Zhou et al. *Boundary Value Problems* 2014, **2014**:210
http://www.boundaryvalueproblems.com/content/2014/1/210

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Multiplicity of solutions for Kirchhoff-type problems involving critical growth

Chenxing Zhou[1], Fenghua Miao[1], Sihua Liang[1,2] and Yueqiang Song[3*]

[*]Correspondence: songyueqiang@sohu.com
[3]Scientific Research Department, Changchun Normal University, Changchun, Jilin 130032, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, by using the concentration-compactness principle and the variational method, we obtain a multiplicity result for Kirchhoff-type problems involving critical growth in bounded domains.

MSC: 35J70; 35B20

Keywords: Kirchhoff-type problems; critical growth; concentration-compactness principle; variational method

1 Introduction

In this paper we deal with the existence and multiplicity of solutions to the following Kirchhoff-type problems involving the critical growth:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u - a[\Delta(u^2)]u = u^{2(2^*)-1} + \lambda h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is an open bounded domain with smooth boundary and λ is a positive parameter. The number $2^* = 2N/(N-2)$ is the critical exponent according to the Sobolev embedding.

Much interest has arisen in problems involving critical exponents, starting from the celebrated paper by Brezis and Nirenberg [1]. For example, Li and Zou [2] obtained infinitely many solutions with odd nonlinearity. Chen and Li [3] obtained the existence of infinitely many solutions by using minimax procedure. For more related results, we refer the interested readers to [4–10] and references therein.

On the one hand, without $a[\Delta(u^2)]$, (1.1) reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, problem (1.2) is a generalization of a model introduced by Kirchhoff [11]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Equation (1.2) is related to the stationary analog of problem (1.3). Problem (1.2) received much attention only after Lions [12] proposed an abstract framework to study the problem. Some important and interesting results can be found; see for example [13–17]. We note that results dealing with problem (1.2) with critical nonlinearity are relatively scarce [18–21].

In [22], by means of a direct variational method, the authors proved the existence and multiplicity of solutions to a class of p-Kirchhoff-type problem with Dirichlet boundary data. In [23], the authors showed the existence of infinite solutions to the p-Kirchhoff-type quasilinear elliptic equation. But they did not give any further information on the sequence of solutions. Recently, Kajikiya [24] established a critical point theorem related to the symmetric mountain-pass lemma and applied to a sublinear elliptic equation. However, there are no such results on Kirchhoff-type problems (1.1).

On the other hand, there are many papers concerned with the following quasilinear elliptic equations:

$$-\Delta u + V(x)u - \left[\Delta\left(u^2\right)\right]u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Such equations arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. In [25], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.4) by the mountain-pass theorem. The same method of a change of variables was used in [26], but the usual Sobolev space $H^1(\Omega)$ framework was used as the working space and one studied a different class of nonlinearity. In [27], the existence of both one sign and nodal ground state-type solutions was established by the Nehari method.

Motivated by the reasons above, the aim of this paper is to show the existence of infinitely many soliton solutions of problem (1.1), and there exists a sequence of infinitely many arbitrarily small soliton solutions converging to zero by using a new version of the symmetric mountain-pass lemma due to Kajikiya [24].

Note that $2(2^*)$ behaves like a critical exponent for the above equations; see [25]. For the subcritical case, the existence of solutions for problem (1.4) was studied in [25–28] and it was left open for the critical exponent case; see [25]. To the best of our knowledge, the existence of non-trivial radial solutions for (1.4) with $h(u) = \mu u^{2(2^*)-1}$ was firstly studied by Moameni [29], where the same Orlicz space as [25] was used. In [30], the authors showed the existence of multiple solutions for problems (1.1) with $a = 1$ and $b = 0$ by minimax methods and the Krasnoselski genus theory. For other interesting results see [31, 32].

To the best of our knowledge, the existence and multiplicity of soliton solutions to problem (1.1) has never been studied by variational methods. As we shall see in the present paper, problem (1.1) can be viewed as an elliptic equation coupled with a non-local term. The competing effect of the non-local term with the critical nonlinearity and the lack of compactness of the embedding of $H^1(\Omega)$ into the space $L^p(\Omega)$ prevent us from using the variational methods in a standard way. Some new estimates for such a Kirchhoff equation involving Palais-Smale sequences, which are key points in the application of this kind of theory, need to be established. We mainly follow the idea of [24, 33]. Let us point out that, although the idea was used before for other problems, the adaptation of the procedure to our problem is not trivial at all; because of the appearance of a non-local term, we must consider our problem for a suitable space and so we need more delicate estimates.

Our main result in this paper is the following.

Theorem 1.1 *Suppose that* $h(x,u)$ *satisfies the following conditions*:

(H_1^*) $h(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $h(x,-u) = -h(x,u)$ *for all* $u \in \mathbb{R}$;
(H_2^*) $\lim_{|u|\to\infty} \frac{h(x,u)}{|u|^{2(2^*)-1}} = 0$ *uniformly for* $x \in \Omega$;
(H_3^*) $\lim_{|u|\to 0^+} \frac{h(x,u)}{u} = \infty$ *uniformly for* $x \in \Omega$.

Then there exists $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *has a sequence of nontrivial solutions* $\{u_n\}$ *and* $u_n \to 0$ *in* $H_0^1(\Omega)$ *as* $n \to \infty$.

