beta + 1, m\}$ , and  $\alpha + \beta > m$ , then the solution of problem (1.5)-(1.7) exists globally for small initial data.
  - (iii) If  $\alpha + \beta > m$ , then the solution of problem (1.5)-(1.7) exists globally for small enough initial data.

The rest of the paper is organized as follows. In Section 2, we give the preliminaries for our research. The proofs of blow-up results and blow-up rate of solutions are given in Section 3. In Section 4, we will deduce the results of global existence.

## 2 Comparison principle and local existence

Since equation (1.2) is degenerate when  $p > 2$ , there is no classical solution in general. Hence, it is reasonable to find a weak solution. To this end, we first give the following definition of nonnegative weak solution of problem (1.5)-(1.7).

**Definition 1** If the nonnegative function  $u(x, t)$  satisfies the following conditions:

$$(1) \quad u \in C(0, T; L^\infty(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)),$$

$$u_t \in L^2(0, T; L^2(\Omega)), \quad u(x, 0) = u_0(x), \quad (2.1)$$

$$(2) \quad \int_{Q_T} u_t \phi \, dx \, dt + \int_{Q_T} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \, dt$$

$$= \int_{Q_T} \phi \left( u^\alpha \int_0^t u^\beta \, ds - k u^m \right) dx \, dt, \quad (2.2)$$

where  $\phi \in L^1(0, T; W^{1,2}(\Omega)) \cap L^2(Q_T)$  is nonnegative,  $\phi(x, t)|_{\partial\Omega} = 0$ , and  $Q_T = \Omega \times (0, T)$ .

$$(3) \quad u(x, t) = \int_{\Omega} f(x, y) u(y, t) \, dy, \quad x \in \partial\Omega, \quad (2.3)$$

then  $u(x, t)$  is called the weak solution of problem (1.5)-(1.7).

If the equalities in equations (2.1)-(2.3) are replaced by ' $\leq$ ' and ' $\geq$ ', we can get  $\underline{u}(x, t)$  and  $\bar{u}(x, t)$  which are called the lower solution and upper solution of problem (1.5)-(1.7), respectively.

The following modified comparison principle plays a crucial role in our proofs, which can be obtained by establishing a suitable test function and Gronwall's inequality.

**Proposition 1** (Comparison principle) *Suppose that  $\underline{u}(x, t)$  and  $\bar{u}(x, t)$  are the lower and upper solutions of problem (1.2)-(1.4), respectively. If  $\underline{u}(x, 0) \geq 0$ ,  $\bar{u}(x, 0) \geq \varepsilon > 0$  and  $\underline{u}(x, 0) \leq \bar{u}(x, 0)$ , where  $\varepsilon$  is any positive constant, then  $\underline{u}(x, t) \leq \bar{u}(x, t)$  in  $\bar{\Omega} \times (0, T)$ .*

*Proof* For  $x \in \Omega$ , since  $\underline{u}(x, t)$  and  $\bar{u}(x, t)$  are the lower and upper solutions of problem (1.5)-(1.7), respectively, it follows that

$$\int_{Q_T} (\underline{u}_t - \bar{u}_t) \phi \, dx \, dt + \int_{Q_T} (|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla \bar{u}|^{p-2} \nabla \bar{u}) \cdot \nabla \phi \, dx \, dt$$

$$\leq \int_{Q_T} \phi \left[ \underline{u}^\alpha \int_0^t \underline{u}^\beta \, ds - \bar{u}^\alpha \int_0^t \bar{u}^\beta \, ds - k(\underline{u}^m - \bar{u}^m) \right] dx \, dt$$

$$= \int_{Q_T} \phi \left[ (\underline{u}^\alpha - \bar{u}^\alpha) \int_0^t \underline{u}^\beta \, ds + \bar{u}^\alpha \int_0^t (\underline{u}^\beta - \bar{u}^\beta) \, ds - k(\underline{u}^m - \bar{u}^m) \right] dx \, dt.$$

Choose a test function  $\phi = \chi_{[0,t]}(\underline{u} - \bar{u})_+ = \max\{\underline{u} - \bar{u}, 0\}$  for  $t > 0$ , where  $\chi_{[0,t]}$  is a characteristic function defined on  $[0, t]$ , then we have

$$\begin{aligned} & \int \int_{Q_T} (\underline{u} - \bar{u})_t (\underline{u} - \bar{u})_+ dx dt + \int \int_{Q_T} (|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla \bar{u}|^{p-2} \nabla \bar{u}) \cdot \nabla (\underline{u} - \bar{u})_+ dx dt \\ & \leq \int \int_{Q_T} (\underline{u} - \bar{u})_+ \left[ (\underline{u}^\alpha - \bar{u}^\alpha) \int_0^t \underline{u}^\beta ds + \bar{u}^\alpha \int_0^t (\underline{u}^\beta - \bar{u}^\beta) ds - k(\underline{u}^m - \bar{u}^m) \right] dx dt \\ & \leq \int \int_{Q_T} \left[ \Phi_1(x, t) \int_0^t \underline{u}^\beta ds + \bar{u}^\alpha T \Phi_2(x, t) + k \Phi_3(x, t) \right] (\underline{u} - \bar{u})_+^2 dx dt, \end{aligned}$$

where

$$\begin{aligned} \Phi_1(x, t) &= \alpha \int_0^1 (\theta \underline{u} + (1 - \theta) \bar{u})^{\alpha-1} d\theta, \\ \Phi_2(x, t) &= \beta \int_0^1 (\theta \underline{u} + (1 - \theta) \bar{u})^{\beta-1} d\theta, \\ \Phi_3(x, t) &= m \int_0^1 (\theta \underline{u} + (1 - \theta) \bar{u})^{m-1} d\theta. \end{aligned}$$

