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Large time behavior of solutions for the porous medium equation with a nonlinear gradient source

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Abstract

This paper deals with the large time behavior of non-negative solutions for the porous medium equation with a nonlinear gradient source $u_t = \Delta u^m + |\nabla u^l|^q$, $(x, t) \in \Omega \times (0, \infty)$, where $l \geq m > 1$ and $1 \leq q < 2$. When $lq = m$, we prove that the global solution converges to the separate variable solution $t^{-\frac{1}{m-1}} f(x)$. While $m < lq \leq m + 1$, we note that the global solution converges to the separate variable solution $t^{-\frac{1}{m-1}} f_0(x)$. Moreover, when $lq > m + 1$, we show that the global solution also converges to the separate variable solution $t^{-\frac{1}{m-1}} f_0(x)$ for the small initial data $u_0(x)$, and we find that the solution $u(x, t)$ blows up in finite time for the large initial data $u_0(x)$.

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1 Introduction

In this paper, we investigate the large time behavior of non-negative solutions for the following initial-boundary value problem:

$$\begin{cases} u_t = \Delta u^m + |\nabla u^l|^q, & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $l \geq m > 1$, $1 \leq q < 2$, Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$, and the initial function is

$$u_0(x) \in C_0(\overline{\Omega}) = \left\{ z \in C(\overline{\Omega}) : z = 0 \text{ on } \partial\Omega \right\}, \quad u_0(x) \geq 0, u_0(x) \not\equiv 0. \tag{1.2}$$

Equation (1.1) arises in the study of the growth of surfaces and it has been considered as a mathematical model for a variety of physical problems (see [1, 2]). For instance, in ballistic deposition processes, the evolution of the profile of a growing interface is described by the diffusive Hamilton-Jacobi type equation (1.1) with $m = l = 1$ (see [3]).

One of the particular feature of problem (1.1) is that the equation is a slow diffusion equation with nonlinear source term depending on the gradient of a power of the solution.

In general, there is no classical solution. Therefore, it turns out that a suitable framework for the well-posedness of the initial-boundary value problem (1.1) is the theory of viscosity solutions (see [4–6]), so we first define the notion of solutions.

Definition 1.1 A non-negative function $u(x,t) \in C(\overline{\Omega} \times (0,\infty))$ is called a solution of (1.1), if $u(x,t)$ is a viscosity solution to (1.1) in $\Omega \times (0,\infty)$ and satisfies

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty) \quad \text{and} \quad u(x,0) = u_0(x) \in C_0(\overline{\Omega}), \quad x \in \overline{\Omega}. \tag{1.3}$$

Under some assumptions, the global (local) existence in time, uniqueness and regularity of solutions to reaction-diffusion equations with gradient terms have been extensively investigated by many authors (see [7–12] and the references therein). In particular, in [1], Andreucci proved the existence of solutions for the following degenerate parabolic equation with initial data measures:

$$\begin{cases} u_t = \Delta u^m + |\nabla u^l|^q, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \mu, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $m \geq 1$, $ql \geq m$, $0 < q < 2$, and $N \geq 1$.

The main purpose of this paper is to further study the large time behavior of non-negative solutions $u(x,t)$ to (1.4) with homogeneous Dirichlet boundary conditions. In recent years, many authors have investigated the asymptotic behavior of solutions to the viscous Hamilton-Jacobi equations (see [3, 4, 7, 8, 10, 13–21] and the references therein). For example, for the special case $m = l = 1$, Gilding [22] studied the large time behavior of solutions to the following Cauchy problem:

$$\begin{cases} u_t = \Delta u + |\nabla u|^q, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

and he gave the temporal decay estimates.

In [23], Stinner investigated the asymptotic behavior of solutions for the following one space dimensional viscous Hamilton-Jacobi equation:

$$\begin{cases} u_t = (|u_x|^{p-2} u_x)_x + |u_x|^q, & (x,t) \in (-R,R) \times (0,\infty), \\ u(\pm R, t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in (-R,R), \end{cases} \tag{1.6}$$

where $R > 0$, $p > 2$, and $1 < q < p - 1$, and he proved that these solutions converge to the steady states by Lyapunov functional.

In higher dimensional case, Barles *et al.* [24] studied the large time behavior of solutions for the following initial-boundary value problem:

$$\begin{cases} u_t = \text{div}(|\nabla u|^{p-2} \nabla u) + |\nabla u|^q, & (x,t) \in B(0,1) \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial B(0,1) \times (0,\infty), \\ u(x,0) = u_0(x), & x \in B(0,1), \end{cases} \tag{1.7}$$

where $p \geq 2$, $0 < q < p - 1$, and they showed that the non-negative radially symmetric solutions converge to the stationary solution.

Recently, in [25], Laurencot *et al.* extended the case $0 < q < p - 1$ of the problem (1.7) to the case $q \geq p - 1$, and derived that these solutions converge to two different separate variables solutions according to the cases $q = p - 1$ and $q > p - 1$ in the general bounded domain $\Omega \subset \mathbb{R}^N$, respectively.

Motivated by the above work, by using the modified comparison argument, the self-similar transformation method, and the half-relaxed limits technique used in [6, 25, 26], we investigate the asymptotic behavior of non-negative solutions to (1.1). Our main results in this paper are stated as follows.

Theorem 1.1 *Let $l \geq m > 1$, $1 \leq q < 2$, and $lq = m$. Assume that $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies* (1.2). *Then there exists a unique solution $u(x, t)$ to (1.1) in the sense of Definition 1.1 such that*

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - f \right\|_\infty = 0, \tag{1.8}$$

where $f(x) \in C_0(\overline{\Omega})$ is the unique positive solution to

$$-\Delta f^m - \left| \nabla f^l \right|^q - \frac{f}{m-1} = 0 \quad in \ \Omega, \qquad f > 0 \quad in \ \Omega, \qquad f = 0 \quad on \ \partial\Omega. \tag{1.9}$$

Moreover, we have $|\nabla u(t)| \in L^\infty(\overline{\Omega})$ for all $t \geq 0$ and

$$\ell[u_0] := \sup_{t \geq 0} \left\{ \left\| \nabla u(t) \right\|_\infty \right\} < \infty. \tag{1.10}$$

Theorem 1.2 *Let $l \geq m > 1$, $1 \leq q < 2$, and $m < lq \leq m + 1$. Assume that $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies* (1.2). *Then there exists a unique solution $u(x, t)$ to (1.1) in the sense of Definition 1.1 such that*

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - f_0 \right\|_\infty = 0, \tag{1.11}$$

where $f_0(x) \in C_0(\overline{\Omega})$ is the unique positive solution to

$$-\Delta f_0^m - \frac{f_0}{m-1} = 0 \quad in \ \Omega, \qquad f_0 > 0 \quad in \ \Omega, \qquad f_0 = 0 \quad on \ \partial\Omega. \tag{1.12}$$

Theorem 1.3 *Let $l \geq m > 1$, $1 \leq q < 2$, and $lq > m + 1$. Assume that $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies* (1.2) *and suppose further that there exists $G_0 \in W^{1,\infty}(\Omega)$ satisfying* (1.2) *such that*

$$u_0(x) \leq \frac{G_0(x)}{\ell[G_0]}, \quad x \in \overline{\Omega}, \tag{1.13}$$

where $\ell[G_0]$ is defined in (1.10). *Then there exists a unique solution $u(x, t)$ to (1.1) in the sense of Definition 1.1 such that*

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - f_0 \right\|_\infty = 0, \tag{1.14}$$

where $f_0(x) \in C_0(\overline{\Omega})$ is defined in (1.12).

