Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Profiles of blow-up solution of a weighted diffusion system

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Abstract
In this paper, we study the blow-up profiles for a coupled diffusion system with a weighted source term involved in a product with local term. We prove that the solutions have a global blow-up and the profile of the blow-up is precisely determined in all compact subsets of the domain.

Keywords: diffusion system; weighted localized source; blow-up profile

1 Introduction

In this paper, we consider the following coupled diffusion system with a weighted nonlinear localized sources:

$$\begin{cases} u_t - \Delta u = a(x)u^p(x,t)v^\alpha(0,t), & x \in B, 0 < t < T^*, \\ v_t - \Delta v = b(x)u^\beta(0,t)v^q(x,t), & x \in B, 0 < t < T^*, \\ u(x,t) = v(x,t) = 0, & x \in \partial B, t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in B, \end{cases} \tag{1.1}$$

where B is an open ball of R^N, $N \geq 2$ with radius R; α, β, p, q are nonnegative constants and satisfy $\alpha + p > 0$ and $\beta + q > 0$.

System (1.2) is usually used as a model to describe heat propagation in a two-component combustible mixture [1]. In this case u and v represent the temperatures of the interacting components, thermal conductivity is supposed constant and equal for both substances, a volume energy release given by some powers of u and v is assumed.

The problem with a nonlinear reaction in a dynamical system taking place only at a single site, of the form

$$\begin{cases} u_t - \Delta u = u^p(0,t)v^\alpha(0,t), & x \in \Omega, 0 < t < T^*, \\ v_t - \Delta v = u^\beta(0,t)v^q(0,t), & x \in \Omega, 0 < t < T^*, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

was studied by Pao and Zheng [2] and they obtained the blow-up rates and boundary layer profiles of the solutions.

As for problem (1.2), it is well known that problem (1.2) has a classical, maximal in time solution and that the comparison principle is true (using the methods of [3]). A number of papers have studied problem (1.2) from the point of view of blow-up and global existence (see [4, 5]).

In [6], Chen studied the following problem:

$$\begin{cases} u_t - \Delta u = u^p v^\alpha, & x \in \Omega, 0 < t < T^*, \\ v_t - \Delta v = u^\beta v^q, & x \in \Omega, 0 < t < T^*, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

assuming $p > 1$, or $q > 1$, or $\alpha\beta > (1-p)(1-q)$, he proved that the solution blows up in finite time if the initial data $u_0(x)$ and $v_0(x)$ are large enough.

In the case of $a(x) = b(x) = 1$, Li and Wang [7] discussed the blow-up properties for this system, and they proved that:

(i) If $m, q \le 1$, this system possesses uniform blow-up profiles.
(ii) If $m, q > 1$, this system presents single point blow-up patterns.

Recently, Zhang and Yang [8] studied the problem of (1.1), but they only obtained the estimation of the blow-up rate, which is not precisely determined. In [9], the authors proved there are initial data such that simultaneous and non-simultaneous blow-up occur for a diffusion system with weighted localized sources, but they did not study the profile of the blow-up solution. There are many known results concerning blow-up properties for parabolic system equations, of which the reaction terms are of a nonlinear localized type. For more details as regards a parabolic system with localized sources, see [10–14].

Our present work is partially motivated by [15–18]. The purpose of this paper is to determine the blow-up rate of solutions for a nonlinear parabolic equation system with a weighted localized source. That is, we prove that the solutions u and v blow up simultaneously and that the blow-up rate is uniform in all compact subsets of the domain. Moreover, the blow-up profiles of the solutions are precisely determined.

In the following section, we will build the profile of the blow-up solution of (1.1).