By Lemma 4.10 in [18], we know  $(|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla \bar{u}|^{p-2} \nabla \bar{u}) \cdot \nabla (\underline{u} - \bar{u})_+ \geq 0$  for  $p \geq 2$ . Moreover, it follows from  $m \geq 1$ ,  $\alpha \geq 1$ ,  $\beta \geq 1$  that  $\Phi_i$  ( $i = 1, 2, 3$ ) is bounded, and if  $0 \leq \alpha < 1$ , or  $\beta < 1$ , we have  $\Phi_1 \leq \varepsilon^{\alpha-1}$ ,  $\Phi_2 \leq \varepsilon^{\beta-1}$  since  $\underline{u}(x, 0) \geq 0$ ,  $\bar{u}(x, 0) \geq \varepsilon > 0$ . Furthermore, because  $\underline{u}(x, t)$  and  $\bar{u}(x, t)$  are bounded functions, we can get

$$\int \int_{Q_T} (\underline{u} - \bar{u})_t (\underline{u} - \bar{u})_+ dx dt \leq C \int \int_{Q_T} (\underline{u} - \bar{u})_+^2 dx dt.$$

That is,

$$\int_{\Omega} (\underline{u} - \bar{u})_+^2 dx \leq \int_{\Omega} (\underline{u}(x, 0) - \bar{u}(x, 0))_+^2 dx + C \int \int_{Q_T} (\underline{u} - \bar{u})_+^2 dx dt.$$

Since  $\underline{u}(x, 0) \leq \bar{u}(x, 0)$ , it follows that

$$\int_{\Omega} (\underline{u} - \bar{u})_+^2 dx \leq C \int \int_{Q_T} (\underline{u} - \bar{u})_+^2 dx dt.$$

By Gronwall's inequality, we can deduce that  $(\underline{u} - \bar{u})_+ = 0$ , and so  $\underline{u}(x, t) \leq \bar{u}(x, t)$  in  $\Omega \times (0, T)$ .

For  $x \in \partial\Omega$ ,  $y \in \Omega$ , we have

$$\underline{u} - \bar{u} \leq \int_{\Omega} f(x, y) (\underline{u}(y, t) - \bar{u}(y, t)) dy \leq 0,$$

in the case of  $\int_{\Omega} (\underline{u} - \bar{u})_+ dy = 0$  in  $\Omega$ . Therefore, we obtain  $\underline{u}(x, t) \leq \bar{u}(x, t)$  on  $\partial\Omega \times (0, T)$ , and  $\underline{u}(x, t) \leq \bar{u}(x, t)$  in  $\bar{\Omega} \times (0, T)$ .  $\square$

Next, we state the theorem of local existence and uniqueness without proof.

**Theorem** (Local existence and uniqueness) *Suppose that  $p \geq 2$ ,  $\alpha \geq 0$ ,  $\beta > 0$ ,  $m > 0$  and  $k > 0$ , the nonnegative initial data  $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$  satisfies the compatibility conditions  $u_t(x, 0) = \operatorname{div}(|\nabla u_0(x)|^{p-2} \nabla u_0(x)) - ku_0^\delta(x)$  for  $x \in \Omega$  and  $u_0(x) = \int_\Omega f(x, y) u_0(y) dy$  for  $x \in \partial\Omega$ . Then there exists a constant  $T^* > 0$  such that problem (1.5)-(1.7) admits a nonnegative solution  $u \in C(0, T; L^\infty(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$  for each  $T < T^*$ . Furthermore, either  $T^* = \infty$  or*

$$\lim_{t \rightarrow T^*} \sup \|u(x, t)\|_\infty = \infty.$$

**Remark 1** The existence of local nonnegative solutions in time to problem (1.5)-(1.7) can be obtained by combining Theorem 1.2 in [4] with Theorem A4' in [19]. By the comparison principle above, we can get the uniqueness of the solutions to problem (1.5)-(1.7) with  $\alpha + \beta \geq 1$ ,  $m \geq 1$ .

### 3 Blow-up solutions and blow-up rate

Comparing with the problem under a general homogeneous Dirichlet boundary condition, the existence of weight function in the boundary condition has a great influence on the global and non-global existence of solutions.

**Theorem 1** *Suppose that  $\alpha + \beta > \max\{p - 1, m\}$ , then the solution of problem (1.5)-(1.7) blows up in finite time for arbitrary  $f(x, y) > 0$  and sufficiently large initial data.*

*Proof* In order to prove the blow-up result, we need to establish a self-similar blow-up solution. Let

$$\underline{u}(x, t) = \frac{\omega(r)}{v^\xi(r, t)}, \quad \text{with } 0 \leq r \leq R, \xi > 1, 0 < t < 3T,$$

where  $\omega(r) = \cos^2(\frac{\pi r}{2R})$ ,  $v(x, t) = 4T^2 - \omega^\delta(t - T)^2 = v_1 v_2$ , and

$$v_1 = 2T - \omega^{\frac{\delta}{2}}(t - T), \quad v_2 = 2T + \omega^{\frac{\delta}{2}}(t - T).$$

It is obvious that  $\underline{u}(r, t)$  blows up at  $r = 0$  as  $t$  approaches  $T$ . Set

$$\delta \geq 2, \quad T \leq \frac{1}{2} \quad \text{such that } v(r, t) \leq 1.$$

An explicit calculation yields

$$\begin{aligned} r_{x_i} &= \frac{x_i}{r}, & r_{x_i x_i} &= -\frac{x_i x_j}{r^3}, \\ \omega_{x_i} &= \omega' r_{x_i} = -\frac{\pi}{2R} \sin \frac{\pi r}{R} r_{x_i}, & \nabla \omega &= -\frac{\pi}{2R} \sin \frac{\pi r}{R}, \\ \omega_{x_i x_j} &= (\omega' r_{x_i})_{x_j} = -\frac{\pi^2}{2R^2} \cos \frac{\pi r}{R} \frac{x_i x_j}{r^2} - \frac{\pi}{2R} \sin \frac{\pi r}{R} \left(-\frac{x_i x_j}{r^3}\right), \\ -\Delta \omega &= \frac{\pi}{2R} \left(\frac{\pi}{R} \cos \frac{\pi r}{R} + \frac{n-1}{r} \sin \frac{\pi r}{R}\right), \\ v_{x_i} &= -\delta(t - T)^2 \omega^{\delta-1} \omega' r_{x_i}, & \nabla v &= -\delta(t - T)^2 \omega^{\delta-1} \nabla \omega, & v_t &= -2\omega^\delta(t - T), \end{aligned}$$

$$\nabla \underline{u} = \frac{\nabla \omega v^\xi - \omega \xi v^{\xi-1} \nabla v}{v^{2\xi}} = \frac{4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2}{v^{\xi+1}},$$

$$\Delta \underline{u} = \nabla \left( \frac{v + \delta\xi \omega^\delta(t-T)^2}{v^{\xi+1}} \nabla \omega \right) \geq \frac{4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2}{v^{\xi+1}} \Delta \omega.$$