2 Preliminary lemmas

The energy functional corresponding to problem (1.1) is defined as follows:

$$\begin{aligned} J(u) :=& \frac{a}{2}\int_\Omega |\nabla u|^2\,dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2\,dx\right)^2 + a\int_\Omega |u|^2|\nabla u|^2\,dx \\ &- \frac{1}{2(2^*)}\int_\Omega |u|^{2(2^*)}\,dx - \lambda\int_\Omega H(x,u)\,dx \\ =& \frac{a}{2}\int_\Omega \bigl(1+2|u|^2\bigr)|\nabla u|^2\,dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2\,dx\right)^2 - \frac{1}{2(2^*)}\int_\Omega |u|^{2(2^*)}\,dx \\ &- \lambda\int_\Omega H(x,u)\,dx, \end{aligned}$$

where $H(x,s) = \int_0^s h(x,\tau)\,d\tau$ for $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. It should be pointed out that the functional J is not well defined in general, for instance, in $H_0^1(\Omega)$. To overcome this difficulty, we employ an argument developed by Colin and Jeanjean [26]. We make the change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \quad \text{and} \quad f(0) = 0$$

on $[0,+\infty)$ and by $f(t) = -f(-t)$ on $(-\infty, 0]$.

The following result is due to Colin and Jeanjean [26] (see also [34]).

Lemma 2.1 *The function* f *satisfies the following properties*:

(f_0) f *is uniquely defined* C^∞ *and invertible.*
(f_1) $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$.
(f_2) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$.
(f_3) $\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ *as* $t \to \infty$.
(f_4) $\frac{1}{2}f(t) \le tf'(t) \le f(t)$ *for all* $t \ge 0$.
(f_5) $|f(t)| \le t$ *for all* $t \in \mathbb{R}$.
(f_6) $|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$.
(f_7) *The function* $f^2(t)$ *is strictly convex.*

(f_8) *There exists a positive constant C such that*

$$\left|f(t)\right| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1. \end{cases}$$

(f_9) *There exist positive constants C_1 and C_2 such that*

$$|t| \leq C_1\left|f(t)\right| + C_2\left|f(t)\right|^2 \quad \text{for all } t \in \mathbb{R}.$$

(f_{10}) $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ *for all $t \in \mathbb{R}$.*

So after this change of variables, we can write $J(u)$ as

$$\begin{aligned} J(v) := {} & \frac{a}{2}\int_\Omega |\nabla v|^2\,dx + \frac{b}{4}\left(\int_\Omega \left|f'(v)\right|^2|\nabla v|^2\,dx\right)^2 - \frac{1}{2(2^*)}\int_\Omega \left|f(v)\right|^{2(2^*)}\,dx \\ & - \lambda\int_\Omega H\big(x, f(v)\big)\,dx. \end{aligned} \tag{2.1}$$

Then $J(v)$ is well defined on $H_0^1(\Omega)$. Standard arguments [35, 36] show that $J(v): H_0^1(\Omega) \to \mathbb{R}$ is of class C^1 with

$$\begin{aligned} \big\langle J'(v), w\big\rangle = {} & a\int_\Omega \nabla v \nabla w\,dx - \int_\Omega f^{2(2^*)-1}(v)f'(v)w\,dx - \lambda\int_\Omega h\big(x, f(v)\big)f'(v)w\,dx \\ & + \frac{b}{2}\left(\int_\Omega \frac{|\nabla v|^2}{1+2f^2(v)}\,dx\right)\left(\int_\Omega \frac{2\nabla v\nabla w(1+2f^2(v)) - 4|\nabla v|^2 f(v)f'(v)w}{[1+2f^2(v)]^2}\,dx\right), \end{aligned}$$

for $v, w \in H_0^1(\Omega)$.

As in [26], we note that if v is a non-trivial critical point of J, then v is a non-trivial solution of the problem

$$-a\Delta v - b\int_{\mathbb{R}^N} \left|f'(v)\right|^2|\nabla v|^2\,dx \cdot \eta(v) = g(x, v), \tag{2.2}$$

where

$$\eta(v) = \big(2f'(v)f''(v)|\nabla v|^2 + f'^2(v)\Delta v + 2f(v)f'^5(v)|\nabla v|^2\big)$$

and

$$g(x, v) = f'(v)\big(\lambda h\big(x, f(v)\big) + f^{2(2^*)-1}(v)\big).$$

Therefore, let $u = f(v)$ and since $(f^{-1})'(t) = [f'(f^{-1}(t))]^{-1} = \sqrt{1+2t^2}$, we conclude that u is a non-trivial solution of the problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u - a\big[\Delta\big(u^2\big)\big]u = u^{2(2^*)-1} + \lambda h(x, u).$$

The auxiliary result of this paper is as follows.

Theorem 2.1 *Suppose that $h(x,s)$ satisfies the following conditions*:

(H_1) $h(x,s) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $h(x,-s) = -h(x,s)$ *for all* $s \in \mathbb{R}$;
(H_2) $\lim_{|s|\to\infty} \frac{h(x,s)}{|s|^{2(2^*)-1}} = 0$ *uniformly for* $x \in \Omega$;
(H_3) $\lim_{|u|\to 0^+} \frac{h(x,s)}{s} = \infty$ *uniformly for* $x \in \Omega$.

Then there exists $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *problem* (2.2) *has a sequence of nontrivial solutions* $\{v_n\}$ *and* $v_n \to 0$ *in* $H_0^1(\Omega)$ *as* $n \to \infty$.

We recall the second concentration-compactness principle of Lions [37].