2 Blow-up profile

Throughout this paper, we assume that the functions $a(x)$, $b(x)$, $u_0(x)$ and $v_0(x)$ satisfy the following three conditions:

(A1) $a(x), b(x), u_0(x), v_0(x) \in C^2(B)$; $a(x), b(x), u_0(x), v_0(x) > 0$ in B and $a(x) = b(x) = u_0(x) = v_0(x) = 0$ on ∂B.
(A2) $a(x)$, $b(x)$, $u_0(x)$ and $v_0(x)$ are radially symmetric; $a(r)$, $b(r)$, $u_0(r)$ and $v_0(r)$ are non-increasing for $r \in (0, R]$ ($r = |x|$).
(A3) $u_0(x)$ and $v_0(x)$ satisfy $\Delta u_0(x) + a(x)u_0^p(x)v^\alpha(0,t) \ge 0$ and $\Delta v_0(x) + b(x)u_0^\beta(x)v^q(0,t) \ge 0$ in B, respectively.

Theorem 2.1 *Assume* (A1), (A2), *and* (A3) *hold. Let* (u, v) *be the blow-up solution of* (1.1), *non-decreasing in time, and let the following limits hold uniformly in all compact subsets of* B:

(i) *If* $p < 1$, $q < 1$ *and* $\alpha\beta > (1-p)(1-q)$, *then*

$$\lim_{t \to T^*} u(x,t)\bigl(T^* - t\bigr)^\theta = a(x)^{1/(1-p)} C_2 \theta^\theta (\sigma/\theta)^{\beta/\alpha\beta-(1-p)(1-q)},$$

$$\lim_{t \to T^*} v(x,t)\bigl(T^* - t\bigr)^\sigma = b(x)^{1/(1-p)} C_1 \sigma^\sigma (\theta/\sigma)^{\alpha/\alpha\beta-(1-p)(1-q)},$$

where

$$\theta = (\alpha + 1 - q)/\big(\alpha\beta - (1-p)(1-q)\big), \qquad \sigma = (\beta + 1 - p)/\big(\alpha\beta - (1-p)(1-q)\big),$$
$$C_1 = \big(a(0)b(0)\big)^{\frac{\beta}{(1-p)(1-q)-\alpha\beta}}\big(b(0)\big)^{\frac{\beta\theta}{1-q}}, \qquad C_2 = \big(a(0)b(0)\big)^{\frac{\alpha}{(1-p)(1-q)-\alpha\beta}}\big(a(0)\big)^{\frac{\alpha\theta}{1-q}}.$$

(ii) *If $p < 1$ and $q = 1$, then*

$$\lim_{t\to T^*} u(x,t)\big(T^* - t\big)^{1/\beta} = a(x)^{1/(1-p)}\big(a(0)\big)^{1/p-1}\left(\frac{\alpha b(0)}{1+\beta-p}\right)^{1/\beta}(1/\beta)^{1/\beta},$$
$$\lim_{t\to T^*} v(x,t)\big(T^* - t\big)^{\frac{(1+p-\beta)b(x)}{\alpha\beta b(0)}} = \big(a(0)\big)^{\frac{-a(x)}{\alpha b(0)}}(1/\beta)^{\frac{(1+\beta-p)b(x)}{\alpha\beta b(0)}}\left(\frac{1+\beta-p}{\alpha b(0)}\right)^{\frac{(1-p)b(x)}{\alpha\beta b(0)}}.$$

(iii) *If $p = 1$ and $q = 1$, then*

$$\lim_{t\to T^*} u(x,t)\big(T^* - t\big)^{\frac{a(x)}{\beta a(0)}} = \left(\frac{1}{\alpha b(0)}\right)^{\frac{a(x)}{\beta a(0)}},$$
$$\lim_{t\to T^*} v(x,t)\big(T^* - t\big)^{\frac{b(x)}{\alpha b(0)}} = \left(\frac{1}{\beta b(0)}\right)^{\frac{b(x)}{\alpha b(0)}}.$$

