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Periodic solution of second-order impulsive delay differential system via generalized mountain pass theorem

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Abstract

In this paper we use variational methods and generalized mountain pass theorem to investigate the existence of periodic solutions for some second-order delay differential systems with impulsive effects. To the authors' knowledge, there is no paper about periodic solution of impulses delay differential systems via critical point theory. Our results are completely new.

Keywords: impulse; delay; variational methods; periodic solution

1 Introduction

In this paper, we study the following second-order delay differential systems with impulsive conditions:

$$\begin{cases} \ddot{u}(t) - u(t) = -f(t, u(t-\pi)), \quad \text{for } t \in (t_{k-1}, t_k), & \text{(1a)} \\ u(0) = u(2\pi), \qquad \dot{u}(0) = \dot{u}(2\pi), & \text{(1b)} \\ \Delta\dot{u}(t_k) = g_k(u(t_k - \pi)), & \text{(1c)} \end{cases}$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^n$, $\Delta\dot{u}(t_k) = \dot{u}(t_k^+) - \dot{u}(t_k^-)$ with $\dot{u}(t_k^\pm) = \lim_{t\to t_k^\pm} \dot{u}(t)$. $g_k(u) = \operatorname{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for each $k \in \mathbb{Z}$; there exists an $m \in \mathbb{N}$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \pi$, $t_{k+m+1} = t_k + \pi$ and $g_{k+m+1} = g_k$ for all $k \in \mathbb{Z}$; $f(t,u)$ is π-periodic in t and $f(t,u) = \operatorname{grad}_u F(t,u)$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, 2\pi]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, 2\pi; \mathbb{R}^+)$ such that

$$|F(t,x)| + |f(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, 2\pi]$. For convenience, we denote (1a)-(1c) as problem (IP).

Impulsive effects are important problems in the world due to the fact that some dynamics of processes will experience sudden changes depending on their states or at certain moments of time. For a second-order differential equation $\ddot{u} = f(t, \dot{u}(t), u)$, one usually considers impulses in the position u and the velocity $\dot{u}$. However, for the motion of spacecraft one has to consider instantaneous impulses depending on the position, that result in jump discontinuities in velocity but with no change in position [1, 2]. Impulses only in the velocity occur also in impulsive mechanics [3]. Such impulsive problems with impulses in the derivative only have been considered in many literatures; see, for instance [4–11].

In recent years, impulsive and periodic boundary value problems have been studied by numerous mathematicians; see, for instance, [4, 12–15] and the references therein. Some classical tools such as fixed point theory, topological degree theory, the comparison method, the upper and lower solutions method and the monotone iterative method have been used to get the solutions of impulsive differential equations; we refer the reader to [5, 16–19] and the references therein.

Recently, some authors studied boundary value problems for second-order impulsive differential equations via variational methods (see [6–9, 20–26]).

On the other hand, in the past two decades, a wide variety of techniques, especially critical point theorem, have been developed to investigate the existence of the periodic solutions to the functional differential equations by several authors (see [10, 27, 28]). In 2009, by applying the critical theory and S^1-index theory, Guo and Guo [28] obtained some results on the existence and multiplicity of periodic solutions for the delay differential equations

$$\ddot{u}(t) = -f\big(u(t-\tau)\big).$$

In [10], the non-autonomous second-order delay differential systems

$$\ddot{u}(t) + \lambda u(t-\tau) = \nabla F\big(t, u(t-\tau)\big)$$

were studied by a new critical point theorem.

Motivated by the above work, in this paper our main purpose is to apply the critical point directly to study problem (IP). To the best of our knowledge, there is no paper studying this delay differential systems under impulsive conditions via variational methods.

The rest of the paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, the main result of this paper is stated, and finally we will give the proof of it.

2 Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we construct a variational structure. With this variational structure, we can reduce the problem of finding solutions of (IP) to that of seeking the critical points of a corresponding functional.

Denote $AC = \{u : \mathbb{R} \to \mathbb{R}^n : u$ is absolutely continuous and $u(t) = u(t+2\pi)\}$. Let

$$H^1 = \big\{u \in AC : \dot{u}(t) \in L^2\big(0, 2\pi; \mathbb{R}^n\big)\big\}$$

with the inner product

$$\langle u, v\rangle = \int_0^{2\pi} \big(u(t)v(t) + \dot{u}(t)\dot{v}(t)\big)\, dt, \quad \forall u, v \in H^1.$$

The corresponding norm is defined by

$$\|u\| = \left(\int_0^{2\pi} \big(\big|u(t)\big|^2 + \big|\dot{u}(t)\big|^2\big)\, dt\right)^{\frac{1}{2}}, \quad \forall u \in H^1.$$

The space H^1 has some important properties: there are constants c such that

$$\|u\|_{L^p} \leq c\|u\| \tag{2}$$

for all $u \in H^1$.

Let $H^2(a,b) = \{u \in C^1(a,b) : \ddot{u} \in L^2(a,b)\}$.

Definition 2.1 A function $u \in \{x \in H^1 : x(t) \in H^2(t_k, t_{k+1}), k \in K \equiv \{0, 1, \dots, 2m+1\}\}$ is said to be a classic periodic solution of (IP), if u satisfies equation in (1a) for all $t \in [0, 2\pi] \setminus \{t_1, t_2, \dots, t_{2m+1}\}$ and (1b), (1c) hold.