Lemma 2.2 *Let* $\{v_n\} \subset H_0^1(\Omega)$ *be a weakly convergent sequence to* v *in* $H_0^1(\Omega)$ *such that* $|v_n|^{2^*} \rightharpoonup \nu$ *and* $|\nabla v_n|^2 \rightharpoonup \mu$ *in the sense of measures. Then, for some at most countable index set* I,

(i) $\nu = |v|^{2^*} + \sum_{j\in I} \delta_{x_j} \nu_j$, $\nu_j > 0$,
(ii) $\mu \ge |\nabla v|^2 + \sum_{j\in I} \delta_{x_j} \mu_j$, $\mu_j > 0$,
(iii) $\mu_j \ge S\nu_j^{2/2^*}$,

where S *is the best Sobolev constant, i.e.* $S = \inf\{\int_\Omega |\nabla v|^2\,dx : \|v\|_{2^*}^{2^*} = 1\}$, $x_j \in \mathbb{R}^N$, δ_{x_j} *are Dirac measures at* x_j *and* μ_j, ν_j *are constants.*

Under assumptions (H_1) and (H_2), we have

$$h(x,s)s = o\big(|s|^{2(2^*)}\big), \qquad H(x,s) = o\big(|s|^{2(2^*)}\big),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon), b(\varepsilon) > 0$ such that

$$\big|h(x,s)s\big| \le a(\varepsilon) + \varepsilon|s|^{2(2^*)}, \tag{2.3}$$

$$\big|H(x,s)\big| \le b(\varepsilon) + \varepsilon|s|^{2(2^*)}. \tag{2.4}$$

Hence,

$$H(x,s) - \frac{1}{4}h(x,s)s \le c(\varepsilon) + \varepsilon|s|^{2(2^*)}, \tag{2.5}$$

for some $c(\varepsilon) > 0$.

Lemma 2.3 *Assume conditions* (H_1) *and* (H_2) *hold. Then for any* $\lambda > 0$, *the functional* J *satisfies the local* $(PS)_c$ *condition in*

$$c \in \left(-\infty, \frac{1}{2N}\big(2^{-1}aS\big)^{\frac{N}{2}} - \lambda c\left(\frac{1}{4N\lambda}\right)|\Omega|\right)$$

in the following sense: *if*

$$J(v_n) \to c < \frac{1}{2N}\big(2^{-1}aS\big)^{\frac{N}{2}} - \lambda c\left(\frac{1}{4N\lambda}\right)|\Omega|$$

and $J'(v_n) \to 0$ *for some sequence in* $H_0^1(\Omega)$, *then* $\{v_n\}$ *contains a subsequence converging strongly in* $H_0^1(\Omega)$.

Proof Let $\{v_n\}$ be a sequence in $H_0^1(\Omega)$ such that

$$\begin{aligned} J(v_n) &= \frac{a}{2}\int_\Omega |\nabla v_n|^2\,dx + \frac{b}{4}\left(\int_\Omega \left|f'(v_n)\right|^2|\nabla v_n|^2\,dx\right)^2 - \frac{1}{2(2^*)}\int_\Omega \left|f(v_n)\right|^{2(2^*)}\,dx \\ &\quad - \lambda\int_\Omega H\bigl(x,f(v_n)\bigr)\,dx = c + o(1), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \bigl\langle J'(v_n),w\bigr\rangle &= a\int_\Omega \nabla v_n\nabla w\,dx - \int_\Omega f^{2(2^*)-1}(v_n)f'(v_n)w\,dx \\ &\quad - \lambda\int_\Omega h\bigl(x,f(v_n)\bigr)f'(v_n)w\,dx + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right) \\ &\quad \times\left(\int_\Omega \frac{2\nabla v_n\nabla w(1+2f^2(v_n)) - 4|\nabla v_n|^2 f(v_n)f'(v_n)w}{[1+2f^2(v_n)]^2}\,dx\right) \\ &= o(1)\|v_n\|. \end{aligned} \tag{2.7}$$

Choose $w = w_n = \sqrt{1+2f^2(v_n)}f(v_n)$, we have $w_n \in H_0^1(\Omega)$ and

$$|\nabla w_n| = \left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|.$$

Thus, we can deduce that $\|w_n\| \le c\|v_n\|$. By (2.7) we have

$$\begin{aligned} \bigl\langle J'(v_n),w_n\bigr\rangle &= a\int_\Omega\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\,dx - \int_\Omega f^{2(2^*)}(v_n)\,dx \\ &\quad + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)^2 - \lambda\int_\Omega h\bigl(x,f(v_n)\bigr)f(v_n)\,dx \\ &= o(1)\|v_n\|. \end{aligned} \tag{2.8}$$

By (2.6) and (2.8), we have

$$\begin{aligned} c + o(1)\|v_n\| &= J(v_n) - \frac{1}{4}\bigl\langle J'(v_n),w_n\bigr\rangle \\ &= \frac{a}{4}\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx + \left(\frac{1}{4}-\frac{1}{2(2^*)}\right)\int_\Omega f^{2(2^*)}(v_n)\,dx \\ &\quad - \lambda\int_\Omega H\bigl(x,f(v_n)\bigr)\,dx + \frac{\lambda}{4}\int_\Omega h\bigl(x,f(v_n)\bigr)f(v_n)\,dx \\ &\ge \frac{1}{2N}\int_\Omega f^{2(2^*)}(v_n)\,dx - \lambda\int_\Omega H\bigl(x,f(v_n)\bigr)\,dx + \frac{\lambda}{4}\int_\Omega h\bigl(x,f(v_n)\bigr)f(v_n)\,dx, \end{aligned}$$

i.e.