(iv) *If $p = 1$ and $q < 1$, then*

$$\lim_{t\to T^*} u(x,t)\big(T^* - t\big)^{\frac{(1+\alpha-q)b(x)}{\alpha\beta a(0)}} = \big(b(0)\big)^{\frac{-a(x)}{\beta a(0)}}(1/\alpha)^{\frac{(1+\alpha-q)b(x)}{\alpha\beta a(0)}}\left(\frac{1+\alpha-q}{\beta a(0)}\right)^{\frac{(1-q)a(x)}{\alpha\beta a(0)}},$$
$$\lim_{t\to T^*} v(x,t)\big(T^* - t\big)^{1/\alpha} = b(x)^{1/(1-q)}\big(b(0)\big)^{1/q-1}\left(\frac{\beta a(0)}{1+\alpha-q}\right)^{1/\alpha}(1/\alpha)^{1/\beta}.$$

Throughout this section, we denote

$$g_1(t) = u^\beta(0,t), \qquad G_1(t) = \int_0^t g_1(s)\,ds, \qquad g_2(t) = v^\alpha(0,t), \qquad G_2(t) = \int_0^t g_2(s)\,ds.$$

Lemma 2.1 *Assume that (u,v) is the positive solution of* (1.1), *which blow up in finite time T^*. Let $p \le 1$ and $q \le 1$, then*

$$\lim_{t\to T^*} g_1(t) = \lim_{t\to T^*} G_1(t) = \infty, \qquad \lim_{t\to T^*} g_2(t) = \lim_{t\to T^*} G_2(t) = \infty.$$

Proof First we claim that $\lim_{t\to T^*} G_2(t) = \infty$. Since $u(0,t) = \max_\Omega u(x,t)$, we have

$$u_t(0,t) \le a(0)u^p(0,t)g_2(t).$$

By integrating the above inequality over $(0,t)$, we get

$$u^{1-p}(0,t) \le (1-p)a(0)\int_0^t g_2(s)\,ds + u_0^{1-p}(0), \quad \text{if } p < 1,$$
$$\ln u(0,t) \le a(0)G_2(t) + \ln u_0(0), \quad \text{if } p = 1.$$

From $\lim_{t\to T^*} u(0,t) = \infty$, it follows that $\lim_{t\to T^*} G_2(t) = \infty$. Applying similar arguments as above to the equation of v in system (1.2), it is reasonable that $\lim_{t\to T^*} g_1(t) = \lim_{t\to T^*} G_1(t) = \infty$. □

The following lemma will play a key role in proving Theorem 2.1, which will give the relationships among u, v, $G_1(t)$, and $G_2(t)$.

Lemma 2.2 *Under the conditions of Theorem* 2.1, *the following statements hold uniformly in any compact subsets of* B:

(i) $p < 1$ *and* $q < 1$, *then*

$$\lim_{t\to T^*} \frac{u^{1-p}(x,t)}{G_2(t)} = (1-p)a(x), \qquad \lim_{t\to T^*} \frac{v^{1-q}(x,t)}{G_1(t)} = (1-q)b(x).$$

(ii) $p = 1$ *and* $q < 1$, *then*

$$\lim_{t\to T^*} \frac{\ln u(x,t)}{G_2(t)} = a(x), \qquad \lim_{t\to T^*} \frac{v^{1-q}(x,t)}{G_1(t)} = (1-q)b(x).$$

(iii) $p = 1$ *and* $q = 1$, *then*

$$\lim_{t\to T^*} \frac{\ln u(x,t)}{G_2(t)} = a(x), \qquad \lim_{t\to T^*} \frac{\ln v(x,t)}{G_1(t)} = b(x).$$

(iv) $p < 1$ *and* $q = 1$, *then*

$$\lim_{t\to T^*} \frac{u^{1-p}(x,t)}{G_2(t)} = (1-p)a(x), \qquad \lim_{t\to T^*} \frac{\ln v(x,t)}{G_1(t)} = b(x).$$