Theorem 1.4 *Let $l \geq m > 1$, $1 \leq q < 2$, and $lq > m + 1$. Assume that there exists a positive constant K_0 depending only on m, l, q, s, and ε such that $\int_\Omega u_0^s(x)\phi_\varepsilon(x)\,dx \geq K_0$, where $s \in (0, \frac{1}{2})$, $\varepsilon \in (0, 1)$, and $\phi_\varepsilon(x) = A_\varepsilon e^{-\varepsilon|x|}$ with $A_\varepsilon = \frac{1}{\int_\Omega e^{-\varepsilon|x|}\,dx}$. Then the solution $u(x, t)$ of the problem* (1.1) *blows up in finite time in the sense of weak solution. Moreover, the upper bound of blow-up time is given as follows*:

$$T \leq \frac{2}{ql-1}\left(\frac{ql}{ql+s-1}\right)^{\frac{q(m-1)}{ql-m}}\left(m\varepsilon^2\right)^{\frac{1-ql}{ql-m}}. \tag{1.15}$$

Remark 1.1 Compared to the results in [25], we extend the results of p-Laplacian equation to the porous medium equation (1.1) with a nonlinear gradient source.

Remark 1.2 In Theorem 1.4, we only give an upper estimate of the blow-up time. But the lower estimate of the blow-up time is an open problem.

This paper is organized as follows. In Section 2, we establish the comparison lemmas to prove the uniqueness of the positive solution to (1.9) and the identification of the half-relaxed limits. In Section 3, using the comparison principle, we construct the global solutions to obtain the upper bound and Hölder estimate of solutions to (1.1), and we prove Theorems 1.1 and 1.2 by the half-relaxed limits method. Moreover, we give the large time behavior of solutions to (1.1) with the small initial data $u_0(x)$ for $lq > m + 1$, and we prove Theorem 1.3 in Section 4. Finally, we obtain the blow-up case, and we prove Theorem 1.4 in Section 5.

2 Comparison lemmas

In this section, we establish the following comparison lemma between positive supersolutions and non-negative subsolutions to the elliptic equation in (1.9): which is an important tool for the uniqueness of the positive solution to (1.9) and the identification of the half-relaxed limits later.

Lemma 2.1 *Let $l \geq m > 1$, $1 \leq q < 2$, and $lq = m$. Assume that $w \in USC(\overline{\Omega})$ and $W \in LSC(\overline{\Omega})$ are respectively a bounded upper semicontinuous (usc) viscosity subsolution and a bounded lower semicontinuous (lsc) viscosity supersolution to*

$$-\Delta\zeta^m - \left|\nabla\zeta^l\right|^q - \frac{\zeta}{m-1} = 0 \quad in\ \Omega, \tag{2.1}$$

such that

$$w(x) = W(x) = 0 \quad for\ x \in \partial\Omega, \tag{2.2}$$

and

$$W(x) > 0 \quad for\ x \in \Omega. \tag{2.3}$$

Then

$$w(x) \leq W(x) \quad for\ x \in \overline{\Omega}. \tag{2.4}$$

Proof The proof is based on the idea as in [25, 27], but with different auxiliary functions.

For $i \geq N_0$ large enough, it is easy to see that $\Omega_i = \{x \in \Omega : d(x, \partial\Omega) > \frac{1}{i}\}$ is a non-empty open subset of $\mathbb{R}^N$.

Since $\overline{\Omega}_i$ is compact and W is lower semicontinuous, the function W has a minimum in $\overline{\Omega}_i$. By the positivity (2.3) of W in $\overline{\Omega}_i$, we have

$$\mu_i = \min_{x \in \overline{\Omega}_i} \{W(x)\} > 0. \tag{2.5}$$

Similarly, the compactness of $\overline{\Omega} \setminus \Omega_i$, the upper semicontinuity and boundedness of w imply that w has at least one point of maximum x_i in $\overline{\Omega} \setminus \Omega_i$ and we set

$$\eta_i = w(x_i) = \max_{x \in \overline{\Omega} \setminus \Omega_i} \{w(x)\}. \tag{2.6}$$

It follows from $\partial\Omega \subset \overline{\Omega} \setminus \Omega_i$ and w vanishes on $\partial\Omega$ by (2.2) that $\eta_i \geq 0$.

Next, we claim that

$$\lim_{i \to \infty} \eta_i = 0. \tag{2.7}$$

Indeed, owing to the compactness of $\overline{\Omega}$ and the definition of Ω_i, there exist $y \in \partial\Omega$ and a subsequence of $\{x_i\}_{i \geq N_0}$ (not relabeled) such that $x_i \to y$ as $i \to \infty$. Since $w(y) = 0$, we deduce from the upper semicontinuity of w that

$$\lim_{\varepsilon \to 0} \sup \{w(x) : x \in B(y, \varepsilon) \cap \overline{\Omega}\} \leq 0.$$

Given $\varepsilon > 0$ small enough, there exists i_ε large enough such that

$$x_i \in B(y, \varepsilon) \cap \overline{\Omega} \quad \text{for } i \geq i_\varepsilon.$$

Therefore, we have

$$0 \leq \eta_i = w(x_i) \leq \sup \{w(x) : x \in B(y, \varepsilon) \cap \overline{\Omega}\}, \quad i \geq i_\varepsilon.$$

Thus

$$0 \leq \lim_{i \to \infty} \sup \eta_i \leq \sup \{w(x) : x \in B(y, \varepsilon) \cap \overline{\Omega}\}.$$

Letting $\varepsilon \to 0$, we conclude that zero is a cluster point of $\{\eta_i\}_{i \geq N_0}$ as $i \to \infty$. The claim (2.7) follows from the monotonicity of $\{\eta_i\}_{i \geq N_0}$.

Now, fix $s \in (0, \infty)$. For $\delta > 0$ and $i \geq N_0$, we define

$$z_i(t, x) = (t + s)^{-\frac{1}{m-1}} w(x) - s^{-\frac{1}{m-1}} \eta_i, \quad (t, x) \in [0, \infty) \times \overline{\Omega}$$

and

$$Z_\delta(t, x) = (t + \delta)^{-\frac{1}{m-1}} W(x), \quad (t, x) \in [0, \infty) \times \overline{\Omega}.$$

It follows from the assumptions on w and W that z_i and Z_δ are, respectively, a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to

$$\partial_t \zeta - \Delta \zeta^m - \left| \nabla \zeta^l \right|^q = 0 \quad \text{in } (0, \infty) \times \Omega,$$

and satisfy

$$Z_\delta(t, x) = 0 \geq -s^{-\frac{1}{m-1}} \eta_i = z_i(t, x), \quad (t, x) \in (0, \infty) \times \partial \Omega.$$

Moreover, if

$$0 < \delta < \left[\frac{\mu_i}{1 + \|w\|_\infty} \right]^{m-1} s, \tag{2.8}$$

then it follows from (2.5) and (2.8) that, for $x \in \Omega_i$,

$$Z_\delta(0, x) = \delta^{-\frac{1}{m-1}} W(x) \geq \delta^{-\frac{1}{m-1}} \mu_i \geq s^{-\frac{1}{m-1}} \|w\|_\infty \geq z_i(0, x).$$

For $x \in \overline{\Omega} \setminus \Omega_i$, we deduce from (2.6) that

$$Z_\delta(0, x) \geq 0 \geq s^{-\frac{1}{m-1}} \left(w(x) - \eta_i \right) = z_i(0, x).$$

By the comparison principle [6], we have

$$z_i(t, x) \leq Z_\delta(t, x), \quad (t, x) \in [0, \infty) \times \overline{\Omega}, \tag{2.9}$$

for $i \geq N_0$ and $\delta > 0$ satisfying (2.8).