Taking $v \in H^1$ and multiplying the two sides of the equality

$$\ddot{u}(t+\pi) - u(t+\pi) = -f\big(t, u(t)\big)$$

by v and integrating between 0 and 2π, we have

$$\int_0^{2\pi} \big[\ddot{u}(t+\pi) - u(t+\pi) + f\big(t, u(t)\big)\big] v(t)\, dt = 0.$$

Thus consider a functional ϕ defined on H^1, given by

$$\phi(u) = \frac{1}{2}\int_0^{2\pi} \big[\dot{u}(t+\pi)\dot{u}(t) + u(t+\pi)u(t)\big]\, dt - \int_0^{2\pi} F\big(t, u(t)\big)\, dt + \sum_{k=1}^{2m+1} G_k\big(u(t_k)\big).$$

Let $L^2[0, 2\pi]$ be the space of square integrable 2π periodic vector-valued functions with dimension n, and $C^\infty[0, 2\pi]$ be the space of 2π-periodic vector-valued functions with dimension n. For any $u \in C^\infty[0, 2\pi]$, it has the following Fourier expansion in the sense that it is convergent in the space $L^2[0, 2\pi]$:

$$u(t) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} (a_j \cos jt + b_j \sin jt),$$

where $a_0, a_j, b_j \in \mathbb{R}^n$. Moreover, we infer from the above decomposition of H^1 that the norm can be written as

$$\|u\| = \left[|a_0|^2 + \sum_{j=1}^{+\infty} \big(1 + j^2\big)\big(|a_j|^2 + |b_j|^2\big) \right]^{\frac{1}{2}}.$$

It is well known that H^1 is compactly embedded in $C[0, 2\pi]$. Let L be an operator from H^1 to H^1 defined by the following form:

$$(Lu)(v) = \int_0^{2\pi} \big[\dot{u}(t+\pi)\dot{v}(t) + u(t+\pi)v(t)\big]\, dt. \tag{3}$$

By the Riesz representation theorem, Lu can also be viewed as an element belonging to H^1 such that $\langle Lu, v\rangle = (Lu)v$ for any $u, v \in H^1$. It is easy to see that L is a bounded linear operator on H^1. Set

$$\psi(u) = -\int_0^{2\pi} F\bigl(t, u(t)\bigr)\,dt + \sum_{k=1}^{2m+1} G_k\bigl(u(t_k)\bigr),$$

then $\phi(u)$ can be rewritten as

$$\phi(u) = \frac{1}{2}\langle Lu, u\rangle + \psi(u). \tag{4}$$

Lemma 2.1 *L is selfadjoint on H^1.*

Proof For any $u, v \in H^1$, we have

$$\begin{aligned}\langle Lu, v\rangle = (Lu)(v) &= \int_0^{2\pi} \bigl[\dot{u}(t+\pi)\dot{v}(t) + u(t+\pi)v(t)\bigr]\,dt \\ &= \int_0^{2\pi} \bigl[\dot{u}(t)\dot{v}(t-\pi) + u(t)v(t-\pi)\bigr]\,dt \\ &= \int_0^{2\pi} \bigl[\dot{v}(t+\pi)\dot{u}(t) + v(t+\pi)u(t)\bigr]\,dt = \langle u, Lv\rangle.\end{aligned}$$

The proof is completed. □

Remark 2.1 It follows from assumption (A) and the continuity of g_k, by a standard argument as in [29], that ϕ is continuously differentiable and weakly lower semi-continuous on H^1. Moreover, we have

$$\begin{aligned}\bigl\langle \dot{\phi}(u), v\bigr\rangle &= \int_0^{2\pi} \bigl[\dot{u}(t+\pi)\dot{v}(t) + u(t+\pi)v(t)\bigr]\,dt \\ &\quad - \int_0^{2\pi} f\bigl(t, u(t)\bigr)v(t)\,dt + \sum_{k=1}^{2m+1} g_k\bigl(u(t_k)\bigr)v(t_k) \\ &= \langle Lu, v\rangle + \bigl\langle \dot{\psi}(u), v\bigr\rangle\end{aligned}$$

for $u, v \in H^1$ and $\dot{\phi}$ is weakly continuous. Moreover, $\dot{\psi} : H^1 \to H^1$ is a compact operator defined by

$$\bigl\langle \dot{\psi}(u), v\bigr\rangle = -\int_0^{2\pi} f\bigl(t, u(t)\bigr)v(t)\,dt + \sum_{k=1}^{2m+1} g_k\bigl(u(t_k)\bigr)v(t_k).$$

Similarly to [8], we introduce the following concept for the solution of problem (IP).

Definition 2.2 We say that a function $u \in H^1$ is a weak solution of problem (IP) if the identity

$$\bigl\langle \dot{\phi}(u), v\bigr\rangle = 0$$

holds for any $v \in H^1$.

Since we have the following result, Definition 2.2 is suitable.

Lemma 2.2 *If $u \in H^1$ is a weak solution of* (IP), *then u is a classical solution of* (IP).