Then there exists  $r_0 \in (0, R)$  satisfying

$$-\Delta \omega \leq 0, \quad r_0 \leq r \leq R,$$

$$0 \leq -\Delta \omega \leq \frac{n\pi^2}{2R^2}, \quad \text{and} \quad \omega \geq \cos^2 \frac{\pi r_0}{2R}, \quad 0 \leq r \leq r_0,$$

and

$$\begin{aligned} \frac{\partial \underline{u}}{\partial x_i} &= \frac{\omega' r_{x_i} v^\xi - \omega \xi v^{\xi-1} v_{x_i}}{v^{2\xi}}, \\ \frac{\partial^2 \underline{u}}{\partial x_i \partial x_j} &= \omega'' v^{-\xi} r_{x_i} r_{x_j} + \xi \delta v^{-\xi-1} (t-T)^2 \omega^{\delta-1} (\omega')^2 r_{x_i} r_{x_j} + \omega' v^{-\xi} r_{x_i x_j} \\ &\quad + \xi \delta^2 v^{-\xi-1} (t-T)^2 \omega^{\delta-1} (\omega')^2 r_{x_i} r_{x_j} + \xi \delta v^{-\xi-1} (t-T)^2 \omega^\delta \omega'' r_{x_i} r_{x_j} \\ &\quad + \xi \delta^2 (\xi+1) v^{-\xi-2} (t-T)^2 (t-T)^2 \omega^{2\delta-1} (\omega')^2 r_{x_i} r_{x_j} \\ &\quad + \xi \delta v^{-\xi-1} (t-T)^2 \omega^\delta (\omega') r_{x_i x_j} \\ &= v^{-\xi-1} \left( 4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2 \omega'' \frac{x_i x_j}{r^2} \right) \\ &\quad + v^{-\xi-1} \left( 4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2 \omega' \left( -\frac{x_i x_j}{r^3} \right) \right) \\ &\quad + \delta(\xi+1) v^{-\xi-2} (t-T)^2 \omega^{\delta-1} (\omega')^2 [4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2] \frac{x_i x_j}{r^2} \\ &\quad + \delta(\delta\xi - 1) v^{-\xi-1} (t-T)^2 \omega^{\delta-1} (\omega')^2 \frac{x_i x_j}{r^2}. \end{aligned}$$

Therefore we have

$$\frac{\partial^2 \underline{u}}{\partial x_i \partial x_j} \geq v^{-\xi-1} (t-T)^2 \omega^{\delta-1} (\omega')^2 [4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2] \Delta \omega \frac{x_i x_j}{r^2},$$

and

$$\sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{u}}{\partial x_i} \frac{\partial^2 \underline{u}}{\partial x_i \partial x_j} \frac{\partial \underline{u}}{\partial x_j} \geq v^{-\xi-1} (t-T)^2 \omega^{\delta-1} (\omega')^2 [4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2] \Delta \omega |\nabla \omega|^2.$$

In view of the above, this gives

$$\begin{aligned} \underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \\ = \underline{u}_t - (|\nabla \underline{u}|^{p-2} \Delta \underline{u} + (p-2)|\nabla \underline{u}|^{p-4} (\nabla \underline{u})^T H_x(\underline{u}) \cdot \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \\ = \underline{u}_t - \left( |\nabla \underline{u}|^{p-2} \Delta \underline{u} + (p-2)|\nabla \underline{u}|^{p-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{u}}{\partial x_i} \frac{\partial^2 \underline{u}}{\partial x_i \partial x_j} \frac{\partial \underline{u}}{\partial x_j} \right) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \end{aligned}$$



$$\begin{aligned}
&\leq \frac{2\xi\omega^{\delta+1}(t-T)}{\nu^{\xi+1}} - \left( \frac{4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2}{\nu^{\xi+1}} \right)^{p-2} \left\{ \frac{4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2}{\nu^{\xi+1}} \right. \\
&\quad \left. + (p-2) \frac{4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2}{\nu^{\xi+1}} \right\} \Delta\omega |\nabla\omega|^{p-2} - \frac{\omega^{\alpha+\beta}}{\nu^{\xi\alpha}(r,t)} \int_0^t \frac{1}{\nu^{\xi\beta}(r,s)} ds + k \frac{\omega^m}{\nu^{m\xi}} \\
&= \frac{2\xi\omega^{\delta+1}(t-T)}{\nu^{\xi+1}} - \frac{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1}}{\nu^{(\xi+1)(p-1)}} \Delta\omega |\nabla\omega|^{p-2} \\
&\quad - \frac{\omega^{\alpha+\beta}}{\nu^{\xi\alpha}(r,t)} \int_0^t \frac{1}{\nu^{\xi\beta}(r,s)} ds + k \frac{\omega^m}{\nu^{m\xi}}.
\end{aligned}$$

We will discuss the problem for two cases.

Case 1.  $0 < t \leq \frac{T}{2}$ ,  $3T^2 < \nu \leq 4T^2$ . We need to show that for sufficiently small  $T$ ,

$$\frac{2\xi\omega^{\delta+1}(t-T)}{\nu^{\xi+1}} + k \frac{\omega^m}{\nu^{m\xi}} - \frac{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1}}{\nu^{(\xi+1)(p-1)}} \Delta\omega |\nabla\omega|^{p-2} \leq 0.$$

That is,

$$\begin{aligned}
-\Delta\omega &\leq \frac{\nu^{(\xi+1)(p-2)-m\xi} [2\xi\omega^{\delta+1}(T-t)\nu^{m\xi} - k\omega^m\nu^{\xi+1}]}{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1} |\nabla\omega|^{p-2}} \\
&\leq \frac{2\xi\omega^{\delta+1}(T-t)\nu^{(\xi+1)(p-2)}}{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1} |\nabla\omega|^{p-2}}.
\end{aligned}$$

Let  $\delta$  be sufficiently large, satisfying  $\xi\delta > 2$ . By the condition  $3T^2 < \nu \leq 4T^2$ ,  $0 < \omega \leq 1$ , and  $|\nabla\omega| < \frac{\pi}{2R}$ , we just have to make the following equality hold:

$$-\Delta\omega \leq \frac{1}{p-1} \frac{2\xi\omega^{\delta+1}(3T^2)^{(\xi+1)(p-2)}}{(3\xi\delta T^2)^{p-1}} \frac{1}{(\frac{\pi}{2R})^{p-2}}.$$

It is obvious that it holds for  $r_0 < r \leq R$ .