$$\frac{1}{2N}\int_\Omega f^{2(2^*)}(v_n)\,dx \le \lambda\int_\Omega\left(H\bigl(x,f(v_n)\bigr) - \frac{1}{4}h\bigl(x,f(v_n)\bigr)f(v_n)\right)dx + c + o(1)\|v_n\|.$$

Then by (2.5), we have

$$\left(\frac{1}{2N} - \lambda\varepsilon\right)\int_\Omega f^{2(2^*)}(v_n)\,dx \le \lambda c(\varepsilon)|\Omega| + c + o(1)\|v_n\|.$$

Setting $\varepsilon = 1/4N\lambda$, we get

$$\int_\Omega f^{2(2^*)}(v_n)\,dx \le M + o(1)\|v_n\|, \tag{2.9}$$

where $o(1) \to 0$ and M is a some positive number. On the other hand, by (2.4) and (2.6), we have

$$\begin{aligned} c + o(1)\|v_n\| &= J(v_n) = \frac{a}{2}\int_\Omega |\nabla v_n|^2\,dx + \frac{b}{4}\left(\int_\Omega \big|f'(v_n)\big|^2|\nabla v_n|^2\,dx\right)^2 \\ &\quad - \frac{1}{2(2^*)}\int_\Omega \big|f(v_n)\big|^{2(2^*)}\,dx - \lambda\int_\Omega H\big(x,f(v_n)\big)\,dx \\ &\ge \frac{a}{2}\|v_n\|^2 - \lambda b(\varepsilon)|\Omega| - \left[\frac{1}{2(2^*)} + \lambda\varepsilon\right]\int_\Omega \big|f(v_n)\big|^{2(2^*)}\,dx. \end{aligned} \tag{2.10}$$

Therefore, the inequalities (2.9) and (2.10) imply that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Then $\{f(v_n)\}$ is also bounded in $H_0^1(\Omega)$. Therefore we can assume that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ a.e. in Ω, since $f \in C^\infty$, then $f^2(v_n) \to f^2(v)$ a.e. in Ω and then $f^2(v_n) \rightharpoonup f^2(v)$ in $H_0^1(\Omega)$. Thus, there exist measures μ and ν such that $|\nabla f^2(v_n)|^2 \rightharpoonup \mu$, $f^{2(2^*)}(v_n) \rightharpoonup \nu$. Let x_j be a singular point of the measures μ and ν. We define a function $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in $B(x_j,\epsilon)$, $\phi(x) = 0$ in $\Omega \setminus B(x_j,2\epsilon)$ and $|\nabla\phi| \le 2/\epsilon$ in Ω. Let $\tilde{w}_n = \sqrt{1+2f^2(v_n)}f(v_n)\phi$, then $\{\tilde{w}_n\}$ is bounded in $H_0^1(\Omega)$. Obviously, $\langle J'(v_n), \tilde{w}_n\phi\rangle \to 0$, *i.e.*

$$\begin{aligned} &-\lim_{\epsilon\to 0}\lim_{n\to\infty}\left[a\int_\Omega \sqrt{1+2f^2(v_n)}f(v_n)\nabla v_n\nabla\phi\,dx \right.\\ &\qquad \left. + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)\left(\int_\Omega \frac{f(v_n)\nabla v_n\nabla\phi}{\sqrt{1+2f^2(v_n)}}\,dx\right)\right] \\ &\quad = \lim_{\epsilon\to 0}\lim_{n\to\infty}\left\{a\int_\Omega \left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\phi\,dx + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)\right.\\ &\qquad \left.\times\left(\int_\Omega \frac{|\nabla v_n|^2\phi}{1+2f^2(v_n)}\,dx\right) - \int_\Omega h\big(x,f(v_n)\big)f(v_n)\phi\,dx - \int_\Omega f^{2(2^*)}(v_n)\phi\,dx\right\}. \end{aligned} \tag{2.11}$$

On the other hand, by the Hölder inequality and (f_4) in Lemma 2.1, we have

$$\begin{aligned} 0 &\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\left|a\int_\Omega \sqrt{1+2f^2(v_n)}f(v_n)\nabla v_n\nabla\phi\,dx\right| \\ &\le C\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |v_n\nabla v_n\nabla\phi|\,dx \\ &\le C\lim_{\epsilon\to 0}\lim_{n\to\infty}\left[\left(\int_\Omega |\nabla v_n|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_n\nabla\phi|^2\,dx\right)^{\frac{1}{2}}\right] \\ &\le C\lim_{\epsilon\to 0}\left(\int_{B(x_j,2\epsilon)} |v|^{2^*}\,dx\right)^{\frac{1}{2^*}} = 0. \end{aligned} \tag{2.12}$$

Similarly, we have

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\left[b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)\left(\int_\Omega \frac{f(v_n)\nabla v_n\nabla\phi}{\sqrt{1+2f^2(v_n)}}\,dx\right)\right] = 0. \tag{2.13}$$

From the inequalities (2.11), (2.12), and (2.13), together with the following facts:

(i)

$$\begin{aligned}\frac{1}{2}\big|\nabla f^2(v_n)\big|^2\phi &= 2\big|f(v_n)\big|^2\big|f'(v_n)\big|^2|\nabla v_n|^2\phi \\ &= \frac{2f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2\phi \\ &\le \left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\phi.\end{aligned}$$

(ii) Similar to the proof of (2.12), it follows that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega h\big(x,f(v_n)\big)f(v_n)\phi\,dx = 0.$$

(iii)