Proof If u is a weak solution of (IP), then for any $v \in H^1$

$$\begin{aligned}\langle\dot{\phi}(u), v\rangle = \int_0^{2\pi} \big[\dot{u}(t+\pi)\dot{v}(t) + u(t+\pi)v(t)\big]\,dt \\ - \int_0^{2\pi} f\big(t,u(t)\big)v(t)\,dt + \sum_{k=1}^{2m+1} g_k\big(u(t_k)\big)v(t_k) = 0.\end{aligned} \tag{5}$$

For any $j \in K$ and $v \in H^1$ such that $v(t) = 0$ if $t \in [t_k, t_{k+1}]$ for $k \in K \setminus \{j\}$, (5) implies

$$\int_{t_j}^{t_{j+1}} \big[\dot{u}(t+\pi)\dot{v}(t) + u(t+\pi)v(t)\big]\,dt - \int_{t_j}^{t_{j+1}} f\big(t,u(t)\big)v(t)\,dt = 0.$$

By the definition of weak derivative, the above equality implies

$$\ddot{u}(t+\pi) - u(t+\pi) = -f\big(t,u(t)\big) \quad \text{a.e. } t \in (t_j, t_{j+1}).$$

Since $f(t,u)$ is π-periodic in t and $t_j + \pi = t_{m+j+1}$, one has

$$\ddot{u}(t) - u(t) = -f\big(t,u(t-\pi)\big), \quad \text{for } t \in (t_j, t_{j+1}). \tag{6}$$

Hence $u \in H^2(t_j, t_{j+1})$. A classical regularity argument shows that u is a classical solution of (6), which implies that $\ddot{u}(t)$ is bounded for $t \in (t_j, t_{j+1})$, and this implies that $\lim_{t\to t_j^+} \dot{u}(t)$ and $\lim_{t\to t_{j+1}^-} \dot{u}(t)$ exist. Thus we obtain

$$\int_{t_j}^{t_{j+1}} (\ddot{u}v + \dot{u}\dot{v})\,dt = (\dot{u}v)\big|_{t_j}^{t_{j+1}}, \tag{7}$$

where $\dot{u}v|_{t_j}^{t_{j+1}} = \dot{u}(t_{j+1}^-)v(t_{j+1}) - \dot{u}(t_j^+)v(t_j)$. Since j is arbitrary in K and $t_j + \pi = t_{j+m+1}$, (7) and (5) imply that

$$\begin{aligned}&\int_0^{2\pi} \big[\ddot{u}(t+\pi) - u(t+\pi) + f\big(t,u(t)\big)\big]v(t)\,dt \\ &\quad = \sum_{k=0}^{2m+1} \dot{u}(t+\pi)v(t)\big|_{t_k}^{t_{k+1}} + \sum_{k=1}^{2m+1} g_k\big(u(t_k)\big)v(t_k).\end{aligned} \tag{8}$$

Therefore

$$\int_0^{2\pi} \big[\ddot{u}(t+\pi) - u(t+\pi) + f\big(t,u(t)\big)\big]v(t)\,dt = 0 \tag{9}$$

for all $v \in H^1$ with $v(t_k) = 0$ for $k \in K$. Since $C_0^\infty((t_k, t_{k+1}), \mathbb{R}^n)$ is dense in $L^2((t_k, t_{k+1}), \mathbb{R}^n)$, (9) holds for all $v \in H^1$. Thus from (8) and (9), we have

$$\begin{aligned}0 &= \sum_{k=1}^{2m+1} g_k\big(u(t_k)\big)v(t_k) + \sum_{k=1}^{2m+2} \big[\dot{u}\big(t_k^- + \pi\big)v(t_k) - \dot{u}\big(t_{k-1}^+ + \pi\big)v(t_{k-1})\big] \\ &= \sum_{k=1}^{2m+1} \big[\dot{u}\big(t_k^- + \pi\big) - \dot{u}\big(t_k^+ + \pi\big) + g_k\big(u(t_k)\big)\big]v(t_k) + \big[\dot{u}(3\pi)v(2\pi) - \dot{u}(\pi)v(0)\big],\end{aligned}$$

which implies

$$\dot{u}(t_k^+ + \pi) - \dot{u}(t_k^- + \pi) = g_k(u(t_k)) \tag{10}$$

for any $k \in \{1, 2, \ldots, 2m+1\}$, since v is arbitrary in H^1. By (10), $\dot{u}(t_k^+) - \dot{u}(t_k^-) = g_k(u(t_k - \pi))$. Therefore u is a classical solution of (IP). The proof is completed. □

Definition 2.3 ([29]) Let E be a real Banach space and $\phi \in C^1(E, \mathbb{R})$. ϕ is said to satisfy the (PS) condition on E if any sequence $\{u_n\} \subseteq E$ for which $\{\phi(u_n)\}$ is bounded and $\dot{\phi}(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in E.

Let E be a Hilbert space with $E = E_1 \oplus E_2$. Let P_1, P_2 be the projections of E onto E_1 and E_2, respectively. Set

$$\Lambda \equiv \{\varphi \in C([0, 2\pi] \times E, E) | \varphi(0, u) = u, P_2\varphi(t, u) = P_2 u - \Phi(t, u)\}, \tag{11}$$

where $\Phi : C[0, 2\pi] \times E \to E_2$ is compact.