Proof (i) When $p < 1$ and $q < 1$. A simple computation shows that

$$\frac{du^{1-p}}{dt} = \Delta u^{1-p} + p(1-p)u^{-1-p}|\nabla u|^2 + (1-p)a(x)g_2(t), \quad x \in \Omega, 0 < t < T^*, \tag{2.1}$$

$$\frac{dv^{1-p}}{dt} = \Delta v^{1-q} + q(1-q)v^{-1-q}|\nabla u|^2 + (1-q)b(x)g_1(t), \quad x \in \Omega, 0 < t < T^*, \tag{2.2}$$

and the initial and boundary conditions are given by

$$\begin{cases} u^{1-p}(x,t) = v^{1-q}(x,t) = 0, \quad x \in \partial B, t > 0, \\ u^{1-p}(x,0) = u_0^{1-p}(x), \qquad v^{1-q}(x,0) = v_0^{1-q}(x), \quad x \in B. \end{cases}$$

Denote λ_2, the first eigenvalue of $-\Delta$ in $H_0^1(B)$ and by $\varphi(x) > 0$ and $\phi(x) > 0$ the corresponding eigenfunction, normalized by $\int_B a(x)\varphi(x)\,dx = 1$ and $\int_B b(x)\phi(x)\,dx = 1$.

Multiplying both sides of (2.1) and (2.2) by φ and ϕ, respectively, and integrating over $B \times (0,t)$, we have, for $0 < t < T^*$

$$\begin{aligned} \int_B u^{1-p}\varphi\,dx - \int_B u_0^{1-p}\varphi\,dx &= -\lambda_2 \int_0^t \int_B u^{1-p}\varphi\,dx\,ds \\ &\quad + \int_0^t \int_B p(1-p)u^{-p-1}|\nabla u|^2\varphi\,dx\,ds + (1-p)G_2(t), \end{aligned}$$

$$\int_B v^{1-p}\phi\,dx - \int_B v_0^{1-p}\phi\,dx = -\lambda_2\int_0^t\int_B v^{1-p}\phi\,dx\,ds + \int_0^t\int_B p(1-p)v^{-p-1}|\nabla v|^2\phi\,dx\,ds + (1-p)G_1(t).$$

We claim that $\lim_{t\to T^*} u^{1-p}(0,t)/g_2(t) = 0$ and $\lim_{t\to T^*} v^{1-q}(0,t)/g_1(t) = 0$. In fact, we have $u_t(0,t) \le u^p(0,t)v^\alpha(0,t)$, for $0 < t < T^*$ that is,

$$\lim_{t\to T^*}\sup\frac{u^{1-p}(0,t)}{G_2(t)} \le (1-p)a(0). \tag{2.3}$$

Since $g_2(t)$ is non-decreasing, it follows that for all $\varepsilon > 0$,

$$0 \le \frac{G_2(t)}{g_2(t)} \le \frac{\int_0^{T^*-\varepsilon} g_2(s)\,ds}{g_2(t)} + \varepsilon,$$

and using $\lim_{t\to T^*} g_2(t) = \infty$, we deduce that $\lim_{t\to T^*} G_2(t)/g_2(t) = 0$, so that (2.3) implies $\lim_{t\to T^*} u^{1-p}(0,t)/g_2(t) = 0$. By a process analogous to above, we arrive at $\lim_{t\to T^*} v^{1-p}(0,t)/g_1(t) = 0$.

Analogous to the proof of Theorem 2.2 in Ref. [18], it can be inferred that

$$\lim_{t\to T^*}\frac{\int_\Omega u^{1-p}\varphi\,dx}{G_2(t)} = (1-p), \qquad \lim_{t\to T^*}\frac{\int_\Omega v^{1-q}\varphi\,dx}{G_1(t)} = (1-q). \tag{2.4}$$

From (2.1) and (2.2), we know (u^{1-p}, v^{1-q}) is a sub-solution of the following problem:

$$\begin{cases} \frac{dw}{dt} = \Delta w + (1-p)a(x)g_2(t), & x\in B, 0<t<T^*,\\ \frac{dz}{dt} = \Delta z + (1-q)b(x)g_1(t), & x\in B, 0<t<T^*,\\ w(x,t) = z(x,t) = 0, & x\in\partial B, t>0,\\ w(x,0) = u_0^{1-p}(x), \quad z(x,0) = v_0^{1-q}(x), & x\in B, \end{cases} \tag{2.5}$$

Equation (2.5) and Lemma 2.1 assert that

$$\lim_{t\to T^*}\frac{w(x,t)}{G_2(t)} = (1-p)a(x), \qquad \lim_{t\to T^*}\frac{z(x,t)}{G_1(t)} = (1-q)b(x), \tag{2.6}$$

uniformly in all compact subsets of B.