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega \big|\nabla f^2(v_n)\big|^2\phi\,dx = \mu_j \quad \text{and} \quad \lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega f^{2(2^*)}(v_n)\phi\,dx = \nu_j.$$

We get

$$\begin{aligned}0 &= \lim_{\epsilon\to 0}\lim_{n\to\infty}\left\{a\int_\Omega\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\phi\,dx + b\left(\int_\Omega\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)\right. \\ &\quad \left.\times\left(\int_\Omega\frac{|\nabla v_n|^2\phi}{1+2f^2(v_n)}\,dx\right) - \int_\Omega h\big(x,f(v_n)\big)f(v_n)\phi\,dx - \int_\Omega f^{2(2^*)}(v_n)\phi\,dx\right\} \\ &\ge \lim_{\epsilon\to 0}\lim_{n\to\infty}\left[\frac{a}{2}\int_\Omega \phi\big|\nabla f^2(v_n)\big|^2\,dx - \int_\Omega h\big(x,f(v_n)\big)f(v_n)\phi\,dx - \int_\Omega f^{2(2^*)}(v_n)\phi\,dx\right] \\ &= \frac{a}{2}\mu_j - \nu_j. \end{aligned} \tag{2.14}$$

Combining this with Lemma 2.2, we obtain $\nu_j \ge 2^{-1}aS\nu_j^{\frac{2}{2^*}}$. This result implies that

$$\text{(I)} \quad \nu_j = 0 \quad \text{or} \quad \text{(II)} \quad \nu_j \ge \big(2^{-1}aS\big)^{\frac{N}{2}}.$$

If the second case $\nu_j \ge (2^{-1}aS)^{\frac{N}{2}}$ holds, for some $j \in I$, then by using the Hölder inequality, we have

$$\begin{aligned}c &= \lim_{n\to\infty}\left(J(v_n) - \frac{1}{4}\big\langle J'(v_n), v_n\big\rangle\right) \\ &= \lim_{n\to\infty}\left[\frac{a}{4}\int_\Omega\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx + \left(\frac{1}{4}-\frac{1}{2(2^*)}\right)\int_\Omega f^{2(2^*)}(v_n)\,dx\right. \\ &\quad \left. - \lambda\int_\Omega H\big(x,f(v_n)\big)\,dx + \frac{\lambda}{4}\int_\Omega h\big(x,f(v_n)\big)f(v_n)\,dx\right] \\ &\ge \lim_{n\to\infty}\left[\frac{1}{2N}\int_\Omega f^{2(2^*)}(v_n)\,dx - \lambda\int_\Omega H\big(x,f(v_n)\big)\,dx + \frac{\lambda}{4}\int_\Omega h\big(x,f(v_n)\big)f(v_n)\,dx\right].\end{aligned}$$

By using inequality (2.4), we get

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle\right) \\ &\geq \lim_{n\to\infty}\left[\left(\frac{1}{2N} - \lambda\varepsilon\right)\int_\Omega f^{2(2^*)}(v_n)\,dx - \lambda c(\varepsilon)|\Omega|\right] \\ &= \left(\frac{1}{2N} - \lambda\varepsilon\right)\lim_{n\to\infty}\int_\Omega f^{2(2^*)}(v_n)\,dx - \lambda c(\varepsilon)|\Omega|. \end{aligned}$$

Since $0 \leq \phi \leq 1$, it follows that

$$c = \lim_{n\to\infty}\left(J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle\right) \geq \left(\frac{1}{2N} - \lambda\varepsilon\right)\lim_{n\to\infty}\int_\Omega f^{2(2^*)}(v_n)\phi\,dx - \lambda c(\varepsilon)|\Omega|.$$

By using $f^{2(2^*)}(v_n) \rightharpoonup f^{2(2^*)}(v)$ in the measure sense and Lemma 2.2(i), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle\right) \\ &\geq \left(\frac{1}{2N} - \lambda\varepsilon\right)\lim_{n\to\infty}\int_\Omega f^{2(2^*)}(v)\phi\,dx + \left(\frac{1}{2N} - \lambda\varepsilon\right)\sum_{j\in I}\delta_{x_j}(\phi)v_j - \lambda c(\varepsilon)|\Omega| \\ &\geq \left(\frac{1}{2N} - \lambda\varepsilon\right)v_j - \lambda c(\varepsilon)|\Omega| \\ &\geq \frac{1}{2N}\left(2^{-1}aS\right)^{\frac{N}{2}} - \lambda c\left(\frac{1}{4N\lambda}\right)|\Omega|, \end{aligned}$$

where $\varepsilon = 1/4N\lambda$. This is impossible. Consequently, $v_j = 0$ for all $j \in I$ and hence

$$\int_\Omega f^{2(2^*)}(v_n)\,dx \to \int_\Omega f^{2(2^*)}(v)\,dx, \quad \text{as } n \to +\infty.$$