For  $0 \leq r < r_0$ , choose  $\xi > \frac{1}{p-2}$ ,  $T < \min\{\frac{1}{2}, [\frac{n\pi^p(\xi\delta)^{p-1}}{2^{p-1}3\xi p-2\xi-1R^p} (\cos \frac{\pi r_0}{2R})^{-2(\delta+1)}]^{\frac{1}{2\xi p-4\xi-1}}\}$  such that

$$\underline{u}_t - \operatorname{div}(|\nabla\underline{u}|^{p-2}\nabla\underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds - k\underline{u}^m \leq 0.$$

Case 2.  $\frac{T}{2} < t < 3T$ , choose  $\xi\beta > 2$  and we can get

$$\begin{aligned}
\underline{u}^\alpha \int_0^t \underline{u}^\beta ds &\geq \frac{\omega^{\alpha+\beta}}{\nu^{\xi\alpha}\nu_2^{\xi\beta}} \int_0^t \frac{1}{\nu_1^{\xi\beta}} ds = \frac{\omega^{\alpha+\beta}}{\omega^{\frac{\delta}{2}}\nu^{\xi\alpha}\nu_2^{\xi\beta}} \frac{1}{\beta\xi-1} \left( \frac{1}{\nu_1^{\xi\beta-1}} - \frac{1}{\nu_1^{\xi\beta-1}(0)} \right) \\
&= \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{\nu^{\xi(\alpha+\beta)-1}\nu_2} \frac{1}{\beta\xi-1} - \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{\nu^{\xi\alpha}\nu_2^{\xi\beta}\nu_1^{\xi\beta-1}(0)} \frac{1}{\beta\xi-1},
\end{aligned}$$

and

$$\begin{aligned}
&\underline{u}_t - \operatorname{div}(|\nabla\underline{u}|^{p-2}\nabla\underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k\underline{u}^m \\
&\leq \frac{4\xi\omega^{\delta+1}T}{\nu^{\xi+1}} - \frac{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1}}{\nu^{(\xi+1)(p-1)}} \Delta\omega |\nabla\omega|^{p-2} \\
&\quad - \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{\nu^{\xi(\alpha+\beta)-1}\nu_2} \frac{1}{\beta\xi-1} + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{\nu^{\xi\alpha}\nu_2^{\xi\beta}\nu_1^{\xi\beta-1}(0)} \frac{1}{\beta\xi-1} + k \frac{\omega^m}{\nu^{m\xi}}. \tag{3.1}
\end{aligned}$$

(i)  $r_0 \leq r \leq R$ ; we have  $-\Delta\omega \leq 0$ , then

$$\begin{aligned} & \underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \\ & \leq \frac{4\xi \omega^{\delta+1} T}{v_1^{\xi+1}} - \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi-1} + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi\alpha} v_2^{\xi\beta} v_1^{\xi\beta-1}(0)} \frac{1}{\beta\xi-1} + k \frac{\omega^m}{v_1^m}. \end{aligned}$$

Since  $\alpha + \beta > \max\{p-1, m\}$ , choose  $\xi \geq \max\{2(\alpha + \beta - 1)^{-1}, 2\beta^{-1}, (\alpha + \beta - m)^{-1}\}$  and  $\delta \geq \max\{\alpha + \beta - 1, 2(\alpha + \beta - m), 2\}$  to satisfy  $\xi(\alpha + \beta - p + 1 - m) > p$ . However,

$$\frac{v_1}{v_1(0)} = \frac{2T - \omega^{\frac{\delta}{2}}(t - T)}{2T + \omega^{\frac{\delta}{2}}T} \leq \frac{2T - \omega^{\frac{\delta}{2}}(\frac{T}{2} - T)}{2T + \omega^{\frac{\delta}{2}}T} = \frac{2 + \frac{1}{2}\omega^{\frac{\delta}{2}}}{2 + \omega^{\frac{\delta}{2}}} < 1,$$

thus we find

$$\frac{v_1^{\xi\beta-1}}{v_1^{\xi\beta-1}(0)} \leq \frac{v_1}{v_1(0)} \leq \frac{2 + \frac{1}{2}\omega^{\frac{\delta}{2}}}{2 + \omega^{\frac{\delta}{2}}} \quad \text{for } \xi\beta - 1 > 1.$$