According to (2.8), the parameter δ can be taken to be arbitrarily small in (2.9). Therefore, we deduce that

$$(t + s)^{-\frac{1}{m-1}} w(x) - s^{-\frac{1}{m-1}} \eta_i \leq t^{-\frac{1}{m-1}} W(x), \quad (t, x) \in (0, \infty) \times \overline{\Omega},$$

for $i \geq N_0$.

Passing to the limit as $i \to \infty$, it follows from (2.7) that

$$(t + s)^{-\frac{1}{m-1}} w(x) \leq t^{-\frac{1}{m-1}} W(x), \quad (t, x) \in (0, \infty) \times \overline{\Omega}.$$

Finally, let $s \to 0$ and take $t = 1$; then we obtain

$$w(x) \leq W(x), \quad (t, x) \in (0, \infty) \times \overline{\Omega}.$$

The proof of Lemma 2.1 is complete. □

A straightforward consequence of Lemma 2.1 is the uniqueness of the solution to (1.9).

Corollary 2.1 *There is at most one positive viscosity solution to* (1.9).

By the similar argument, we have the following result to (1.12).

Lemma 2.2 *Let $w \in USC(\overline{\Omega})$ and $W \in LSC(\overline{\Omega})$ be respectively a bounded upper semi-continuous (usc) viscosity subsolution and a bounded lower semicontinuous (lsc) viscosity supersolution to*

$$-\Delta \zeta^m - \frac{\zeta}{m-1} = 0 \quad in \ \Omega, \tag{2.10}$$

satisfying (2.2) *and* (2.3). *Then*

$$w(x) \leq W(x) \quad for \ x \in \overline{\Omega}. \tag{2.11}$$

Proof The proof is similar as in Lemma 2.1, so we omit it here. □

3 Proofs of Theorems 1.1 and 1.2

In this section, we obtain the well-posedness and large time behavior of solutions to (1.1) for $lq \in [m, m+1]$, and we prove Theorems 1.1-1.2. To do this, we first obtain the well-posedness to (1.1) by the following proposition.

Proposition 3.1 *Assume that $l \geq m > 1$, $1 \leq q < 2$, $lq \in [m, m+1]$, and $u_0(x) \in C_0(\overline{\Omega})$ satisfies* (1.2). *Then there exists a unique solution $u(x,t)$ to* (1.1) *in the sense of Definition* 1.1.

Proof The idea of the proof is same as in [25, 28], so we omit here. □

Next, in order to prove the large time behavior of the solution to (1.1), we shall need several steps: Step 1, we will find that the temporal decay rate of $\|u(t)\|_\infty$ is indeed $t^{-\frac{1}{m-1}}$. Step 2, we prove the boundary estimates for the large time which guarantee that no loss of boundary condition occurs throughout the time evolution. Step 3, the half-relaxed limits technique is applied to show the expected convergence after introducing self-similar variables. The approach is developed by Laurençot and Stinner in [25, 27]. To do this, we need the following lemmas.

Lemma 3.1 (Upper bound) *Assume that $l \geq m > 1$, $1 \leq q < 2$, $lq = m$, and the initial data $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies* (1.2). *Then there exists $C_1 > 0$ depending only on m, l, q, Ω, and $\|u_0\|_\infty$ such that*

$$u(t,x) \leq C_1(1+t)^{-\frac{1}{m-1}}, \quad (t,x) \in (0,\infty) \times \overline{\Omega}. \tag{3.1}$$

Proof Assume that $x_0 \notin \overline{\Omega}$ and $R_0 > 0$ such that $\Omega \subset B(x_0, R_0)$. For $(t,x) \in (0,\infty) \times \overline{\Omega}$, we define the function

$$S_1(t,x) = A_1(1+t)^{-\frac{1}{m-1}}\varphi(r), \qquad \varphi(r) = \left[\frac{m}{m+1}\left(e^{\frac{R(m+1)}{m}} - e^{\frac{r(m+1)}{m}}\right)\right]^{\frac{1}{m}} \quad \text{with } r = |x - x_0|,$$

where $R > R_0$ and $A_1 > 0$ satisfies the following condition:

$$A_1 \geq \max\left\{\left(\frac{m}{m-1}e^{\frac{R(m+1)}{m^2}}\right)^{\frac{1}{m-1}}, \frac{\|u_0\|_\infty}{\varphi(R_0)}\right\}. \tag{3.2}$$

Since $x_0 \notin \overline{\Omega}$, the function $S_1(t,x)$ is C^∞-smooth in $[0,\infty) \times \overline{\Omega}$. Moreover, according the condition $l \geq m > 1$, $1 \leq q < 2$, and $lq = m$, we have $l = m > 1$ and $q = 1$. Therefore, for $(t,x) \in (0,\infty) \times \Omega$ and $r = |x - x_0| < R_0 < R$, it follows from (3.2) that

$$(1+t)^{\frac{m}{m-1}}\left\{ \partial_t S_1 - \triangle S_1^m - \left| \nabla S_1^l \right|^q \right\}(t,x)$$

$$= -\frac{A_1}{m-1}\varphi(r) - A_1^m\left[\frac{N-1}{r}\left(\varphi^m\right)_r + \left(\varphi^m\right)_{rr} \right] - A_1^m\left|\left(\varphi^l\right)_r\right|^q$$

$$= -\frac{A_1}{m-1}\varphi(r) + A_1^m\left[\frac{N-1}{r}e^{\frac{r(m+1)}{m}} + \frac{m+1}{m}e^{\frac{r(m+1)}{m}} \right] - A_1^m e^{\frac{r(m+1)}{m}}$$

$$\geq A_1\left[A_1^{m-1}\left(\frac{N-1}{r} + \frac{1}{m} \right)e^{\frac{r(m+1)}{m}} - \frac{1}{m-1}e^{\frac{R(m+1)}{m^2}} \right]$$

$$\geq A_1\left(\frac{1}{m}A_1^{m-1} - \frac{1}{m-1}e^{\frac{R(m+1)}{m^2}} \right)$$

$$\geq 0. \tag{3.3}$$

Hence, the condition (3.2) guarantees that $S_1(t,x)$ is a supersolution to (1.1) in $(0,\infty) \times \Omega$. In addition, since $r = |x - x_0| < R_0 < R$ for $x \in \Omega$, for $(t,x) \in (0,\infty) \times \partial\Omega$, we deduce from (3.2) that

$$u(t,x) = 0 \leq A_1(1+t)^{-\frac{1}{m-1}}\varphi(R_0) \leq S_1(t,x),$$

and

$$u_0(x) \leq \left\| u_0(x) \right\|_\infty \leq A_1\varphi(R_0) \leq S_1(0,x), \quad x \in \overline{\Omega}.$$

By the comparison principle, we have

$$u(t,x) \leq S_1(t,x), \quad (t,x) \in [0,\infty) \times \overline{\Omega}.$$

The proof of Lemma 3.1 is complete. $\qquad\square$

Lemma 3.2 (Upper bound) *Assume that $l \geq m > 1$, $1 \leq q < 2$, $lq \in (m, m+1]$, and the initial data $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies (1.2). Then there exists $C_2 > 0$ depending only on m, l, q, Ω, and $\|u_0\|_\infty$ such that*

$$u(t,x) \leq C_2(1+t)^{-\frac{1}{m-1}}, \quad (t,x) \in (0,\infty) \times \overline{\Omega}. \tag{3.4}$$