Thus, from the weak lower semicontinuity of the norm and $f \in C^\infty$ we have

$$\begin{aligned} o(1)\|v_n\| &= \langle J'(v_n), w_n\rangle = a\int_\Omega\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\,dx + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)^2 \\ &\quad - \lambda\int_\Omega h\big(x,f(v_n)\big)f(v_n)\,dx - \int_\Omega f^{2(2^*)}(v_n)\,dx \\ &= a\|v_n\|^2 + a\int_\Omega \frac{2f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2\,dx + b\left(\int_\Omega \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx\right)^2 \\ &\quad - \lambda\int_\Omega h\big(x,f(v_n)\big)f(v_n)\,dx - \int_\Omega f^{2(2^*)}(v_n)\,dx \\ &\geq a\|v_n - v\|^2 + a\|v\|^2 + a\int_\Omega \frac{2f^2(v)}{1+2f^2(v)}|\nabla v|^2\,dx + b\left(\int_\Omega \frac{|\nabla v|^2}{1+2f^2(v)}\,dx\right)^2 \\ &\quad - \lambda\int_\Omega h\big(x,f(v)\big)f(v)\,dx - \int_\Omega f^{2(2^*)}(v)\,dx \\ &= a\|v_n - v\|^2 + o(1)\|v\|, \end{aligned}$$

since $J'(v) = 0$. Thus we prove that $\{v_n\}$ strongly converges to v in $H_0^1(\Omega)$. □

3 Existence of a sequence of arbitrarily small solutions

In this section, we prove the existence of infinitely many solutions of (1.1) which tend to zero. Let X be a Banach space and denote

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in N : \exists \varphi \in C(A, R^m \setminus \{0\}, -\varphi(x) = \varphi(-x))\}.$$

If there is no mapping φ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus (see [24, 35]).

Proposition 3.1 *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

(1) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.*
(2) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
(3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*
(4) *Then n-dimensional sphere S^n has a genus of $n + 1$ by the Borsuk-Ulam theorem.*
(5) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [24].

Lemma 3.1 *Let E be an infinite-dimensional space and $J \in C^1(E, R)$ and suppose the following conditions hold.*

(C_1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais-Smale condition, i.e. for some $\bar{c} > 0$, in the case when every sequence $\{u_k\}$ in E satisfying $\lim_{k\to\infty} J(u_k) = c < \bar{c}$ and $\lim_{k\to\infty} \|J'(u_k)\|_{E^*} = 0$ has a convergent subsequence.*
(C_2) *For each $k \in N$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u\in A_k} J(u) < 0$.*

Then either (R_1) *or* (R_2) *below holds.*

(R_1) *There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$ and $\{u_k\}$ converges to zero.*
(R_1) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $J'(v_k) = 0$, $J(v_k) < 0$, $\lim_{k\to\infty} v_k = 0$, and $\{v_k\}$ converges to a non-zero limit.*

Remark 3.1 From Lemma 3.1 we have a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. Let $\varepsilon = \frac{1}{2(2^*)\lambda}$, from (2.4) we have

$$\begin{aligned} J(v) := {}& \frac{a}{2}\int_\Omega |\nabla v|^2\,dx + \frac{b}{4}\left(\int_\Omega |f'(v)|^2|\nabla v|^2\,dx\right)^2 - \frac{1}{2(2^*)}\int_\Omega |f(v)|^{2(2^*)}\,dx \\ & - \lambda\int_\Omega H(x, f(v))\,dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{a}{2}\int_\Omega |\nabla v|^2\,dx - \left(\frac{1}{2(2^*)} + \varepsilon\lambda\right)\int_\Omega \big|f(v)\big|^{2(2^*)}\,dx - \lambda b(\varepsilon)|\Omega| \\
&= \frac{a}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega \big|f(v)\big|^{2(2^*)}\,dx - \lambda b\left(\frac{1}{2(2^*)\lambda}\right)|\Omega| \\
&\geq L_1\|v\|^2 - L_2\|v\|^{2^*} - L_3\lambda,
\end{aligned}$$

where L_1, L_2, L_3 are some positive constants.

Let $Q(t) = L_1t^2 - L_2t^{2^*} - L_3\lambda$. Then

$$J(v) \geq Q\big(\|v\|\big).$$

Furthermore, there exists $\lambda_* := \frac{2L_1}{NL_3}(\frac{2L_1}{2^*L_2})^{(N-2)/2}$ such that for $\lambda \in (0,\lambda_*)$, $Q(t)$ attains its positive maximum, that is, there exists

$$R_1 = \left(\frac{2L_1}{2^*L_2}\right)^{(N-2)/4}$$

such that

$$e_1 = Q(R_1) = \max_{t\geq 0} Q(t) > 0.$$

Therefore, for $e_0 \in (0,e_1)$, we may find $R_0 < R_1$ such that $Q(R_0) = e_0$. Now we define

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq R_0, \\ \frac{L_1t^2-\lambda L_3-e_1}{L_2t^{2^*}}, & t \geq R_1, \\ C^\infty, \quad \chi(t) \in [0,1], & R_0 \leq t \leq R_1. \end{cases}$$

Then it is easy to see $\chi(t) \in [0,1]$ and $\chi(t)$ is C^∞. Let $\varphi(v) = \chi(\|v\|)$ and consider the perturbation of $J(v)$:

$$\begin{aligned}
G(v) := {}& \frac{a}{2}\int_\Omega |\nabla v|^2\,dx + \frac{b}{4}\left(\int_\Omega \big|f'(v)\big|^2|\nabla v|^2\,dx\right)^2 - \frac{1}{2(2^*)}\varphi(v)\int_\Omega \big|f(v)\big|^{2(2^*)}\,dx \\
&- \lambda\varphi(v)\int_\Omega H\big(x,f(v)\big)\,dx.
\end{aligned} \tag{3.1}$$

Then

$$G(v) \geq L_1\|v\|^2 - L_2\varphi(v)\|v\|^{2^*} - L_3\lambda = \overline{Q}\big(\|v\|\big),$$

where $\overline{Q}(t) = L_1t^2 - L_2\chi(t)t^{2^*} - L_3\lambda$ and

$$\overline{Q}(t) = \begin{cases} Q(t), & 0 \leq t \leq R_0, \\ e_1, & t \geq R_1. \end{cases}$$

From the above arguments, we have the following.