Then

$$\begin{aligned} & \underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \\ & \leq \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi-1} \left[ 4\xi T(\xi\beta-1) v_2 v_1^{\xi(\alpha+\beta-1)-2} \omega^{\frac{3\delta}{2}-(\alpha+\beta-1)} \right. \\ & \quad \left. + \frac{v_1^{\xi\beta-1}}{v_1^{\xi\beta-1}(0)} - 1 + k \frac{\omega^{m-\alpha-\beta+\frac{\delta}{2}} v_2}{v_1^{\xi(m-\alpha-\beta)+1}} (\xi\beta-1) \right] \\ & \leq \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi-1} \left[ 8\xi T(\xi\beta-1) \omega^{\frac{3\delta}{2}-(\alpha+\beta-1)} \right. \\ & \quad \left. + \frac{2 + \frac{1}{2}\omega^{\frac{\delta}{2}}}{2 + \omega^{\frac{\delta}{2}}} - 1 + 4Tk(\xi\beta-1) \omega^{m-\alpha-\beta+\frac{\delta}{2}} \right] \\ & \leq \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi(\alpha+\beta)-1} v_2 (2 + \omega^{\frac{\delta}{2}})} \frac{1}{\beta\xi-1} \left[ 8\xi T(\xi\beta-1) \omega^{\frac{3\delta}{2}-(\alpha+\beta-1)} (2 + \omega^{\frac{\delta}{2}}) + \left( 2 + \frac{1}{2}\omega^{\frac{\delta}{2}} \right) \right. \\ & \quad \left. - (2 + \omega^{\frac{\delta}{2}}) + 4kT(2 + \omega^{\frac{\delta}{2}})(\xi\beta-1) \omega^{m-\alpha-\beta+\frac{\delta}{2}} \right] \\ & \leq \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v_1^{\xi(\alpha+\beta)-1} v_2 (2 + \omega^{\frac{\delta}{2}})} \frac{1}{\beta\xi-1} \left[ 24\xi T(\xi\beta-1) \omega^{\frac{3\delta}{2}-(\alpha+\beta-1)} \right. \\ & \quad \left. - \frac{1}{2}\omega^{\frac{\delta}{2}} + 12kT(\xi\beta-1) \omega^{m-\alpha-\beta+\frac{\delta}{2}} \right] \end{aligned}$$

in the sense of  $0 < \omega \leq 1$  and  $v_2 \leq 4T \leq 2$ .

In order to get the result, we have to show that

$$24\xi T(\xi\beta-1) \omega^{\frac{3\delta}{2}-(\alpha+\beta-1)} + 12kT(\xi\beta-1) \omega^{m-\alpha-\beta+\frac{\delta}{2}} \leq \frac{1}{2} \omega^{\frac{\delta}{2}}.$$

Note that  $0 < \omega \leq 1$  and  $\delta \geq \max\{\alpha + \beta - 1, 2(\alpha + \beta - m), 2\}$ , we choose

$$T \leq \min\left\{\frac{1}{2}, \frac{1}{24(2\xi + k)(\xi\beta - 1)}\right\},$$

in which case we can get the result.

(ii)  $0 \leq r < r_0$ ; we have  $0 \leq \Delta\omega \leq \frac{n\pi^2}{2R^2}$  and  $\omega \geq \cos^2 \frac{\pi r_0}{2R}$ . Since  $0 < \omega \leq 1$  and  $v(r, t) \leq 1$ , we know that  $\frac{4\xi\omega^{\delta+1}T}{v^{\xi+1}} \leq \frac{4\xi T}{v^{(\xi+1)(p-1)}}$ , and

$$\begin{aligned} & \frac{(p-1)[4T^2 + (\delta\xi - 1)\omega^\delta(t-T)^2]^{p-1}}{v^{(\xi+1)(p-1)}}(-\Delta\omega)|\nabla\omega|^{p-2} \\ & \leq \frac{(p-1)(4\xi\delta T^2)^{p-1}}{v^{(\xi+1)(p-1)}} \frac{n\pi^2}{2R^2} \left(\frac{\pi}{2R}\right)^{p-2} \\ & = \frac{(p-1)(2\xi\delta T^2)^{p-1}}{v^{(\xi+1)(p-1)}} \frac{n\pi^p}{R^p}. \end{aligned} \quad (3.2)$$

Let  $A = (\cos^2 \frac{\pi r_0}{2R})^\delta$ ,  $B = \frac{2+\frac{A}{2}}{2+A}$ , it is obvious that  $\frac{1}{2} < B < 1$  and  $\omega^\delta \geq A$ . Substituting equation (3.2) into equation (3.1) gives

$$\begin{aligned} & \underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds + k \underline{u}^m \\ & \leq \frac{4\xi T}{v^{(\xi+1)(p-1)}} + \frac{(p-1)(2\xi\delta T^2)^{p-1}}{v^{(\xi+1)(p-1)}} \frac{n\pi^p}{R^p} + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi\alpha} v_2^{\xi\beta} v_1^{\xi\beta-1}(0)} \frac{1}{\beta\xi - 1} \\ & \quad - \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi - 1} + k \frac{\omega^m}{v^{m\xi}} \\ & \leq \frac{4\xi T + (p-1)(2\xi\delta T^2)^{p-1} n\pi^p}{v^{(\xi+1)(p-1)} R^p} + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi\alpha} v_2^{\xi\beta} v_1^{\xi\beta-1}(0)} \frac{1}{\beta\xi - 1} - \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi - 1} + \frac{k}{v^{m\xi}} \\ & = \frac{4\xi T + (p-1)(2\xi\delta T^2)^{p-1} n\pi^p + kR^p}{v^{(\xi+1)(p-1)+m\xi} R^p} + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi\alpha} v_2^{\xi\beta}} \frac{1}{\beta\xi - 1} \left( \frac{1}{v_1^{\xi\beta-1}(0)} - \frac{B^{\xi\delta-1}}{v_1^{\xi\delta-1}} \right) \\ & \quad - \frac{(1-B^{\xi\delta-1})\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi(\alpha+\beta)-1} v_2} \frac{1}{\beta\xi - 1} \\ & \leq \frac{4\xi T + (p-1)(2\xi\delta T^2)^{p-1} n\pi^p + kR^p}{v^{(\xi+1)(p-1)+m\xi} R^p} - \frac{(1-B^{\xi\delta-1})A^{\frac{\alpha+\beta}{\delta}-\frac{1}{2}}}{v^{\xi(\alpha+\beta)-1}} \frac{1}{\beta\xi - 1} \\ & \quad + \frac{\omega^{\alpha+\beta-\frac{\delta}{2}}}{v^{\xi\alpha} v_2^{\xi\beta}} \frac{1}{\beta\xi - 1} \left( \frac{1}{v_1^{\xi\beta-1}(0)} - \frac{B^{\xi\delta-1}}{v_1^{\xi\delta-1}} \right). \end{aligned}$$

However, since  $v_1 = 2T - \omega^{\frac{\delta}{2}}(t-T) \leq 2T + \frac{T}{2}\omega^{\frac{\delta}{2}}$  and  $\frac{1}{2} < B < 1$ , it follows that

$$\frac{1}{v_1^{\xi\beta-1}(0)} \leq \frac{B^{\xi\delta-1}}{v_1^{\xi\delta-1}}.$$