Proof Assume that $x_0 \notin \overline{\Omega}$ and $R_0 > 0$ such that $\Omega \subset B(x_0, R_0)$. For $(t,x) \in (0,\infty) \times \overline{\Omega}$, we define the function

$$S_2(t,x) = A_2(1 + \delta t)^{-\frac{1}{m-1}}\sigma(r), \qquad \sigma(r) = \left[\frac{m}{m+1}\left(R^{\frac{m+1}{m}} - r^{\frac{m+1}{m}} \right) \right]^{\frac{1}{m}} \quad \text{with } r = |x - x_0|,$$

where the positive constants A_2, R, and δ satisfy the following condition:

$$A_2 = \left[\frac{m^{q-1}(1+m(N-1))}{2l^q R^{\frac{(m+1)(l-m)q+m(q+m-1)}{m^2}}}\right]^{\frac{1}{lq-m}}, \qquad R = \left[R_0^{\frac{m+1}{m}} + \frac{(m+1)\|u_0\|_\infty^m}{mA_2^m}\right]^{\frac{m}{m+1}} > R_0$$

$$\text{and} \quad \delta = \frac{(m-1)(1+m(N-1))A_2^{m-1}}{2mR^{\frac{1+m^2}{m^2}}}.$$

(3.5)

Since $x_0 \notin \overline{\Omega}$, the function $S_2(t,x)$ is C^∞-smooth in $[0,\infty) \times \overline{\Omega}$. Moreover, for $(t,x) \in (0,\infty) \times \Omega$ and $r = |x - x_0| < R_0 < R$, it follows from (3.5) and $lq > m$ that

$$(1+\delta t)^{\frac{m}{m-1}}\left\{\partial_t S_2 - \triangle S_2^m - \left|\nabla S_2^l\right|^q\right\}(t,x)$$

$$= -\frac{A_2\delta}{m-1}\sigma(r) - A_2^m\left[\frac{N-1}{r}(\sigma^m)_r + (\sigma^m)_{rr}\right] - A_2^{lq}(1+\delta t)^{-\frac{lq-m}{m-1}}\left|(\sigma^l)_r\right|^q$$

$$= -\frac{A_2\delta}{m-1}\sigma(r) + A_2^m\left[\frac{1}{m}+N-1\right]r^{-\frac{m-1}{m}} - A_2^{lq}(1+\delta t)^{-\frac{lq-m}{m-1}}\left|\left((\sigma^m)^{\frac{l}{m}}\right)_r\right|^q$$

$$= -\frac{A_2\delta}{m-1}\sigma(r) + A_2^m\left[\frac{1}{m}+N-1\right]r^{-\frac{m-1}{m}} - A_2^{lq}\left(\frac{l}{m}\right)^q R^{\frac{(m+1)(l-m)q+mq}{m^2}}$$

$$\geq A_2^m\left[\left(N-1+\frac{1}{m}\right)R^{-\frac{m-1}{m}} - A_2^{lq-m}\left(\frac{l}{m}\right)^q R^{\frac{(m+1)(l-m)q+mq}{m^2}} - \frac{\delta R^{\frac{m+1}{m^2}}}{(m-1)A_2^{m-1}}\right]$$

$$\geq A_2^m\left[\left(\frac{1}{2}\left(N-1+\frac{1}{m}\right)R^{-\frac{m-1}{m}} - A_2^{lq-m}\left(\frac{l}{m}\right)^q R^{\frac{(m+1)(l-m)q+mq}{m^2}}\right)\right.$$

$$\left. + \left(\frac{1}{2}\left(N-1+\frac{1}{m}\right)R^{-\frac{m-1}{m}} - \frac{\delta R^{\frac{m+1}{m^2}}}{(m-1)A_2^{m-1}}\right)\right]$$

$$\geq 0.$$

(3.6)

Therefore, the condition (3.5) guarantees that $S_2(t,x)$ is a supersolution to (1.1) in $(0,\infty) \times \Omega$. In addition, since $r = |x - x_0| < R_0 < R$ for $x \in \Omega$, for $(t,x) \in (0,\infty) \times \partial\Omega$, we deduce from (3.5) that

$$u(t,x) = 0 \leq A_2(1+t)^{-\frac{1}{m-1}}\sigma(R_0) \leq S_2(t,x),$$

and

$$u_0(x) \leq \|u_0(x)\|_\infty \leq A_2\sigma(R_0) \leq S_2(0,x), \quad x \in \overline{\Omega}.$$

By the comparison principle, we have

$$u(t,x) \leq S_2(t,x), \quad (t,x) \in [0,\infty) \times \overline{\Omega}.$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3 (Hölder estimate) *Assume that $l \geq m > 1$, $1 \leq q < 2$, $lq \in [m, m+1]$, and the initial data $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies (1.2). Then there exists $L_1 > 0$ depending only on m,*

l, q, Ω, *and* $\|u_0\|_{W^{1,\infty}(\Omega)}$ *such that*

$$\left|u(t,x)\right| = \left|u(t,x) - u(t,x_0)\right| \leq \frac{L_1}{(1+t)^{\frac{1}{m-1}}} |x - x_0|^{\frac{1}{m}},$$

$$(t,x,x_0) \in [0,\infty) \times \overline{\Omega} \times \partial\Omega. \tag{3.7}$$

Proof Since the boundary $\partial\Omega$ of Ω is smooth, there exists $R_\Omega > 0$ such that for each $x_0 \in \partial\Omega$, there exists $y_0 \in \mathbb{R}^N$ satisfying $|x_0 - y_0| = R_\Omega$ and $B(y_0, R_\Omega) \cap \Omega = \emptyset$. It follows from the initial data condition $u_0(x) \in W^{1,\infty}(\Omega)$ that $u_0(x)$ is Lipschitz continuous, *i.e.*, there exists $L_0 > 0$ such that

$$\left|u_0(x) - u_0(y)\right| \leq L_0|x - y|, \quad (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{3.8}$$

Next, we define the open subset U_{κ,x_0} of $\mathbb{R}^N$ by

$$U_{\kappa,x_0} = \left\{x \in \Omega : R_\Omega < |x - y_0| < R_\Omega + \kappa\right\},$$

where $0 < \kappa < 1$ satisfies $\Omega_\kappa = \{x \in \Omega : d(x, \partial\Omega) > \kappa\} \neq \emptyset$, and denote the function

$$S_{\kappa,x_0}(t,x) = A_3(1+t)^{-\frac{1}{m-1}}\psi(r), \qquad \psi(r) = \left(1 - e^{-\frac{r}{\kappa}}\right)^{\frac{1}{m}} \quad \text{with } r = |x - y_0| - R_\Omega,$$

for $(t,x) \in (0,\infty) \times \overline{U}_{\kappa,x_0}$ and $A_3 > 0$.

Moreover, we assume that

$$A_3 \geq \max\left\{eL_0, \frac{e^{\frac{1}{m}}C}{(e-1)^{\frac{1}{m}}}, \left(\frac{3e}{m-1}\right)^{\frac{1}{m-1}}\right\} \tag{3.9}$$

and

$$0 < \kappa < \min\left\{1, \frac{R_\Omega}{3(N-1)}, \left(\frac{m^q}{3l^q A^{lq-m}}\right)^{\frac{1}{2-q}}\right\}, \tag{3.10}$$

where $C = \max\{C_1, C_2\} > 0$, C_1, and C_2 are defined in Lemmas 3.1 and 3.2, respectively.