Definition 2.4 Let $S, Q \subset E$, and Q be boundary. We call S and ∂Q link if whenever $\varphi \in \Lambda$ and $\varphi(t, \partial Q) \cap S = \emptyset$ for all t, then $\varphi(t, Q) \cap S \neq \emptyset$ for all t.

Then [30] Theorem 5.29 can be stated as follows.

Theorem A *Let E be a real Hilbert space with $E = E_1 \oplus E_2$, $E_2 = E_1^\perp$ and inner product $\langle \cdot, \cdot \rangle$. Suppose $\phi \in C^1(E, \mathbb{R})$ satisfies (PS) condition, and*

- (I_1) *$\phi(u) = \frac{1}{2}\langle Lu, u\rangle + \psi(u)$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : E_i \to E_i$ is bounded and selfadjoint $(i = 1, 2)$, where P_1, P_2 be the projections of E onto E_1 and E_2, respectively,*
- (I_2) *$\dot{\psi}(u)$ is compact, and*
- (I_3) *there exist a subspace $\tilde{E} \subset E$, sets $S \subset E$, $Q \subset \tilde{E}$ and constants $\tau > \omega$ such that*
    - (i) *$S \subset E_1$ and $\phi|_S \geq \tau$,*
    - (ii) *Q is bounded and $\phi|_{\partial Q} \leq \omega$,*
    - (iii) *S and ∂Q link.*

Then ϕ possesses a critical value $c \geq \tau$.

3 Main results

In order to state our main results, we have to further assume the following hypotheses.

(H_1) g_k $(k = 1, 2, \ldots, 2m+1)$ satisfy

$$2G_k(u) - g_k(u)u \geq 0, \qquad G_k(u) \geq 0$$

for all $u \in \mathbb{R}^n$.

(H_2) For any $k \in \{1, 2, \ldots, 2m+1\}$, there exist numbers $a > 0$ and $\gamma \in [0, 1)$ such that

$$|g_k(u)| \leq a|u|^\gamma$$

for all $u \in \mathbb{R}^n$.

(H_3) There are constants $\beta > 1$, $1 < d < 1 + \frac{\beta-1}{\beta}$, $\theta > 0$, and $L > 0$ such that

$$uf(t,u) - 2F(t,u) \geq \theta |u|^\beta, \qquad \big|f(t,u)\big| \leq \theta |u|^d$$

for all $t \in [0, 2\pi]$ and $u \in \mathbb{R}^n$ with $|u| \geq L$.

(H_4) $\frac{|F(t,u)|}{|u|^2} \to +\infty$ as $|u| \to \infty$ and $\frac{|F(t,u)|}{|u|^2} \to 0$ as $|u| \to 0$ uniformly for all t.

(H_5) $F(t,u) \geq 0$ for all $(t,u) \in [0,2\pi] \times \mathbb{R}^n$.

Theorem 3.1 *Assume that* (H_1)-(H_5) *hold. Then problem* (IP) *has at least one periodic solution.*

Example There are many examples which satisfy (H_1)-(H_5). For example,

$$F(t,x) = |x|^2 \ln\big(1 + 2|x|^4\big)$$

and $G_k(x) = |x|$, for $k = 1, 2, \ldots, 2m+1$.

Obviously, $G_k(u)$ satisfy (H_1)-(H_2) and $F(t,u)$ satisfies (H_4)-(H_5). Note that

$$uF_u(t,u) - 2F(t,u) = \frac{8|u|^6}{1 + 2|u|^4} \geq c|u|^2, \quad \forall u \geq L,$$

$$\big|F_u(t,u)\big| \leq 2|u| \ln\big(1 + 2|u|^4\big) + |u|^2 \frac{8|u|^3}{1 + 2|u|^4} \leq c|u|^{\frac{5}{4}}, \quad \forall u \geq L,$$

for L being large enough. This implies (H_3).

We will use Theorem A to prove Theorem 3.1.

Set $E_1 = \{u \in H^1 : u(t+\pi) = u(t)\}$ and $E_2 = \{u \in H^1 : u(t+\pi) = -u(t)\}$.

Lemma 3.1 $H^1 = E_1 \oplus E_2$ *and* $E_2 = E_1^\perp$.

Proof For any $v \in E_1$ and $w \in E_2$, we have

$$\begin{aligned}
\langle v, w\rangle &= \int_0^{2\pi} v(t)w(t)\,dt + \int_0^{2\pi} \dot{v}(t)\dot{w}(t)\,dt \\
&= \int_0^{2\pi} v(t+\pi)w(t+\pi)\,dt + \int_0^{2\pi} \dot{v}(t+\pi)\dot{w}(t+\pi)\,dt \\
&= \int_0^{2\pi} v(t)\big(-w(t)\big)\,dt + \int_0^{2\pi} \dot{v}(t)\big(-\dot{w}(t)\big)\,dt \\
&= -\langle v, w\rangle,
\end{aligned}$$

which implies that $\langle v, w\rangle = 0$, that is, $E_2 \perp E_1$.