Lemma 3.2 *Let $G(v)$ is defined as in* (3.1). *Then*

(i) *$G \in C^1(H_0^1(\Omega),R)$ and G is even and bounded from below.*

(ii) *If* $G(v) < e_0$, *then* $\overline{Q}(\|v\|) < e_0$, *consequently,* $\|v\| < R_0$ *and* $I(v) = G(v)$.

(iii) *There exists* λ^* *such that, for* $\lambda \in (0, \lambda^*)$, G *satisfies a local* $(PS)_c$ *condition for*

$$c < e_0 \in \left(0, \min\left\{e_1, \frac{1}{2N}\left(2^{-1}aS\right)^{\frac{N}{2}} - \lambda c\left(\frac{1}{4N\lambda}\right)|\Omega|\right\}\right).$$

Lemma 3.3 *Assume that* (H_3) *of Theorem* 1.1 *holds. Then for any* $k \in N$, *there exists* $\delta = \delta(k) > 0$ *such that* $\gamma(\{v \in H_0^1(\Omega) : G(v) \leq -\delta(k)\} \setminus \{0\}) \geq k$.

Proof Firstly, by (H_3) of Theorem 1.1, for any fixed $v \in D_0^{1,2}(\Omega)$, $v \neq 0$, we have

$$H(x, \rho v) \geq M(\rho)(\rho v)^2 \quad \text{with } M(\rho) \to \infty \text{ as } \rho \to 0.$$

Secondly, given any $k \in N$, let E_k be a k-dimensional subspace of $H_0^1(\Omega)$. Then there exists a constant σ_k such that

$$\|v\| \leq \sigma_k |v|_2, \quad \forall v \in E_k.$$

Therefore for any $v \in E_k$ with $\|v\| = 1$ and ρ small enough, by (f_1) in Lemma 2.1 we have

$$\begin{aligned} G(\rho v) &= \frac{a\rho^2}{2}\int_\Omega |\nabla v|^2\,dx + \frac{b\rho^4}{4}\left(\int_\Omega \left|f'(\rho v)\right|^2 |\nabla v|^2\,dx\right)^2 \\ &\quad - \frac{1}{2(2^*)}\varphi(\rho v)\int_\Omega \left|f(\rho v)\right|^{2(2^*)}\,dx - \lambda\varphi(\rho v)\int_\Omega H\bigl(x, f(\rho v)\bigr)\,dx \\ &\leq \frac{a\rho^2}{2} + \frac{b\rho^4}{4} - \frac{\lambda M(\rho)\varphi(\rho v)}{\sigma_k^2}\rho^2 \\ &\leq \left(\frac{a}{2} + \frac{b\rho^2}{4} - \frac{\lambda M(\rho)\varphi(\rho v)}{\sigma_k^p}\right)\rho^2 \\ &= -\delta(k) < 0, \end{aligned}$$

since $\lim_{|\rho|\to 0} M(\rho) = +\infty$. That is,

$$\bigl\{v \in E_k : \|v\| = \rho\bigr\} \subset \bigl\{v \in H_0^1(\Omega) : G(v) \leq -\delta(k)\bigr\} \setminus \{0\}.$$

This completes the proof. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 2.1 Recall that

$$\Sigma_k = \bigl\{A \in H_0^1(\Omega) \setminus \{0\} : A \text{ is closed and } A = -A, \gamma(A) \geq k\bigr\}$$

and define

$$c_k = \inf_{A\in\Sigma_k} \sup_{u\in A} G(v).$$

By Lemmas 3.2(i) and Lemmas 3.3, we know that $-\infty < c_k < 0$. Therefore, assumptions (C_1) and (C_2) of Lemma 3.1 are satisfied. This means that G has a sequence of solutions $\{v_n\}$ converging to zero. Hence, Theorem 2.1 follows by Lemma 3.2(ii). □

Proof of Theorem 1.1 This follows from Theorem 2.1, since $u_m = f(v_m) \neq u_n = f(v_n)$ if $v_m \neq v_n$ and $f \in C^\infty$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CZ carried out the theoretical studies, and participated in the sequence alignment and drafted the manuscript. YS participated in the design of the study and performed the statistical analysis. SL and FM conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Author details
[1]College of Mathematics, Changchun Normal University, Changchun, Jilin 130032, P.R. China. [2]Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University, Changchun, 130012, P.R. China. [3]Scientific Research Department, Changchun Normal University, Changchun, Jilin 130032, P.R. China.

Acknowledgements
The authors would like to express their appreciation of the referees for their precious comments and suggestions as regards the original manuscript. The authors are supported by NSFC (Grant No. 11301038), Youth Foundation for Science and Technology Department of Jilin Province (20130522100JH), Research Foundation during the 12st Five-Year Plan Period of Department of Education of Jilin Province, China (Grant [2013] No. 252), The open project program of Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University (Grant No. 93K172013K03).