Since $y_0 \notin \overline{U}_{\kappa,x_0}$, the function $S_{\kappa,x_0}(t,x)$ is C^∞-smooth in $[0,\infty) \times \overline{U}_{\kappa,x_0}$. For $(t,x) \in (0,\infty) \times U_{\kappa,x_0}$, we have $r = |x - y_0| - R_\Omega \in (0, \kappa)$. By a direct computation, we infer from (3.9)-(3.10) that

$$\frac{(1+t)^{\frac{m}{m-1}}}{A_3^m}\left\{\partial_t S_{\kappa,x_0} - \Delta S_{\kappa,x_0}^m - \left|\nabla S_{\kappa,x_0}^l\right|^q\right\}(t,x)$$

$$= -\frac{1}{(m-1)A_3^{m-1}}\psi(r) - \left[\frac{N-1}{r+R_\Omega}\left(\psi^m\right)_r + \left(\psi^m\right)_{rr}\right] - A_3^{lq-m}(1+t)^{-\frac{lq-m}{m-1}}\left|\left(\psi^l\right)_r\right|^q$$

$$= -\frac{1}{(m-1)A_3^{m-1}}\left(1 - e^{-\frac{r}{\kappa}}\right)^{\frac{1}{m}} + \frac{1}{\kappa^2}e^{-\frac{r}{\kappa}} - \frac{N-1}{(r+R_\Omega)\kappa}e^{-\frac{r}{\kappa}}$$

$$\quad - A_3^{lq-m}(1+t)^{-\frac{lq-m}{m-1}}\left|\left(\left(\psi^m\right)^{\frac{l}{m}}\right)_r\right|^q$$

$$\geq -\frac{1}{(m-1)A_3^{m-1}} + \frac{1}{\kappa^2}e^{-\frac{r}{\kappa}} - \frac{N-1}{\kappa R_\Omega}e^{-\frac{r}{\kappa}} - A_3^{lq-m}\left|\left(\left(\psi^m\right)^{\frac{l}{m}}\right)_r\right|^q$$

$$\geq \frac{1}{\kappa^2} e^{-\frac{r}{\kappa}} \left[1 - \frac{N-1}{R_\Omega}\kappa - A_3^{lq-m}\left(\frac{l}{m}\right)^q \kappa^{2-q} - \frac{e}{(m-1)A_3^{m-1}} \right]$$

$$\geq \frac{1}{\kappa^2} e^{-\frac{r}{\kappa}} \left[\left(\frac{1}{3} - \frac{N-1}{R_\Omega}\kappa\right) + \left(\frac{1}{3} - A_3^{lq-m}\left(\frac{l}{m}\right)^q \kappa^{2-q}\right) + \left(\frac{1}{3} - \frac{e}{(m-1)A_3^{m-1}}\right) \right]$$

$$\geq 0.$$

Therefore, $S_{\kappa,x_0}(t,x)$ is a supersolution to (1.1) in $(0,\infty) \times U_{\kappa,x_0}$. In addition, it follows from (3.8)-(3.10) and the mean value theorem that

$$S_{\kappa,x_0}(0,x) = A_3\left(1 - e^{-\frac{|x-y_0|-R_\Omega}{\kappa}}\right)^{\frac{1}{m}} \geq A_3\left(1 - e^{-\frac{|x-y_0|-R_\Omega}{\kappa}}\right)$$

$$\geq A_3 \frac{|x-y_0|-R_\Omega}{e\kappa} = \frac{A_3}{e\kappa} \operatorname{dist}\left(x, \partial B(y_0, R_\Omega)\right)$$

$$\geq \frac{A_3}{e\kappa} \operatorname{dist}(x, \partial\Omega) \geq L_0 \operatorname{dist}(x, \partial\Omega) \geq u_0(x), \quad x \in \overline{U}_{\delta,x_0}.$$

Moreover, for $(t,x) \in (0,\infty) \times \partial U_{\kappa,x_0}$, we have either $x \in \partial\Omega$ or $r = |x-y_0| - R_\Omega = \delta$. If $x \in \partial\Omega$, then we have

$$S_{\kappa,x_0}(t,x) = A_3(1+t)^{-\frac{1}{m-1}}\left(1 - e^{-\frac{r}{\kappa}}\right)^{\frac{1}{m}} \geq 0 = u(x,t), \quad (t,x) \in (0,\infty) \times \partial\Omega.$$

If $r = |x-y_0| - R_\Omega = \delta$, by Lemmas 3.1-3.2 and (3.9), we have

$$S_{\kappa,x_0}(t,x) = A_3(1+t)^{-\frac{1}{m-1}}\left(1 - e^{-1}\right)^{\frac{1}{m}} \geq C(1+t)^{-\frac{1}{m-1}} \geq u(t,x).$$

By the comparison principle [6] we have

$$u(t,x) \leq S_{\kappa,x_0}(t,x), \quad (t,x) \in [0,\infty) \times \overline{U}_{\kappa,x_0}.$$

Consequently,

$$0 \leq u(t,x) - u(t,x_0) = u(t,x) \leq A_3(1+t)^{-\frac{1}{m-1}}\left(1 - e^{-\frac{|x-y_0|-R_\Omega}{\kappa}}\right)^{\frac{1}{m}},$$

for $(t,x) \in [0,\infty) \times \overline{U}_{\kappa,x_0}$.

Since $|x_0 - y_0| - R_\Omega = 0$, we have

$$0 \leq \left|u(t,x) - u(t,x_0)\right| \leq \frac{A_3}{\kappa^{\frac{1}{m}}(1+t)^{\frac{1}{m-1}}} |x-x_0|^{\frac{1}{m}}, \quad x \in \overline{U}_{\kappa,x_0}. \tag{3.11}$$

Finally, we consider $x \in \Omega$ and $x_0 \in \partial\Omega$. If $|x-x_0| \geq \frac{\kappa}{2}$, it follows from Lemmas 3.1-3.2 that

$$\left|u(t,x) - u(t,x_0)\right| = u(t,x) \leq \frac{2^{\frac{1}{m}} C}{\kappa^{\frac{1}{m}}(1+t)^{\frac{1}{m-1}}} |x-x_0|^{\frac{1}{m}},$$

where $C = \max\{C_1, C_2\} > 0$, and C_1 and C_2 are defined in Lemmas 3.1 and 3.2, respectively.

If $|x-x_0| < \frac{\kappa}{2}$, let $y_0 \in \mathbb{R}^N$ satisfy $|x_0 - y_0| = R_\Omega$ and $B(y_0, R_\Omega) \cap \Omega = \emptyset$. Since $x \in \Omega$, we have

$$|x-y_0| > R_\Omega \quad \text{and} \quad |x-y_0| \leq |x-x_0| + |x_0-y_0| < R_\Omega + \kappa,$$

which implies $x \in U_{\kappa,x_0}$. Therefore, we deduce from (3.11) that

$$\left| u(t,x) - u(t,x_0) \right| = u(t,x) \leq \frac{A_3}{\kappa^{\frac{1}{m}}(1+t)^{\frac{1}{m-1}}} |x - x_0|^{\frac{1}{m}}.$$

Choosing $L_1 = \max\{\frac{2^{\frac{1}{m}}C}{\kappa^{\frac{1}{m}}}, \frac{A_3}{\kappa^{\frac{1}{m}}}\}$, we have

$$\left| u(t,x) \right| = \left| u(t,x) - u(t,x_0) \right| \leq \frac{L_1}{(1+t)^{\frac{1}{m-1}}} |x - x_0|^{\frac{1}{m}}, \quad (t,x,x_0) \in [0,\infty) \times \overline{\Omega} \times \partial\Omega.$$

The proof of Lemma 3.3 is complete. □

We next proceed as in [29] to deduce the Hölder continuity of $u(x,t)$ from Lemma 3.3. Therefore, we obtain the following corollary.