The rest of the proof of case (i) is similar to Lemma 2.2(i). Cases (ii), (iii), and (iv) can be treated similarly. Now we prove Theorem 2.1 by using Lemma 2.2. □

Proof of Theorem 2.1 (i) If $p < 1$ and $q < 1$. By Lemma 2.2(i), we know that for choosing positive constants $\delta < 1 < \tau$, there exists $t_0 < T^*$ such that

$$\big(\delta(1-p)a(0)G_2(t)\big)^{\beta/(1-p)} \le G_1'(t) \le \big(\tau(1-p)a(0)G_2(t)\big)^{\beta/(1-p)}, \quad t\in\big[t_0, T^*\big),$$
$$\big(\delta(1-q)b(0)G_1(t)\big)^{\alpha/(1-q)} \le G_2'(t) \le \big(\tau(1-q)b(0)G_1(t)\big)^{\alpha/(1-q)}, \quad t\in\big[t_0, T^*\big).$$

Therefore,

$$\frac{(\delta(1-p)a(0)G_2(t))^{\beta/(1-p)}}{(\tau(1-q)b(0)G_1(t))^{\alpha/(1-q)}} \le \frac{dG_1(t)}{dG_2(t)} \le \frac{(\tau(1-p)a(0)G_2(t))^{\beta/(1-p)}}{(\delta(1-q)b(0)G_1(t))^{\alpha/(1-q)}}. \tag{2.7}$$

From the right-hand side of (2.7),

$$\bigl(\delta(1-q)b(0)G_1(t)\bigr)^{\alpha/(1-q)}dG_1(t) \le \bigl(\tau(1-p)a(0)G_2(t)\bigr)^{\beta/(1-p)}dG_2(t), \quad t\in\bigl[t_0,T^*\bigr).$$

Integrating the above inequality over $[0,t)$ yields

$$\begin{aligned}
&\frac{(1-q)(\delta(1-q)b(0))^{\alpha/(1-q)}}{1+\alpha-q}G_1^{(1+\alpha-q)/(1-q)}(t)\Big|_{t_0}^{t}\\
&\quad\le \frac{(1-p)(\tau(1-p)b(0))^{\beta/(1-p)}}{1+\beta-p}G_2^{(1+\beta-p)/(1-p)}(t)\Big|_{t_0}^{t}\\
&\quad\le \frac{(1-p)(\tau(1-p)b(0))^{\beta/(1-p)}}{1+\beta-p}G_2^{(1+\beta-p)/(1-p)}(t).
\end{aligned}\tag{2.8}$$

Since $\lim_{t\to T^*}G_1(t)=\infty$ and $q<1$, for any constant $0<\varepsilon<1$, there exists $\bar{t}_0: t_0\le \bar{t}_0\le T^*$ such that $G_1^{(1+\alpha-q)/(1-q)}(t_0)\le(1-\varepsilon)G_1^{(1+\alpha-q)/(1-q)}(t)$ for $t\in[\bar{t}_0,T^*)$. Hence, from (2.8) it can be deduced that for $t\in[\bar{t}_0,T^*)$,

$$\begin{aligned}
&\varepsilon\bigl(\delta b(0)\bigr)^{\alpha/(1-q)}(1+\beta-p)\bigl((1-q)G_1(t)\bigr)^{(1+\alpha-q)/(1-q)}\\
&\quad\le \bigl(\tau a(0)\bigr)^{\beta/(1-p)}(1+\partial-q)\bigl((1-p)G_2(t)\bigr)^{(1+\beta-p)/(1-p)}.
\end{aligned}\tag{2.9}$$