For every $u \in H^1$, set

$$u^+(t) = \frac{1}{2}\big(u(t) + u(t+\pi)\big), \qquad u^-(t) = \frac{1}{2}\big(u(t) - u(t+\pi)\big).$$

Then a simple calculation shows that $u^+ \in E_1$ and $u^- \in E_2$ and $u(t) = u^+(t) + u^-(t)$. Then $H^1 = E_1 + E_2$. Combining with $E_2 \perp E_1$, one has $H^1 = E_1 \oplus E_2$ and $E_2 = E_1^\perp$. □

Remark 3.1 Lemma 3.1 is a new orthogonal decomposition different from the one in [10]. We will show that it is a useful result.

By (4) and Lemma 3.1, we have

$$\begin{aligned}
\phi(u) &= \frac{1}{2}\langle Lu,u\rangle + \psi(u) \\
&= \frac{1}{2}\int_0^{2\pi}\big[\dot{u}(t+\pi)\dot{u}(t) + u(t+\pi)u(t)\big]\,dt + \psi(u) \\
&= \frac{1}{2}\int_0^{2\pi}\big[\big(\dot{u}^+(t+\pi) + \dot{u}^-(t+\pi)\big)\big(\dot{u}^+(t) + \dot{u}^-(t)\big) \\
&\quad + \big(u^+(t+\pi) + u^-(t+\pi)\big)\big(u^+(t) + u^-(t)\big)\big]\,dt + \psi(u) \\
&= \frac{1}{2}\int_0^{2\pi}\big[\big(\dot{u}^+(t) - \dot{u}^-(t)\big)\big(\dot{u}^+(t) + \dot{u}^-(t)\big) + \big(u^+(t) - u^-(t)\big)\big(u^+(t) + u^-(t)\big)\big]\,dt \\
&\quad + \psi(u) \\
&= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \psi(u)
\end{aligned}$$

for every $u = u^+ + u^-$, where $u^+ \in E_1$, $u^- \in E_2$. Combining this with Remark 2.1 and Lemma 3.1, (I_1) and (I_2) of Theorem A hold for ϕ.

Now we prove that ϕ satisfies (PS) condition.

Lemma 3.2 *Under the assumptions of Theorem* 3.1, ϕ *satisfies* (PS) *condition.*

Proof Suppose $\{u_n\} \subset H^1$ is such a sequence that $\{\phi(u_n)\}$ is bounded and $\lim_{n\to\infty}\dot{\phi}(u_n) = 0$. We shall prove that $\{u_n\}$ has a convergent subsequence. We now prove that $\{u_n\}$ is bounded in H^1. If $\{u_n\}$ is unbounded, we may assume that, going to a subsequence if necessary, $\|u_n\| \to \infty$ as $n \to \infty$. In view of (H_3), there exists $c_1 > 0$ such that

$$uf(t,u) - 2F(t,u) \ge \theta|u|^\beta - c_1$$

for all $(t,u) \in [0,2\pi] \times \mathbb{R}^n$, and combing (H_1), we have

$$\begin{aligned}
2\phi(u_n) - \langle\dot{\phi}(u_n),u_n\rangle &= \sum_{k=1}^{2m+1}\big[2G_k\big(u_n(t_k)\big) - g_k\big(u_n(t_k)\big)u_n(t_k)\big] \\
&\quad + \int_0^{2\pi}\big(f(t,u_n)u_n - 2F(t,u_n)\big)\,dt \\
&\ge \int_0^{2\pi}\big(\theta|u_n|^\beta - c_1\big)\,dt \\
&= \theta\int_0^{2\pi}|u_n|^\beta\,dt - 2\pi c_1.
\end{aligned}$$

This implies

$$\frac{\int_0^{2\pi}|u_n|^\beta\,dt}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Let $\alpha = \frac{\beta-1}{\beta(d-1)}$, then

$$\alpha > 1, \qquad \alpha d - 1 = \alpha - \frac{1}{\beta}. \tag{13}$$

By (H_3), there exists $c_2 > 0$ such that

$$\left|f(t,u)\right|^{\alpha} \le \theta^{\alpha}|u|^{\alpha d} + c_2 \tag{14}$$

for $(t,u) \in [0,2\pi] \times \mathbb{R}^n$. Define $u_n = u_n^+ + u_n^- \in E_1 \oplus E_2$. We have

$$\begin{aligned}\langle \dot{\phi}(u_n), u_n^+\rangle &= \langle Lu_n^+, u_n^+\rangle + \sum_{k=1}^{2m+1} g_k\bigl(u_n(t_k)\bigr)u_n^+(t_k) - \int_0^{2\pi} f(t,u_n)u_n^+\,dt \\ &\ge \|u_n^+\|^2 - (2m+1)c^{\gamma+1}a\|u_n\|^{\gamma}\|u_n^+\| - c_{\alpha}\left(\int_0^{2\pi}\left|f(t,u_n)\right|^{\alpha}dt\right)^{\frac{1}{\alpha}}\|u_n^+\|, \end{aligned} \tag{15}$$

where c, c_α are constants independent of n. By (14) we have

$$\begin{aligned}\int_0^{2\pi}\left|f(t,u_n)\right|^{\alpha}dt &\le \int_0^{2\pi}\left(\theta^{\alpha}|u_n|^{\alpha d} + c_2\right)dt \\ &\le c_3\left(\int_0^{2\pi}|u_n|^{\beta}\,dt\right)^{\frac{1}{\beta}}\left(\int_0^{2\pi}|u_n|^{\frac{\beta(\alpha d-1)}{\beta-1}}\,dt\right)^{1-\frac{1}{\beta}} + 2\pi c_2 \\ &\le c_3\left(\int_0^{2\pi}|u_n|^{\beta}\,dt\right)^{\frac{1}{\beta}}\|u_n\|^{\alpha d-1} + 2\pi c_2.\end{aligned}$$