Corollary 3.1 *Assume that $l \geq m > 1$, $1 \leq q < 2$, $lq \in [m, m + 1]$, and the initial data $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies (1.2). Then there exists $L_2 > 0$ depending only on m, l, q, Ω, and $\|u_0\|_{W^{1,\infty}(\Omega)}$ such that*

$$\left| u(t,x) - u(t,y) \right| \leq \frac{L_2}{(1+t)^{\frac{1}{m-1}}} |x - y|^{\frac{1}{m}}, \quad (t,x,y) \in [0,\infty) \times \overline{\Omega} \times \overline{\Omega}. \tag{3.12}$$

Proof The proof is similar to the argument in [25, 29], so we omit here. □

Proofs of Theorems 1.1 *and* 1.2 The proofs are based on the ideas in [25], but we give the details of the argument for the reader's convenience. Let $U(x,t)$ be the solution to the porous medium equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} \partial_t U - \Delta U^m = 0, & (t,x) \in (0,\infty) \times \Omega, \\ U(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ U(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{3.13}$$

According to the nonnegativity of $|\nabla u^l|^q$, it follows from the comparison principle [6] that

$$0 \leq U(t,x) \leq u(t,x), \quad (t,x) \in (0,\infty) \times \overline{\Omega}. \tag{3.14}$$

We introduce the scaling variable $s = \ln t$ for $t > 0$ and denote the new unknown function v and V by

$$u(t,x) = t^{-\frac{1}{m-1}} v(\ln t, x), \quad (t,x) \in (0,\infty) \times \overline{\Omega}$$

and

$$U(t,x) = t^{-\frac{1}{m-1}} V(\ln t, x), \quad (t,x) \in (0,\infty) \times \overline{\Omega}.$$

Then v is a viscosity solution to the following problem:

$$\begin{cases} \partial_s v - \Delta v^m - e^{-\frac{(lq-m)s}{m-1}}|\nabla v^l|^q - \frac{v}{m-1} = 0, & (s,x) \in (0,\infty) \times \Omega, \\ v(s,x) = 0, & (s,x) \in (0,\infty) \times \partial\Omega, \\ v(0,x) = u(1,x), & x \in \Omega. \end{cases} \tag{3.15}$$

In addition, owing to Lemmas 3.1-3.3 and (3.14), we have

$$V(s,x) \le v(s,x) \le C, \quad (s,x) \in [0,\infty) \times \overline{\Omega}, \tag{3.16}$$

and

$$\left| v(s,x) - v(s,y) \right| \le L_1 |x-y|^{\frac{1}{m}}, \quad (s,x,y) \in [0,\infty) \times \overline{\Omega} \times \partial\Omega. \tag{3.17}$$

Next, for $\varepsilon \in (0,1)$, we define

$$w_\varepsilon(s,x) = v\left(\frac{s}{\varepsilon}, x\right), \quad (s,x) \in [0,\infty) \times \overline{\Omega},$$

and the half-relaxed limits

$$w_*(x) = \lim_{(\sigma,y,\varepsilon)\to(s,x,0)} \inf w_\varepsilon(\sigma,y) \quad \text{and} \quad w^*(x) = \lim_{(\sigma,y,\varepsilon)\to(s,x,0)} \sup w_\varepsilon(\sigma,y),$$

for $(s,x) \in (0,\infty) \times \overline{\Omega}$.

By (3.16), it is easy to see that $w_*(x)$ and $w^*(x)$ are well-defined and do not depend on $s > 0$. Moreover, it readily follows from (3.15) and (3.17) that

$$w_*(x) = w^*(x) = 0, \quad x \in \partial\Omega. \tag{3.18}$$

By a direct computation, $w_\varepsilon(s,x)$ is a solution to the following initial-boundary problem:

$$\begin{cases} \varepsilon \partial_s w_\varepsilon - \Delta w_\varepsilon^m - e^{-\frac{(lq-m)s}{(m-1)\varepsilon}}|\nabla w_\varepsilon^l|^q - \frac{w_\varepsilon}{m-1} = 0, & (s,x) \in (0,\infty) \times \Omega, \\ w_\varepsilon(s,x) = 0, & (s,x) \in (0,\infty) \times \partial\Omega, \\ w_\varepsilon(0,x) = u(1,x), & x \in \Omega. \end{cases} \tag{3.19}$$

Next, we shall give proofs of Theorems 1.1 and 1.2. To do this, we distinguish the two cases $lq = m$ and $lq \in (m, m+1]$.

Case 1. $lq = m$. We use the stability of semicontinuous viscosity solutions [6] to deduce from (3.19) that

$$w_*(x) \text{ is a supersolution to (2.1) in } \Omega \tag{3.20}$$

and

$$w^*(x) \text{ is a subsolution to (2.1) in } \Omega. \tag{3.21}$$

In addition, by [30], $V(s) \to f_0$ in $L^\infty(\Omega)$ as $s \to \infty$. Moreover, it follows from (3.16) and the definition of $w_*(x)$ and $w^*(x)$ that

$$f_0(x) \leq w_*(x) \leq w^*(x) \leq C, \quad x \in \overline{\Omega}. \tag{3.22}$$

Since $f_0 > 0$ in Ω by [30], we deduce from (3.22) that $w_*(x)$ and $w^*(x)$ are positive and bounded in Ω and vanish on $\partial\Omega$ by (3.18). Owing to (3.20) and (3.21), we infer from Lemma 2.1 that

$$w^*(x) \leq w_*(x), \quad x \in \overline{\Omega}.$$

By (3.22), we have

$$w^*(x) = w_*(x), \quad x \in \overline{\Omega}.$$

Setting $f(x) = w_*(x) = w^*(x)$, we deduce from (3.18), (3.20), (3.21), and (3.22) that $f(x) \in C_0(\overline{\Omega})$ is a positive viscosity solution to (2.1) and solves (1.9). Therefore, the existence of a positive solution to (1.9) is proved. Moreover, by Corollary 2.1, we obtain the uniqueness of solution to (1.9).

Furthermore, it follows from the equality $w_*(x) = w^*(x)$ that

$$\left\| w_\varepsilon(1) - f \right\|_\infty \to 0 \quad \text{as } \varepsilon \to 0,$$

i.e.,

$$\lim_{s \to \infty} \left\| v(s) - f \right\|_\infty = 0. \tag{3.23}$$

Therefore, we infer from the scaling transformation that

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - f \right\|_\infty = 0. \tag{3.24}$$

Finally, Corollary 3.1 gives the last statement of Theorem 1.1. The proof of Theorem 1.1 is complete.