Finally, we need to show that

$$\frac{4\xi T + (p-1)(2\xi\delta T^2)^{p-1} n\pi^p + kR^p}{v^{(\xi+1)(p-1)+m\xi} R^p} \leq \frac{(1-B^{\xi\delta-1})A^{\frac{\alpha+\beta}{\delta}-\frac{1}{2}}}{v^{\xi(\alpha+\beta)-1}} \frac{1}{\beta\xi - 1}.$$

Because  $T \leq \frac{1}{2}$  and  $v \leq 4T^2 \leq 1$ , we have to show

$$\left[ \frac{2\xi + (p-1)(2\xi\delta)^{p-1}n\pi^p + kR^p}{R^p} \right] (4T^2)^{\xi(\alpha+\beta-p+1-m)-p} \leq \frac{(1-B^{\xi\delta-1})A^{\frac{\alpha+\beta}{\delta}-\frac{1}{2}}}{\beta\xi-1}.$$

In other words, we just need the following inequality:

$$4T^2 \leq \left[ \frac{(1-B^{\xi\delta-1})A^{\frac{\alpha+\beta}{\delta}-\frac{1}{2}}R^p}{(\xi\beta-1)(4\xi + (2\xi\delta)^{p-1})n\pi^p} \right]^{\frac{1}{\xi(\alpha+\beta-p+1-m)-p}}$$

to hold. So choose  $T$  to be small enough, and we can get

$$\underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) - \underline{u}^\alpha \int_0^t \underline{u}^\beta ds - k\underline{u}^m \leq 0.$$

For  $x \in \partial\Omega$ ,  $\omega(R) = 0$ ,

$$\underline{u}(R, t) = \frac{\omega(R)}{v^\xi(R, t)} = 0 \leq \int_\Omega f(x, y) \underline{u} dy,$$

choose  $\underline{u}(r, 0) = \frac{\omega(r)}{v^\xi(r, 0)} = \frac{\omega(r)}{4T^2 - \omega^\delta T^2} \leq u_0$ , then  $\underline{u}(r, t)$  is the lower solution of problem (1.5)-(1.7). This implies that the solution blows up in finite time for large enough initial data.  $\square$

**Theorem 2** Suppose that  $\int_\Omega f(x, y) dy \geq 1$  for  $x \in \partial\Omega$ . If  $\alpha + \beta > m \geq 1$ , then the solution of problem (1.5)-(1.7) blows up in finite time for all strictly positive initial data with  $T$  sufficiently large.

*Proof* Consider the following problem:

$$v'(t) = Tv^{\alpha+\beta} - kv^m, \quad v(0) = v_0. \quad (3.3)$$

As  $\alpha + \beta > m$ , we know  $v^{\alpha+\beta} + 1 > v^m$ , and  $Tv^{\alpha+\beta} - kv^m \geq (T-k)v^{\alpha+\beta} - k$ . Therefore, the solution of equation (3.3) is an upper solution of the following problem:

$$v'(t) = (T-k)v^{\alpha+\beta} - k, \quad v(0) = v_0.$$

Since  $T > k$  and  $\alpha + \beta > 1$ , the solution of this problem blows up in finite time if  $v_0 > \left(\frac{k}{T-k}\right)^{\frac{1}{\alpha+\beta}}$ .

It is obvious that the solution of problem (3.3) is a lower solution of problem (1.5)-(1.7) when  $\int_\Omega \varphi(x, y) dy \geq 1$  and  $u_0(x) > v_0$ . By Proposition 1,  $u(x, t)$  is a blow-up solution.  $\square$

Suppose that the solution  $u(x, t)$  of problem (1.5)-(1.7) with  $p = 2$  blows up in finite time, and let  $U(t) = \max_{x \in \overline{\Omega}} u(x, t)$ . We suppose that the initial data satisfies the following assumptions:

$$(H_1) \quad \Delta u_0(x) + u_0^\alpha \int_0^t u_0^\beta ds - ku_0^m > 0.$$

$$(H_2) \quad \text{There exists a constant } \delta > 0 \text{ such that } \Delta u_0(x) + u_0^\alpha \int_0^t u_0^\beta ds - ku_0^m - \delta u_0^{\alpha+\beta} \geq 0.$$

**Theorem 3** Suppose that  $\int_{\Omega} f(x, y) dy \leq 1$  for  $x \in \partial\Omega$ . If  $\alpha + \beta > 1$ ,  $u_0(x)$  satisfies condition  $(H_1)$ – $(H_2)$ , and  $u(x, t)$  is the blow-up solution of problem (1.5)–(1.7) in finite time  $T$  with  $p = 2$ , then the blow-up rate is

$$c(T-t)^{-\frac{1}{\alpha+\beta-1}} \leq u(x, t) \leq C(T-t)^{-\frac{1}{\alpha+\beta-1}},$$

where  $c = [(\alpha + \beta - 1)T]^{-\frac{1}{\alpha+\beta-1}}$  and  $C = [\delta(\alpha + \beta - 1)]^{-\frac{1}{\alpha+\beta-1}}$ .

**Remark 2** Choose  $\Omega = (1, 2)$ ,  $\alpha = 2$ ,  $\beta = 1$ ,  $m = 2$  and  $f(x, y) = \frac{1}{8}xy$ , one can easily verify that  $u_0(x) = x^2$  satisfies  $(C_1)$ – $(C_2)$ , conditions in Theorem 3 are thus valid.

**Lemma 1** If  $u_0(x)$  satisfies condition  $(H_1)$ – $(H_2)$ ,  $p = 2$ , then there exists a positive constant  $c = [(\alpha + \beta - 1)T]^{-\frac{1}{\alpha+\beta-1}}$  such that  $U(t) \geq c(T-t)^{-\frac{1}{\alpha+\beta-1}}$ .