Combining this inequality with (12) and (13) yields

$$\frac{(\int_0^{2\pi}|f(t,u_n)|^{\alpha}\,dt)^{\frac{1}{\alpha}}}{\|u_n\|} \le \left[\frac{c_3(\int_0^{2\pi}|u_n|^{\beta}\,dt)^{\frac{1}{\beta}}}{\|u_n\|^{\frac{1}{\beta}}}\frac{\|u_n\|^{\alpha d-1}}{\|u_n\|^{\alpha-\frac{1}{\beta}}} + \frac{2\pi c_2}{\|u_n\|^{\alpha}}\right]^{\frac{1}{\alpha}} \to 0$$

as $n \to \infty$. Since $\gamma < 1$, by (15), we have

$$\begin{aligned}\frac{\|u_n^+\|^2}{\|u_n^+\|\|u_n\|} &\le \frac{\langle \phi'(u_n), u_n^+\rangle}{\|u_n^+\|\|u_n\|} + \frac{(2m+1)c^{\gamma+1}a\|u_n\|^{\gamma}\|u_n^+\|}{\|u_n^+\|\|u_n\|} \\ &\quad + \frac{c_{\alpha}(\int_0^{2\pi}|f(t,u_n)|^{\alpha}\,dt)^{\frac{1}{\alpha}}\|u_n^+\|}{\|u_n^+\|\|u_n\|} \to 0\end{aligned}$$

as $n \to \infty$. This implies

$$\frac{\|u_n^+\|}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{16}$$

Similarly, we have

$$\frac{\|u_n^-\|}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{17}$$

Therefore, combining (16) and (17), we have

$$1 = \frac{\|u_n\|}{\|u_n\|} \leq \frac{\|u_n^+\| + \|u_n^-\|}{\|u_n\|} \to 0 \quad \text{as } n \to \infty$$

which gives a contradiction. Therefore, $\{u_n\}$ is bounded in H^1 and, going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in H^1 and $u_n(t) \to u(t)$ in $C[0, 2\pi]$. Write $u_n = u_n^+ + u_n^-$ and $u = u^+ + u^-$, then $u_n^\pm \rightharpoonup u^\pm$ in H^1, and $u_n^\pm \to u^\pm$ in $C[0, 2\pi]$.

By (4), we have

$$\begin{aligned}
&\langle \dot{\phi}(u_n) - \dot{\phi}(u), u_n^+ - u^+ \rangle \\
&\quad = \langle L(u_n^+ - u^+), u_n^+ - u^+ \rangle - \int_0^{2\pi} [f(t, u_n) - f(t, u)](u_n^+ - u^+)\, dt \\
&\qquad + \sum_{k=1}^{2m+1} [g_k(u_n(t_k)) - g_k(u(t_k))](u_n^+ - u^+) \\
&\quad \geq \|u_n^+ - u^+\|^2 - \int_0^{2\pi} [f(t, u_n) - f(t, u)](u_n^+ - u^+)\, dt \\
&\qquad + \sum_{k=1}^{2m+1} [g_k(u_n(t_k)) - g_k(u(t_k))](u_n^+ - u^+).
\end{aligned} \tag{18}$$

Since $u_n^+ \to u^+$ in $C[0, 2\pi]$, it is then easy to verify

$$\int_0^{2\pi} [f(t, u_n) - f(t, u)](u_n^+ - u^+)\, dt \to 0 \quad \text{and} \quad [g_k(u_n(t_k)) - g_k(u(t_k))](u_n^+ - u^+) \to 0.$$

Combining this with $\langle \dot{\phi}(u_n) - \dot{\phi}(u), u_n^+ - u^+ \rangle \to 0$, as $n \to \infty$ and (18), we have $u_n^+ \to u^+$ in H^1. Similarly, $u_n^- \to u^-$ in H^1 and hence $u_n \to u$ in H^1, that is, ϕ satisfies the (PS) condition. □

Proof of Theorem 3.1 We prove that ϕ satisfies the other conditions of Theorem A.

Step 1: By (H_3) and (H_4), we have

$$F(t, u) \leq a_1 + a_2 |u|^{d+1}.$$

By (H_4), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(t, u) \leq \varepsilon |u|^2, \quad \forall t \in [0, 2\pi], |u| < \delta.$$

Therefore, there exists $M = M(\varepsilon) > 0$ such that

$$F(t, u) \leq \varepsilon |u|^2 + M |u|^{d+1}, \quad \forall (t, u) \in [0, 2\pi] \times \mathbb{R}^n.$$

Combining this with (2), we have

$$\int_0^{2\pi} F(t, u)\, dt \leq \varepsilon \|u\|_{L^2}^2 + M \|u\|_{L^{d+1}}^{d+1} \leq (\varepsilon a_3 + a_4 M \|u\|^{d-1}) \|u\|^2.$$

Consequently, by (H_1), for $u \in E_1$,

$$\phi(u) \geq \frac{1}{2}\|u\|^2 - \left(\varepsilon a_3 + a_4 M\|u\|^{d-1}\right)\|u\|^2.$$

Choose $\varepsilon = (6a_3)^{-1}$ and ρ such that $6Ma_4\rho^{d-1} = 1$. Then for any $u \in \partial B_\rho \cap E_1$,

$$\phi(u) \geq \frac{1}{6}\rho^2. \tag{19}$$

Thus ϕ satisfies (i) of (I_3) with $S = \partial B_\rho \cap E_1$ and $\tau = \frac{1}{6}\rho^2$.