Case 2. $lq \in (m, m+1]$. We use once more the stability of semicontinuous viscosity solutions [6] to deduce from (3.19) that

$$w_*(x) \text{ is a supersolution to (2.10) in } \Omega \tag{3.25}$$

and

$$w^*(x) \text{ is a subsolution to (2.10) in } \Omega. \tag{3.26}$$

In addition, by [30], $V(s) \to f_0$ in $L^\infty(\Omega)$ as $s \to \infty$. Moreover, it follows from (3.16) and the definition of $w_*(x)$ and $w^*(x)$ that

$$f_0(x) \leq w_*(x) \leq w^*(x) \leq C, \quad x \in \overline{\Omega}. \tag{3.27}$$

Since $f_0 > 0$ in Ω by [30] and is a solution to (2.10), we can apply Lemma 2.2 to conclude that $w_*(x) \leq f_0$ in $\overline{\Omega}$. Owing to (3.27), we have

$$w_*(x) = w^*(x) = f_0(x), \quad x \in \overline{\Omega}.$$

Therefore, we deduce from (3.19) that

$$\left\| w_\varepsilon(1) - f_0 \right\|_\infty \to 0 \quad \text{as } \varepsilon \to 0,$$

i.e.,

$$\lim_{s \to \infty} \left\| v(s) - f_0 \right\|_\infty = 0. \tag{3.28}$$

Thus, we infer from the scaling transformation that

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - f_0 \right\|_\infty = 0. \tag{3.29}$$

The proof of Theorem 1.2 is complete. □

4 Proof of Theorem 1.3

In this section, we shall consider the well-posedness and the large time behavior of solutions to (1.1) with the small initial data $u_0(x)$ for $lq > m + 1$ by the method used in [25]. To do this, we need the following lemma.

Lemma 4.1 *Let $l \geq m > 1$ and $1 \leq q < 2$. Assume that G is the corresponding solution to* (1.1) *with the initial data $G_0 \in W^{1,\infty}(\Omega)$ satisfying* (1.2) *for the case $lq = m$, and denote*

$$\mathcal{F}(t,x) = \frac{1}{\ell[G_0]} G\left(\frac{t}{\ell[G_0]^{m-1}}, x \right), \quad (t,x) \in [0,\infty) \times \overline{\Omega}, \tag{4.1}$$

where the parameter $\ell[G_0]$ is defined in (1.10). *Then $\mathcal{F}(t,x)$ is a solution to* (1.1) *with the initial data $G_0/\ell[G_0]$ and $lq = m$ such that $|\nabla \mathcal{F}^l| \leq 1$ for $(t,x) \in [0,\infty) \times \overline{\Omega}$. Moreover, $\mathcal{F}(t,x)$ is a supersolution to* (1.1) *for $lq > m$.*

Proof According to the definition of $\ell[G_0]$ in (1.10), it is easy to see that $|\nabla \mathcal{F}^l| \leq 1$ for $(t,x) \in [0,\infty) \times \overline{\Omega}$.

Next, let $\varphi(t,x) \in C^2((0,\infty) \times \Omega)$ and $\mathcal{F} - \varphi$ has a local minimum at $(t_0,x_0) \in (0,\infty) \times \Omega$. Since $\mathcal{F}$ is smooth with respect to the time variable and Hölder continuous with respect to the space variable, we obtain

$$\left| \nabla \varphi^l(t_0,x_0) \right| \leq 1. \tag{4.2}$$

Moreover, introducing $\psi(t,x) := \ell[G_0]\varphi(\ell[G_0]^{m-1}t,x)$ for $(t,x) \in [0,\infty) \times \overline{\Omega}$, the function $G - \psi$ has a local minimum at $(t_0\ell[G_0]^{1-m}, x_0)$ such that

$$\partial_t \psi \left(t_0\ell[G_0]^{1-m}, x_0 \right) - \Delta \psi^m \left(t_0\ell[G_0]^{1-m}, x_0 \right) - \left| \nabla \psi^l \left(t_0\ell[G_0]^{1-m}, x_0 \right) \right|^{\frac{m}{l}} \geq 0,$$

i.e.,

$$\partial_t \varphi(t_0, x_0) - \Delta \varphi^m(t_0, x_0) - \left| \nabla \varphi^l(t_0, x_0) \right|^{\frac{m}{l}} \geq 0. \tag{4.3}$$

Therefore, $\mathcal{F}(t, x)$ is a supersolution to (1.1) with $lq = m$. In a similar way, it can be shown that $\mathcal{F}(t, x)$ is also a subsolution. Hence, $\mathcal{F}(t, x)$ is a solution to (1.1) with $lq = m$.

Furthermore, we deduce from (4.2), (4.3), and $lq > m$ that

$$\partial_t \varphi(t_0, x_0) - \Delta \varphi^m(t_0, x_0) - \left| \nabla \varphi^l(t_0, x_0) \right|^q$$
$$\geq \left| \nabla \varphi^l(t_0, x_0) \right|^{\frac{m}{l}} \left(1 - \left| \nabla \varphi^l(t_0, x_0) \right|^{q - \frac{m}{l}} \right) \geq 0. \tag{4.4}$$

The proof of Lemma 4.1 is complete. $\qquad\square$

Proposition 4.1 *Let $l \geq m > 1$, $1 \leq q < 2$, and $lq > m + 1$. Assume that the initial data $u_0(x) \in W^{1,\infty}(\Omega)$ satisfies (1.2), moreover, there exists $G_0 \in W^{1,\infty}(\Omega)$ satisfying (1.2) such that*

$$u_0(x) \leq \frac{G_0(x)}{\ell[G_0]}, \quad x \in \overline{\Omega}, \tag{4.5}$$

where the parameter $\ell[G_0]$ is defined in (1.10). Then there exists a unique solution u to (1.1) in the sense of Definition 1.1 and it satisfies

$$u(t, x) \leq \mathcal{F}(t, x), \quad (t, x) \in [0, \infty) \times \overline{\Omega}, \tag{4.6}$$

where $\mathcal{F}(t, x)$ is defined in (4.1).

Proof On the one hand, the solution U to the porous medium equation (3.13) is clearly a subsolution to (1.1) in $(0, \infty) \times \Omega$.

On the other hand, it follows from Lemma 4.1 that the function $\mathcal{F}(t, x)$ is a supersolution to (1.1) in $(0, \infty) \times \Omega$. Therefore, $\mathcal{F}(t, x)$ is a supersolution to (3.13).

Since $U = \mathcal{F} = 0$ on $(0, \infty) \times \partial\Omega$ and $U(0, x) = u_0(x) \leq \mathcal{F}(0, x)$ for $x \in \overline{\Omega}$ by (4.5), we infer from the comparison principle [6] that

$$U(t, x) \leq \mathcal{F}(t, x), \quad (t, x) \in [0, \infty) \times \overline{\Omega}.$$

This property and the simultaneous vanishing of U and $\mathcal{F}$ on $(0, \infty) \times \partial\Omega$ allow us to use the classical Perron method to establish the existence of a solution $u(x, t)$ to (1.1) in the sense of Definition 1.1 which satisfies (4.6). The uniqueness next follows from the comparison principle [6]. The proof of Proposition 4.1 is complete. $\qquad\square$

Proof of Theorem 1.3 We notice that Lemma 3.2 is also valid in that case. It readily follows from Lemma 3.3 and Proposition 4.1 that

$$0 \leq u(t, x) = u(t, x) - u(t, x_0) \leq \mathcal{F}(t, x) \leq \frac{L_1}{(1 + t)^{\frac{1}{m-1}}} |x - x_0|^{\frac{1}{m}},$$

$$(t, x, x_0) \in [0, \infty) \times \overline{\Omega} \times \partial\Omega.$$

The convergence proof is similar to that performed in the proof of Theorem 1.2 for $lq \in (m, m + 1]$. The proof of Theorem 1.3 is complete. □

5 Proof of Theorem 1.4

In this section, when $lq > m + 1$, we shall prove that the solution $u(x, t)$ of (1.1) blows up in finite time for the large initial data $u_0(x)$ in the sense of weak solution by the method used in [26].