By an argument similar to above, there exists $\tilde{t}_0<T^*$ such that $\tilde{t}_0<t<T^*$,

$$\begin{aligned}
&\varepsilon\bigl(\delta a(0)\bigr)^{\beta/(1-p)}(1+\partial-q)\bigl((1-p)G_2(t)\bigr)^{(1+\alpha-q)/(1-q)}\\
&\quad\le \bigl(\tau b(0)\bigr)^{\alpha/(1-q)}(1+\beta-p)\bigl((1-q)G_1(t)\bigr)^{(1+\alpha-q)/(1-q)}.
\end{aligned}\tag{2.10}$$

Set $t^*=\max\{\bar{t}_0,\tilde{t}_0\}$, then (2.9) and (2.10) hold simultaneously for all $t\in[t^*,T^*)$. Next we choose $\{\delta_i\}_{i=1}^\infty$, $\{\varepsilon_i\}_{i=1}^\infty$, $\{\tau_i\}_{i=1}^\infty$, satisfying $0<\delta_i,\varepsilon_i<1$ and $\tau_i>1$ with $\delta_i,\varepsilon_i,\tau_i\to1$ as $i\to\infty$. Let $t^*<T^*$ such that (2.9) and (2.10) hold for $t_i^*\le t<T^*$. From Lemma 2.2(i), it follows that for such sequences $\{\delta_i\}_{i=1}^\infty$, and $\{\tau_i\}_{i=1}^\infty$, there exists $\{t_i\}_{i=1}^\infty: t_i<T^*$ with $t_i\to T^*$, as $i\to\infty$ such that

$$\bigl(\delta_i(1-p)a(0)G_2(t)\bigr)^{\beta/(1-p)}\le G_1'(t)\le\bigl(\tau_i(1-p)a(0)G_2(t)\bigr)^{\beta/(1-p)},\quad t\in\bigl[t_i,T^*\bigr).\tag{2.11}$$

Taking $T_i=\max\{t_i^*,t_i\}$, in terms of (2.9), (2.10), and (2.11), we deduce that for $T\le t<T^*$

$$\begin{aligned}
G_1'(t)&\ge\bigl(\delta_i(1-p)a(0)G_2(t)\bigr)^{\beta/(1-p)}\\
&\ge\bigl(\delta_i b(0)\bigr)^{\frac{\beta\theta}{\sigma(1-q)}}\left(\frac{\delta_i b(0)}{\tau_i a(0)}\right)^{\frac{\beta^2}{(1-p)(1+\beta-p)}}(\varepsilon_i\sigma/\theta)^{\frac{\beta}{1+\beta-p}}\bigl((1-q)G_1(t)\bigr)^{\frac{\beta\theta}{\sigma(1-q)}},
\end{aligned}\tag{2.12}$$

$$G_1'(t)\le\bigl(\tau_i b(0)\bigr)^{\frac{\beta\theta}{\sigma(1-q)}}\left(\frac{\tau_i b(0)}{\delta_i a(0)}\right)^{\frac{\beta^2}{(1-p)(1+\beta-p)}}(\sigma/\varepsilon_i\theta)^{\frac{\beta}{1+\beta-p}}\bigl((1-q)G_1(t)\bigr)^{\frac{\beta\theta}{\sigma(1-q)}},\tag{2.13}$$

where $C=(a(0)/b(0))^{\beta/(1-p)}$.