Step 2: Let $e \in E_1$ with $\|e\| = 1$ and $\tilde{E} = E_2 \oplus \operatorname{span}\{e\}$. We denote

$$J = \left\{u \in \tilde{E} : \|u\| = 1\right\}.$$

For $u \in J$, we write $u = u^+ + u^-$, where $u^+ \in \operatorname{span}\{e\}$, $u^- \in E_2$.

(i) If $\|u^-\| \geq 2\|u^+\|$, one has $\|u^-\|^2 \leq \|u\|^2 = 1 \leq \frac{5}{4}\|u^-\|^2$. By (H_2) and (H_5) there exists $r_1 > 0$, for any $r > r_1$,

$$\begin{aligned}
\phi(ru) &= \frac{1}{2}r^2\left\|u^+\right\|^2 - \frac{1}{2}r^2\left\|u^-\right\|^2 - \int_0^{2\pi} F\left(t, ru(t)\right)dt + \sum_{k=1}^{2m+1} G_k\left(ru(t_k)\right) \\
&\leq -\frac{3}{10}r^2\|u\|^2 + a_5 r^{\gamma+1}\|u\|^{\gamma+1} \\
&= -\frac{3}{10}r^2 + a_5 r^{\gamma+1} \leq 0.
\end{aligned}$$

(ii) If $\|u^-\| \leq 2\|u^+\|$, one has $\|u\|^2 = 1 = \|u^-\|^2 + \|u^+\|^2 \leq 5\|u^+\|^2$, which implies that

$$\left\|u^+\right\|^2 \geq \frac{1}{5} > 0. \tag{20}$$

Denote $\tilde{J} = \{u \in J : \|u^-\| \leq 2\|u^+\|\}$.

Claim: There exists $\varepsilon_1 > 0$ such that, $\forall u \in \tilde{J}$,

$$\operatorname{meas}\left\{t \in [0, 2\pi] : \left|u(t)\right| \geq \varepsilon_1\right\} \geq \varepsilon_1. \tag{21}$$

For otherwise, $\forall j > 0$, $\exists u_j \in \tilde{J}$ such that

$$\operatorname{meas}\left\{t \in [0, 2\pi] : \left|u(t)\right| \geq \frac{1}{j}\right\} < \frac{1}{j}. \tag{22}$$

Write $u_j = u_j^+ + u_j^- \in \tilde{E}$. Notice that $\dim(\operatorname{span}\{e\}) < +\infty$ and $\|u_j^+\| \leq 1$. In the sense of subsequence, we have

$$u_j^+ \to u_0^+ \in \operatorname{span}\{e\} \quad \text{as } j \to \infty.$$

Then (20) implies that

$$\left\|u_0^+\right\|^2 \geq \frac{1}{5} > 0. \tag{23}$$

Note that $\|u_j^-\| \le 1$, in the sense of subsequence $u_j^- \rightharpoonup u_0^- \in E_2$ as $j \to \infty$. Thus in the sense of subsequences,

$$u_j \rightharpoonup u_0 = u_0^- + u_0^+ \quad \text{as } j \to \infty.$$

This means that $u_j \to u_0$ in L^2, *i.e.*,

$$\int_0^{2\pi} |u_j - u_0|^2 \, dt \to 0 \quad \text{as } j \to \infty. \tag{24}$$

By (23) we know that $\|u_0\| > 0$. Therefore, $\int_0^{2\pi} |u_0|^2 \, dt > 0$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\text{meas}\bigl\{t \in [0, 2\pi] : \bigl|u_0(t)\bigr| \ge \delta_1\bigr\} \ge \delta_2. \tag{25}$$

Otherwise, for all $n > 0$, we must have

$$\text{meas}\left\{t \in [0, 2\pi] : \bigl|u_0(t)\bigr| \ge \frac{1}{n}\right\} = 0,$$

i.e.,

$$\text{meas}\left\{t \in [0, 2\pi] : \bigl|u_0(t)\bigr| < \frac{1}{n}\right\} = 2\pi.$$

We have

$$0 < \int_0^{2\pi} |u_0|^2 \, dt \le \frac{1}{n^2} \cdot 2\pi \to 0 \quad \text{as } n \to \infty.$$

We get a contradiction. Thus (25) holds. Let $\Omega_0 = \{t \in [0, 2\pi] : |u_0(t)| \ge \delta_1\}$, $\Omega_j = \{t \in [0, 2\pi] : |u_j(t)| < \frac{1}{j}\}$, and $\Omega_j^\perp = [0, 2\pi] \setminus \Omega_j$. By (22), we have

$$\text{meas}(\Omega_j \cap \Omega_0) = \text{meas}\bigl(\Omega_0 - \Omega_0 \cap \Omega_j^\perp\bigr) \ge \text{meas}(\Omega_0) - \text{meas}\bigl(\Omega_0 \cap \Omega_j^\perp\bigr) \ge \delta_2 - \frac{1}{j}.$$