In order to obtain a blow-up condition corresponding to (1.1), we have to modify the function $e^{-\varepsilon|x|^2}$ used in [31–33], and introduce a test function $\phi_\varepsilon(x)$ as follows:

$$\phi_\varepsilon(x) = A_\varepsilon e^{-\varepsilon|x|} \quad \text{with } A_\varepsilon = \frac{1}{\int_\Omega e^{-\varepsilon|x|} \, dx}.$$

Proof of Theorem 1.4 Suppose that $u(x, t)$ is the solution of the problem (1.1) and T is the blow-up time of the solution. For $s \in (0, \frac{1}{2})$, we denote

$$W(t) = \frac{1}{s} \int_\Omega u^s \phi_\varepsilon(x) \, dx, \quad t \in (0, T). \tag{5.1}$$

By a direct calculation, we have

$$\begin{aligned}
W'(t) &= \int_\Omega u^{s-1} \phi_\varepsilon u_t \, dx \\
&= \int_\Omega u^{s-1} \phi_\varepsilon \Delta u^m \, dx + \int_\Omega u^{s-1} \phi_\varepsilon |\nabla u^l|^q \, dx \\
&\geq -(s-1) \int_\Omega u^{s-2} \phi_\varepsilon \nabla u^m \cdot \nabla u \, dx + \varepsilon \int_\Omega u^{s-1} \phi_\varepsilon \nabla u^m \cdot \frac{x}{|x|} \, dx \\
&\quad + \int_\Omega u^{s-1} \phi_\varepsilon |\nabla u^l|^q \, dx \\
&\geq m(1-s) \int_\Omega u^{m+s-3} \phi_\varepsilon |\nabla u|^2 \, dx - m\varepsilon \int_\Omega u^{m+s-2} \phi_\varepsilon |\nabla u| \, dx \\
&\quad + \int_\Omega u^{s-1} \phi_\varepsilon |\nabla u^l|^q \, dx.
\end{aligned} \tag{5.2}$$

By Young's inequality, we obtain

$$\varepsilon \int_\Omega u^{m+s-2} \phi_\varepsilon |\nabla u| \, dx \leq \frac{1}{2} \int_\Omega u^{m+s-3} \phi_\varepsilon |\nabla u|^2 \, dx + \frac{\varepsilon^2}{2} \int_\Omega u^{m+s-1} \phi_\varepsilon \, dx. \tag{5.3}$$

Since $0 < s < \frac{1}{2}$, it follows from (5.2), (5.3), and Poincaré's inequality that

$$\begin{aligned}
W'(t) &\geq \int_\Omega u^{s-1} \phi_\varepsilon |\nabla u^l|^q \, dx - \frac{m\varepsilon^2}{2} \int_\Omega u^{m+s-1} \phi_\varepsilon \, dx \\
&= \left(\frac{ql}{ql+s-1} \right)^q \int_\Omega \phi_\varepsilon |\nabla u^{\frac{ql+s-1}{q}}|^q \, dx - \frac{m\varepsilon^2}{2} \int_\Omega u^{m+s-1} \phi_\varepsilon \, dx \\
&\geq \left(\frac{ql}{ql+s-1} \right)^q \int_\Omega \phi_\varepsilon u^{ql+s-1} \, dx - \frac{m\varepsilon^2}{2} \int_\Omega u^{m+s-1} \phi_\varepsilon \, dx.
\end{aligned} \tag{5.4}$$

According to $ql > m + 1$, $0 < s < \frac{1}{2}$, $\int_\Omega \phi_\varepsilon(x)\,dx = 1$ and Hölder's inequality, we have

$$\int_\Omega u^{m+s-1}\phi_\varepsilon\,dx = \int_\Omega u^{m+s-1}\phi_\varepsilon^{\frac{m+s-1}{ql+s-1}}\phi_\varepsilon^{\frac{ql-m}{ql+s-1}}\,dx \leq \left(\int_\Omega u^{ql+s-1}\phi_\varepsilon\,dx\right)^{\frac{m+s-1}{ql+s-1}}. \tag{5.5}$$

Thus, by (5.4) and (5.5), we obtain

$$\frac{dW}{dt} \geq \left(\int_\Omega u^{ql+s-1}\phi_\varepsilon\,dx\right)^{\frac{m+s-1}{ql+s-1}}\left[\left(\frac{ql}{ql+s-1}\right)^q\left(\int_\Omega u^{ql+s-1}\phi_\varepsilon\,dx\right)^{\frac{ql-m}{ql+s-1}} - \frac{m\varepsilon^2}{2}\right]. \tag{5.6}$$

Owing to $ql > m + 1$, $0 < s < \frac{1}{2}$, $\int_\Omega \phi_\varepsilon(x)\,dx = 1$, and Jensen's inequality, we have

$$\left(\int_\Omega u^{ql+s-1}\phi_\varepsilon\,dx\right)^{\frac{ql-m}{ql+s-1}} \geq \left(\int_\Omega u^s\phi_\varepsilon\,dx\right)^{\frac{ql-m}{s}}. \tag{5.7}$$

Therefore, it follows from (5.6) and (5.7) that

$$\frac{dW}{dt} \geq \frac{1}{2}\left(\frac{ql}{ql+s-1}\right)^q\left(\int_\Omega u^s\phi_\varepsilon\,dx\right)^{\frac{ql+s-1}{s}} = \frac{1}{2}\left(\frac{ql}{ql+s-1}\right)^q s^{\frac{ql+s-1}{s}} W^{\frac{ql+s-1}{s}}(t) \tag{5.8}$$

as long as

$$W(t) \geq \frac{1}{s}\left(\frac{ql+s-1}{ql}\right)^{\frac{qs}{ql-m}}\left(m\varepsilon^2\right)^{\frac{s}{ql-m}} \quad \text{for all } t \in [0, T). \tag{5.9}$$

Taking

$$K_0 = \left(\frac{ql+s-1}{ql}\right)^{\frac{qs}{ql-m}}\left(m\varepsilon^2\right)^{\frac{s}{ql-m}}.$$

Since the initial data $u_0(x)$ satisfies

$$\int_\Omega u_0^s(x)\phi_\varepsilon(x)\,dx \geq K_0,$$

we have

$$W(0) \geq \frac{1}{s}K_0 = \frac{1}{s}\left(\frac{ql+s-1}{ql}\right)^{\frac{qs}{ql-m}}\left(m\varepsilon^2\right)^{\frac{s}{ql-m}}. \tag{5.10}$$

Therefore, $w(t)$ increases and remains above $\frac{1}{s}K_0$ for all $t \in [0, T]$.

By (5.8), integrating over $(0, t)$ yields

$$W(t) \geq \left(W^{\frac{1-ql}{s}}(0) - K_1 t\right)^{-\frac{s}{ql-1}} \quad \text{with } K_1 = \frac{ql-1}{2}\left(\frac{ql}{ql+s-1}\right)^q s^{\frac{ql-1}{s}}. \tag{5.11}$$

Hence, it follows from (5.10) and (5.11) that the solution $u(x, t)$ of (1.1) blows up in finite time, $T = \frac{1}{K_1}W^{\frac{1-ql}{s}}(0)$.

Moreover, by (5.10), we obtain the upper estimate on the blow-up time T of the solution $u(x, t)$ as follows:

$$T \leq \frac{2}{ql-1} \left(\frac{ql}{ql+s-1} \right)^{\frac{q(m-1)}{ql-m}} \left(m\varepsilon^2 \right)^{\frac{1-ql}{ql-m}}. \tag{5.12}$$

The proof of Theorem 1.4 is complete. $\qquad\square$

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