*Proof* It is obvious that  $U(t)$  is Lipschitz continuous and differentiable almost everywhere. By equation (1.5) with  $p = 2$  and  $\Delta U(t) \leq 0$ , it yields

$$U'(t) \leq U^\alpha \int_0^t U^\beta ds - kU^m \leq TU^{\alpha+\beta},$$

and thus

$$-(U^{1-\alpha-\beta}(t))' \leq T(\alpha + \beta - 1).$$

Integrating the result above over  $(t, T)$ , we can obtain the conclusion.  $\square$

*Proof of Theorem 3* Let  $J = u_t - \delta u^{\alpha+\beta}$ , where  $\delta > 0$  is sufficiently small. Since  $\alpha + \beta > 1$ , we have

$$\begin{aligned} J_t &= u_{tt} - \delta(\alpha + \beta)u^{\alpha+\beta-1}u_t \\ &= \Delta u_t + \alpha u^{\alpha-1}u_t \int_0^t u^\beta ds + u^{\alpha+\beta} - kmu^{m-1}u_t - \delta(\alpha + \beta)u^{\alpha+\beta-1}u_t, \\ \Delta J &= \Delta u_t - \delta(\alpha + \beta)u^{\alpha+\beta-1}u_t + \delta(\alpha + \beta)u^{2\alpha+\beta-1} \\ &\quad \times \int_0^t u^\beta ds - \delta(\alpha + \beta)(\alpha + \beta - 1)u^{\alpha+\beta-2}|\nabla u|^2, \\ J_t - \Delta J &\geq \alpha u^{\alpha-1}u_t \int_0^t u^\beta ds + u^{\alpha+\beta} - \delta(\alpha + \beta)u^{2\alpha+\beta-1} \int_0^t u^\beta ds - kmu^{m-1}u_t \\ &= \left( \alpha u^{\alpha-1} \int_0^t u^\beta ds - kmu^{m-1} \right) J + u^{\alpha+\beta} \left( 1 - \delta\beta u^{\alpha-1} \int_0^t u^\beta ds + kmu^{m-1}\delta \right), \end{aligned}$$

so we can choose  $\delta > 0$  to be small enough and thus obtain

$$J_t - \Delta J \geq \alpha u^{\alpha-1}J \int_0^t u^\beta ds.$$

On the other hand, as  $(x, t) \in \partial\Omega \times (0, T)$ , we get

$$\begin{aligned} J &= u_t - \delta u^{\alpha+\beta} = \int_{\Omega} f(x, y) u_t dy - \delta \left( \int_{\Omega} f(x, y) u dy \right)^{\alpha+\beta} \\ &= \int_{\Omega} f(x, y) J dy + \delta \int_{\Omega} f(x, y) u^{\alpha+\beta} dy - \delta \left( \int_{\Omega} f(x, y) u dy \right)^{\alpha+\beta}. \end{aligned}$$

Let  $0 < F(x) = \int_{\Omega} f(x, y) dy \leq 1$ . By Jensen's inequality, this gives

$$\begin{aligned} J &\geq \int_{\Omega} f(x, y) J dy + \delta F(x) \left[ \int_{\Omega} f(x, y) u \frac{dy}{F(x)} \right]^{\alpha+\beta} - \delta \left( \int_{\Omega} f(x, y) u dy \right)^{\alpha+\beta} \\ &\geq \int_{\Omega} f(x, y) J dy, \end{aligned}$$

and

$$J(x, t) = u_t - \delta u^{\alpha+\beta} = \Delta u + u^{\alpha} \int_0^t u^{\beta} ds - \delta u^{\alpha+\beta}.$$

It follows from the assumptions of  $(H_1)$ - $(H_2)$  that  $J(x, 0) \geq 0$ . Therefore, it is easy to deduce that  $J(x, t) \geq 0$  for  $(x, t) \in \overline{\Omega} \times [0, T]$ . That is,  $u_t \geq \delta u^{\alpha+\beta}$  and integrating this over  $(t, T)$  yields  $u \leq C(T - t)^{-\frac{1}{\alpha+\beta-1}}$ . Combining the results with Lemma 1, we obtain the desired result.  $\square$

#### 4 Global existence of solutions

In this section, we give sufficient conditions of the global existence of solutions.

**Theorem 4** Suppose that  $\int_{\Omega} f(x, y) dy < 1$  for  $x \in \partial\Omega$ . If  $\alpha + \beta \leq \max\{p - 1, m\}$ , then the solution of problem (1.5)-(1.7) exists globally for small initial data.

*Proof* Let  $\bar{u}(x, t) = qe^{nt} \psi(x)$ , where  $q, n > 0$  are determined later, and  $\psi(x)$  solves the following problem:

$$-\operatorname{div} |\nabla \psi|^{p-2} \nabla \psi = 1, \quad x \in \Omega,$$

$$\psi(x) = \int_{\Omega} f(x, y) dy, \quad x \in \partial\Omega,$$

where  $0 < \psi(x) < 1$ . Let  $\eta_1 = \max_{\overline{\Omega}} \psi(x)$ ,  $\eta_2 = \max_{\overline{\Omega}} \psi(x)$ ,

$$\begin{aligned} \bar{u}_t - \operatorname{div} (|\nabla \bar{u}|^{p-2} \nabla \bar{u}) - \bar{u}^{\alpha} \int_0^t \bar{u}^{\beta} ds + k \bar{u}^m \\ = qne^{nt} \psi + q^{p-1} e^{n(p-1)t} - \frac{1}{n\beta} q^{\alpha+\beta} e^{n(\alpha+\beta)t} \psi^{\alpha+\beta} + kq^m e^{mnt} \psi^m \\ \geq q^{p-1} e^{n(p-1)t} - \frac{1}{n\beta} q^{\alpha+\beta} e^{n(\alpha+\beta)t} \psi^{\alpha+\beta} + kq^m e^{mnt} \psi^m, \end{aligned}$$

since  $\alpha + \beta < \max\{p - 1, m\}$ , choosing

$$n > \max \left\{ \frac{1}{k\beta} (q\eta_1)^{\alpha+\beta-m}, \frac{1}{\beta} q^{\alpha+\beta-p+1} \eta_2^{\alpha+\beta} \right\},$$

we can infer that

$$\bar{u}_t - \operatorname{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) - \bar{u}^\alpha \int_0^t \bar{u}^\beta ds + k \bar{u}^m \geq 0.$$

For  $x \in \partial\Omega$ ,  $0 < \psi(x) < 1$

$$\bar{u}(x, t) = q e^{nt} \int_\Omega f(x, y) dy \geq \int_\Omega f(x, y) \bar{u} dy.$$

Selecting  $u_0(x) < q\eta_1$ , we can deduce that the result holds.  $\square$

**Theorem 5** Suppose that  $\int_\Omega f(x, y) dy \leq \rho < 1$  for  $x \in \partial\Omega$ . If  $p - 1 < \min\{\alpha, \beta + 1, m\}$ , and  $\alpha + \beta > m$ , then the solution of problem (1.5)-(1.7) exists globally for small initial data.