Received: 13 May 2014 Accepted: 26 August 2014 Published online: 25 September 2014

References

1. Brezis, H, Nirenberg, L: Positive solutions of nonlinear elliptic equations involving critical exponents. Commun. Pure Appl. Math. **34**, 437-477 (1983)
2. Li, S, Zou, W: Remarks on a class of elliptic problems with critical exponents. Nonlinear Anal. **32**, 769-774 (1998)
3. Chen, J, Li, S: On multiple solutions of a singular quasi-linear equation on unbounded domain. J. Math. Anal. Appl. **275**, 733-746 (2002)
4. Chabrowski, J: On multiple solutions for the nonhomogeneous *p*-Laplacian with a critical Sobolev exponent. Differ. Integral Equ. **8**, 705-716 (1995)
5. Chabrowski, J: Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents. Calc. Var. **3**, 493-512 (1995)
6. Ferrero, A, Gazzola, F: Existence of solutions for singular critical growth semilinear elliptic equations. J. Differ. Equ. **177**, 494-522 (2001)
7. Garcia Azorero, J, Peral, I: Hardy inequalities and some critical elliptic and parabolic problems. J. Differ. Equ. **144**, 441-476 (1998)
8. Ghoussoub, N, Yuan, C: Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Am. Math. Soc. **352**, 5703-5743 (2000)
9. He, XM, Zou, WM: Infinitely many arbitrarily small solutions for singular elliptic problems with critical Sobolev-Hardy exponents. Proc. Edinb. Math. Soc. **52**, 97-108 (2009)
10. Silva, EA, Xavier, MS: Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **20**, 341-358 (2003)
11. Kirchhoff, G: Mechanik. Teubner, Leipzig (1883)
12. Lions, JL: On some equations in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977). North-Holland Math. Stud., vol. 30, pp. 284-346. North-Holland, Amsterdam (1978)
13. Corrêa, FJSA, Nascimento, RG: On a nonlocal elliptic system of *p*-Kirchhoff-type under Neumann boundary condition. Math. Comput. Model. **49**, 598-604 (2009)
14. Corrêa, FJSA, Figueiredo, GM: On a elliptic equation of *p*-Kirchhoff type via variational methods. Bull. Aust. Math. Soc. **74**, 263-277 (2006)
15. He, X, Zou, W: Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Anal. **70**, 1407-1414 (2009)
16. Ma, TF, Munoz Rivera, JE: Positive solutions for a nonlinear nonlocal elliptic transmission problem. Appl. Math. Lett. **16**, 243-248 (2003)
17. Perera, K, Zhang, Z: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246-255 (2006)
18. Aouaoui, S: Existence of multiple solutions to elliptic problems of Kirchhoff type with critical exponential growth. Electron. J. Differ. Equ. **2014**, 107 (2014)
19. Figueiredo, GM, Santos, JR Jr.: Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth. Differ. Integral Equ. **25**, 853-868 (2012)

20. Figueiredo, GM: Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument. J. Math. Anal. Appl. **401**, 706-713 (2013)
21. Wang, J, Tian, L, Xu, J, Zhang, F: Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. J. Differ. Equ. **253**, 2314-2351 (2012)
22. Dai, GW, Hao, RF: Existence of solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. **359**, 275-284 (2009)
23. Liu, DC: On a p-Kirchhoff equation via Fountain theorem and dual fountain theorem. Nonlinear Anal. **72**, 302-308 (2010)
24. Kajikiya, R: A critical-point theorem related to the symmetric mountain-pass lemma and its applications to elliptic equations. J. Funct. Anal. **225**, 352-370 (2005)
25. Liu, JQ, ang, YQW, Wang, ZQ: Soliton solutions to quasilinear Schrödinger equations. II. J. Differ. Equ. **187**, 473-493 (2003)
26. Colin, M, Jeanjean, L: Solutions for a quasilinear Schrödinger equations: a dual approach. Nonlinear Anal. **56**, 213-226 (2004)
27. Liu, JQ, Wang, YQ, Wang, ZQ: Solutions for quasilinear Schrödinger equations via the Nehari method. Commun. Partial Differ. Equ. **29**(5/6), 879-901 (2004)
28. Liu, JQ, Wang, ZQ: Soliton solutions for quasilinear Schrödinger equations. I. Proc. Am. Math. Soc. **131**, 441-448 (2003)
29. Moameni, A: Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$. J. Differ. Equ. **229**, 570-587 (2006)
30. Wang, YJ, Zhang, YM, Shen, YT: Multiple solutions for quasilinear Schrödinger equations involving critical exponent. Appl. Math. Comput. **216**, 849-856 (2010)
31. He, X, Zou, W: Multiplicity of solutions for a class of Kirchhoff type problems. Acta Math. Appl. Sin. **26**, 387-394 (2010)
32. Poppenberg, M, Schmitt, K, Wang, ZQ: On the existence of soliton solutions to quasilinear Schrodinger equations. Calc. Var. Partial Differ. Equ. **14**, 329-344 (2002)
33. Garcia Azorero, J, Peral, I: Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. Trans. Am. Math. Soc. **323**, 877-895 (1991)
34. do Ó, JM, Severo, U: Quasilinear Schrödinger equations involving concave and convex nonlinearities. Commun. Pure Appl. Anal. **8**, 621-644 (2009)
35. Rabinowitz, PH: Minimax Methods in Critical-Point Theory with Applications to Differential Equations. CBME Regional Conference Series in Mathematics, vol. 65. Am. Math. Soc., Providence (1986)
36. Willem, M: Minimax Theorems. Birkhäuser Boston, Boston (1996)
37. Lions, PL: The concentration compactness principle in the calculus of variations. The locally compact case. Part II. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 223-283 (1984)

doi:10.1186/s13661-014-0210-7
Cite this article as: Zhou et al.: **Multiplicity of solutions for Kirchhoff-type problems involving critical growth.** *Boundary Value Problems* 2014 **2014**:210.