Let j be large enough such that $\delta_2 - \frac{1}{j} > \frac{\delta_2}{2}$ and $\delta_1 - \frac{1}{j} > \frac{\delta_1}{2}$. Then we have

$$\bigl|u_j(t) - u_0(t)\bigr|^2 \ge \left(\delta_1 - \frac{1}{j}\right)^2 \ge \left(\frac{\delta_1}{2}\right)^2, \quad \forall t \in \Omega_j \cap \Omega_0.$$

This implies that

$$\begin{aligned} \int_0^{2\pi} |u_j - u_0|^2 \, dt &\ge \int_{\Omega_j \cap \Omega_0} |u_j - u_0|^2 \, dt \ge \left(\frac{\delta_1}{2}\right)^2 \cdot \text{meas}(\Omega_j \cap \Omega_0) \\ &\ge \left(\frac{\delta_1}{2}\right)^2 \cdot \left(\delta_2 - \frac{1}{j}\right) \ge \left(\frac{\delta_1}{2}\right)^2 \frac{\delta_2}{2} > 0. \end{aligned}$$

This is a contradiction to (24). Therefore the claim is true and (21) holds. For $u = u^+ + u^- \in \tilde{J}$, let $\Omega_u = \{t \in [0, 2\pi] : |u(t)| \geq \varepsilon_1\}$. By ($H_4$), for $a_6 = \frac{1}{\varepsilon_1^3} > 0$, there exists $L_1 > 0$ such that

$$F\big(t, u(t)\big) \geq a_6 |u|^2, \quad \forall |u| \geq L_1, \text{ uniformly in } t.$$

Choose $r_2 \geq \frac{L_1}{\varepsilon_1}$. For $r \geq r_2$,

$$F\big(t, ru(t)\big) \geq a_6 \big|ru(t)\big|^2 \geq a_6 r^2 \varepsilon_1^2, \quad \forall t \in \Omega_u.$$

By (H_5), for $r > r_2$,

$$\begin{aligned}
\phi(ru) &= \frac{1}{2}\big\|ru^+\big\|^2 - \frac{1}{2}\big\|ru^-\big\|^2 - \int_0^{2\pi} F\big(t, u(t)\big)\,dt + \sum_{k=1}^{2m+1} G_k\big(u(t_k)\big) \\
&\leq \frac{1}{2}r^2 - \int_{\Omega_u} F(t, ru)\,dt + \sum_{k=1}^{2m+1} a|ru|^{\gamma+1} \\
&\leq \frac{1}{2}r^2 - a_6 \varepsilon_1^3 r^2 + a_7 r^{\gamma+1} \\
&= -\frac{1}{2}r^2 + a_7 r^{\gamma+1},
\end{aligned}$$

which implies that there exists $r_3 > r_2$ such that for $r > r_3$

$$\phi(ru) \leq 0 \quad \forall u \in \tilde{J}.$$

Setting $r_4 = \max\{r_1, r_3\}$, we have proved that for any $u \in J$ and $r \geq r_4$

$$\phi(ru) \leq 0. \tag{26}$$

Let $Q = \{re : 0 \leq r \leq 2r_4\} \oplus \{u \in E_2 : \|u\| \leq 2r_4\}$. By (26) we have $\phi|_{\partial Q} \leq 0$, *i.e.*, ϕ satisfies (ii) of (I_3) in Theorem A.

Finally, by Lemma 3.2, ϕ satisfies the (PS) condition. Similar to the proof of [30], we prove that S and ∂Q link. By Theorem A, there exists a critical point $u \in H^1$ of ϕ such that $\phi(u) \geq \tilde{a} > 0$. Moreover, u is a classical solution of (IP) and u is nonconstant by (H_5). The proof is completed. □

Remark 3.2 In order to seek $2T$-periodic solutions of more general systems

$$\begin{cases}
\ddot{u}(t) - u(t) = -f(t, u(t-T)), & \text{for } t \in (t_{k-1}, t_k), \\
u(0) = u(2T), \qquad \dot{u}(0) = \dot{u}(2T), \\
\Delta \dot{u}(t_k) = g_k(u(t_k - T)),
\end{cases}$$

where f and impulsive effects are T-periodic in t, we make the substitution: $s = \frac{\pi}{T}t$ and $\lambda = \frac{T}{\pi}$. Thus the above systems transforms to

$$\begin{cases} \ddot{u}(t) - \lambda^2 u(t) = -\lambda^2 f(\lambda t, u(t-\pi)), & \text{for } t \in (t_{k-1}, t_k), \\ u(0) = u(2\pi), \qquad \dot{u}(0) = \dot{u}(2\pi), \\ \Delta \dot{u}(t_k) = \lambda g_k(u(t_k - \pi)). \end{cases}$$

This implies that a 2π-periodic solution of the second systems corresponds to a $2T$-periodic solution of the first one. Hence we will only look for the 2π-periodic solutions in the sequel.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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