Since $1 - \beta\theta/(\sigma(1-q)) = -1/(\sigma(1-q)) < 0$ and $\lim_{t\to T^*} G_1(t) = \infty$, integrating (2.12) and (2.13) over (t, T^*) we have, for $T \le t < T^*$,

$$D_i^{-1}\sigma(\sigma/\theta)^{-\frac{\beta}{1+\beta-p}} \le \left(T^* - t\right)\left((1-q)G_1(t)\right)^{1/\sigma(1-q)} \le d_i^{-1}\sigma(\sigma/\theta)^{-\frac{\beta}{1+\beta-p}}, \tag{2.14}$$

where

$$d_i = \left(\frac{a(0)}{b(0)}\right)^{\beta/(1-p)} \left(\delta_i b(0)\right)^{\frac{\beta\theta}{\sigma(1-q)}} \left(\frac{\delta_i b(0)}{\tau_i a(0)}\right)^{\frac{\beta^2}{(1-p)(1+\beta-p)}} (\varepsilon_i)^{\frac{\beta}{1+\beta-p}},$$

$$D_i = \left(\frac{a(0)}{b(0)}\right)^{\beta/(1-p)} \left(\tau_i b(0)\right)^{\frac{\beta\theta}{\sigma(1-q)}} \left(\frac{\tau_i b(0)}{\delta_i a(0)}\right)^{\frac{\beta^2}{(1-p)(1+\beta-p)}} (\varepsilon_i)^{\frac{\beta}{1+\beta-p}}.$$

Clearly,

$$d_i, D_i \to \left(\frac{a(0)}{b(0)}\right)^{\beta/(1-p)} \left(b(0)\right)^{\frac{\beta\theta}{\sigma(1-q)}} \left(\frac{b(0)}{a(0)}\right)^{\frac{\beta^2}{(1-p)(1+\beta-p)}}, \quad \text{as } \varepsilon_i, \in ?, \tau_i \to 1.$$

By plugging $i \to \infty$ into (2.14) we get

$$\left((1-q)G_1(t)\right)^{1/(1-q)} \sim C_1\sigma^\sigma(\theta/\sigma)^{\beta/\alpha\beta-(1-p)(1-q)}\left(T^* - t\right)^{-\sigma}, \tag{2.15}$$

where $C_1 = (a(0))^{\frac{\beta}{(1-p)(1-q)-\alpha\beta}}(b(0))^{\frac{\beta\theta}{1-q}+\frac{\beta}{(1-p)(1-q)-\alpha\beta}}$.

Applying a similar proof to the one above, we can conclude that

$$\left((1-q)G_2(t)\right)^{1/(1-p)} \sim C_2\theta^\theta(\sigma/\theta)^{\beta/\alpha\beta-(1-p)(1-q)}\left(T^* - t\right)^{-\theta}, \tag{2.16}$$

where $C_2 = (b(0))^{\frac{\alpha}{(1-p)(1-q)-\alpha\beta}}(a(0))^{\frac{\alpha\theta}{1-q}+\frac{\alpha}{(1-p)(1-q)-\alpha\beta}}$.

According to Lemma 2.2(i), (2.15), and (2.16), it follows that uniformly in all compact subsets of B

$$\lim_{t\to T^*} u(x,t)\left(T^* - t\right)^\theta = a(x)^{1/(1-p)}C_2\theta^\theta(\sigma/\theta)^{\beta/\alpha\beta-(1-p)(1-q)},$$

$$\lim_{t\to T^*} v(x,t)\left(T^* - t\right)^\sigma = b(x)^{1/(1-p)}C_1\sigma^\sigma(\theta/\sigma)^{\alpha/\alpha\beta-(1-p)(1-q)}.$$

The arguments of cases (ii), (iii), and (iv) are very similar to the above, we omit the details. Therefore, we have completed the proof of Theorem 2.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors typed, read, and approved the final manuscript.

Acknowledgements
This work was supported by the National Natural Science Foundation of China under Grant 61374194, the China Postdoctoral Science Foundation Founded Project under Grant 2013M540405, the National Key Technologies R&D Program of China under Grant 2009BAG13A06, the National High-tech R&D Program of China (863 Program) under Grant 2008AA040202, and the Natural Science Foundation of Jiangsu Province under grant BK20140638.

Received: 8 May 2014 Accepted: 27 May 2014 Published online: 11 September 2014

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doi:10.1186/s13661-014-0143-1
Cite this article as: Zeng et al.: **Profiles of blow-up solution of a weighted diffusion system.** *Boundary Value Problems* 2014 **2014**:143.