*Proof* Suppose that  $\eta(x)$  solves the following problem:

$$\begin{aligned} -\operatorname{div}|\nabla \eta|^{p-2} \nabla \eta &= 1, \quad x \in \Omega, \\ \eta(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

in which  $0 < \eta(x) < 1$ . Let  $C = \max_{\bar{\Omega}} \eta(x)$ .

Set  $\bar{u}(x, t) = k_1(A + t)^{-r} \left( \frac{\rho}{1-\rho} + \frac{\eta(x)}{C} \right)$ , where  $A, k_1 \geq 1$  and  $0 \leq r \leq \frac{1}{p-2}$ , then

$$\bar{u}_t = -k_1 r (A + t)^{-r-1} \left( \frac{\rho}{1-\rho} + \frac{\eta(x)}{C} \right) \geq -k_1 r (A + t)^{-r-1} \frac{1}{1-\rho} \geq -k_1 r (A + t)^{-r(p-1)} \frac{1}{1-\rho},$$

and

$$\begin{aligned} -\bar{u}^\alpha \int_0^t \bar{u}^\beta ds &= -k_1^{\alpha+\beta} (A + t)^{-r\alpha} \left( \frac{\eta(x)}{C} + \frac{\rho}{1-\rho} \right)^{\alpha+\beta} \int_0^t (A + t)^{-r\beta} ds \\ &\geq -\frac{k_1^{\alpha+\beta}}{1-r\beta} (A + t)^{1-r\alpha-r\beta} \left( \frac{\eta(x)}{C} + \frac{\rho}{1-\rho} \right)^{\alpha+\beta}. \end{aligned}$$

For  $p - 1 < \min\{\alpha, \beta + 1, m\}$ , we choose  $0 \leq r \leq \frac{1}{p-2}$  such that  $1 - r(\alpha + \beta - m) < 0$  and  $r\beta \leq 1$ . It follows that

$$-\bar{u}^\alpha \int_0^t \bar{u}^\beta ds \geq -\frac{k_1^{\alpha+\beta}}{1-r\beta} (A + t)^{1-r\alpha-r\beta} \left( \frac{1}{1-\rho} \right)^{\alpha+\beta},$$

and

$$\begin{aligned} \bar{u}_t - \operatorname{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) - \bar{u}^\alpha \int_0^t \bar{u}^\beta ds + k \bar{u}^m &\geq -k_1 r (A + t)^{-r(p-1)} \frac{1}{1-\rho} + k_1^{p-1} (A + t)^{-r(p-1)} - \frac{k_1^{\alpha+\beta}}{1-r\beta} (A + t)^{1-r\alpha-r\beta} \left( \frac{1}{1-\rho} \right)^{\alpha+\beta} \\ &\quad + k k_1^m (A + t)^{-mr} \left( \frac{\eta(x)}{C} + \frac{\rho}{1-\rho} \right)^m \\ &= k_1 (A + t)^{-r(p-1)} \left( \frac{-r}{1-\rho} + \frac{k_1^{p-2}}{2} \right) + k_1^{p-1} (A + t)^{-r(p-1)} \end{aligned}$$

$$\times \left[ \frac{1}{2} - \frac{k_1^{\alpha+\beta-p+1}}{1-r\beta} (A+t)^{1-r(\alpha+\beta-p+1)} \left( \frac{1}{1-\rho} \right)^{\alpha+\beta} \right. \\ \left. + k k_1^{m-p+1} (A+t)^{-r(m-p+1)} \left( \frac{\eta(x)}{C} + \frac{\rho}{1-\rho} \right)^m \right].$$

Since  $A, k_1 \geq 1$ , choosing

$$k_1 \geq \max \left\{ \left( \frac{2r}{1-\rho} \right)^{p-2}, 1 \right\}$$

and

$$A \geq \max \left\{ \left[ k(1-r\beta) \left( \frac{1-\rho}{k_1} \right)^{\alpha+\beta-m} \right]^{\frac{1}{1-r(\alpha+\beta-m)}}, 1 \right\},$$

we can get

$$\bar{u}_t - \operatorname{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) - \bar{u}^\alpha \int_0^t \bar{u}^\beta ds + k \bar{u}^m \geq 0.$$

On the other hand, for  $x \in \partial\Omega$  and sufficiently large  $A$ , we have

$$\begin{aligned} \bar{u}(x, t) &= k(A+t)^{-r} \frac{\rho}{1-\rho} \geq k(A+t)^{-r} \frac{1}{1-\rho} \int_{\Omega} f(x, y) dy \\ &\geq \int_{\Omega} k(A+t)^{-r} \left( \frac{\eta(x)}{C} + \frac{\rho}{1-\rho} \right) f(x, y) dy \\ &= \int_{\Omega} \bar{u} f(x, y) dy. \end{aligned}$$

Choosing  $u_0$  to be sufficiently small such that  $u_0 \leq kA^{-r}(\frac{\eta(x)}{C} + \frac{\rho}{1-\rho})$ , we can conclude that  $\bar{u}$  is an upper solution of problem (1.5)-(1.7). The proof is completed.  $\square$

**Theorem 6** Suppose that  $\int_{\Omega} f(x, y) dy < 1$  for  $x \in \partial\Omega$ . If  $\alpha + \beta > m$ , then the solution of problem (1.5)-(1.7) exists globally for small enough initial data.

*Proof* Choosing  $\bar{u}(x, t) = A$  and  $A > \max\{(\frac{k}{\rho})^{\frac{1}{\alpha+\beta-1}}, u_0(x)\}$ , it is easy to see that the result holds.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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