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Global nonexistence of solutions for nonlinear coupled viscoelastic wave equations with damping and source terms

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Abstract

In this paper, we are concerned with a nonlinear coupled viscoelastic wave equations with initial-boundary value conditions and nonlinear damping and source terms. Under suitable assumptions on relaxation functions, damping terms, and source terms, by using the energy method we proved a global nonexistence result for certain solutions with negative initial energy.

Keywords: coupled viscoelastic wave equations; relaxation functions; damping terms; source terms; global nonexistence

1 Introduction

In this paper, we study the following initial-boundary value coupled viscoelastic problem:

$$
\begin{cases}
|u_t|^j u_{tt} - \Delta u_{tt} - \text{div}(|\nabla u|^{\alpha-2}\nabla u) - \Delta u + \int_0^t g(t-s)\Delta u\, ds \\
\quad + |u_t|^{m-1} u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\
|v_t|^j v_{tt} - \Delta v_{tt} - \text{div}(|\nabla v|^{\beta-2}\nabla v) - \Delta v + \int_0^t h(t-s)\Delta v\, ds \\
\quad + |v_t|^{r-1} v_t = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\
u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega,
\end{cases}
\tag{1.1}
$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, the constants $j > 0$, $\alpha \geq 2$, $\beta \geq 2$, $m \geq 1$, $r \geq 1$, the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ are

$$f_1(u,v) = a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2},$$

$$f_2(u,v) = a|u+v|^{2(p+1)}(u+v) + b|u|^{p+2}|v|^p v,$$

in which the constants $a > 0$, $b > 0$, and p satisfies

$$
\begin{cases}
p > -1, & n = 1, 2, \\
-1 < p \leq \frac{4-n}{n-2}, & n \geq 3.
\end{cases}
$$

The pioneering work of Dafermos [1] studied a one-dimensional viscoelastic problem, established some existence and asymptotic stability results for smooth monotone decreas-

ing relaxation functions. Hrusa [2] established several global existence and asymptotic stability results for a semilinear hyperbolic Volterra equation which allows the initial data to be large. Muñoz Rivera [3] considered equations for linear isotropic viscoelastic solids of integral type, and established exponential decay and polynomial decay in a bounded domain and in the whole space, respectively. After that, many results of existence and long-term behavior have been established. Messaoudi [4] considered a nonlinear viscoelastic wave equation with source and damping terms of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t|u_t|^{m-1} = u|u|^{p-1}.$$ (1.2)

He established blow-up result for solutions with negative initial energy and $m < p$, and gave a global existence result for arbitrary initial (in the appropriate space) if $m \geq p$. This work was later improved by Messaoudi [5].

Cavalcanti and Oquendo [6] discussed the equation

$$u_{tt} - k_0\Delta u + \int_0^t \mathrm{div}\big[a(x)g(t-\tau)\nabla u(\tau)\big]\,d\tau + b(x)h(u_t) + f(u) = 0,$$ (1.3)

under some conditions on the relaxation function g and $a(x) + b(x) \geq \delta > 0$. They proved an exponential stability result when the relaxation function g is decaying exponentially and the function h is linear and a polynomial stability when g is decaying polynomially and h is nonlinear.

Cavalcanti *et al.* [7] discussed the wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\,\mathrm{div}\big[a(x)\nabla u(\tau)\big]\,d\tau + b(x)f(u_t) = 0$$ (1.4)

on a compact Riemannian manifold $(M, \mathbf{g})$ subject to a combination of locally distributed viscoelastic and frictional dissipations. It is shown that the solutions decay according to the law dictated by the decay rates corresponding to the slowest damping.

Muñoz Rivera and Naso [8] studied a viscoelastic systems with nondissipative kernels and showed that if the kernel function decays exponentially to zero, then the solution decays exponentially to zero. On the other hand, if the kernel function decays polynomially as t^{-p}, then the corresponding solution also decays polynomially to zero with the same rate of decay.

Wang and Wang [9] studied a one-dimensional wave equation with viscoelastic damping under the Dirichlet boundary condition, where the kernel was taken for the finite sum of exponential polynomials. Using the asymptotic analysis technique, the authors proved an exponential stability result. Zhao and Wang [10] considered a coupled system of an ODE and a wave equation with Kelvin-Voigt damping, where the velocity of the wave equation entered the ODE and the output was fed into the boundary of the wave equation. They presented the asymptotic expressions of eigenvalues and eigenfunctions and established the exponential stability result.

Cavalcanti *et al.* [11] discussed a quasilinear initial-boundary value problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(\tau)\,d\tau - \gamma\Delta u_t = bu|u|^{p-2},$$ (1.5)

with Dirichlet boundary condition, where $\rho > 0$, $\gamma \geq 0$, $p \geq 2$, $b = 0$. An exponential decay result for $\gamma > 0$ and $b = 0$ has been obtained. For $\gamma = 0$ and $b > 0$, Messaoudi and Tatar [12, 13] showed that there exists an appropriate set, called stable set, such that if the initial data are in stable set, the solution continues to live there forever, and the solution goes to zero with an exponential or polynomial rate depending on the decay rate of relaxation function.

Pata [14] studied the abstract integrodifferential equations

$$u_{tt} + \alpha Au + \beta u_t - \int_0^t g(\tau) Au(t - \tau) \, d\tau = 0, \tag{1.6}$$

for $\alpha > 0$, $\beta \geq 0$, g is a positive summable kernel, H be a real Hilbert space and the operator $A : D(A) \to H$ be a self-adjoint linear positive definite operator with domain $D(A) \subset H$ such that the embedding is dense and compact. He introduced some new concepts such as the flatness of a kernel and gave the asymptotic result.

Lasiecka *et al.* [15] discussed (1.6) with $\alpha = 1$, $\beta = 0$ with memory kernel satisfying the inequality $g' + H(g) \leq 0$, $s \geq 0$ where $H(s)$ is a given continuous positive increasing and convex function such that $H(0) = 0$. They developed an intrinsic method for determining decay rates of the energy given in terms of the function $H(s)$.

Han and Wang [16] studied the following initial-boundary value problem for a coupled system of nonlinear viscoelastic equations:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t - \tau)\Delta u(\tau) \, d\tau + |u_t|^{m-1} u_t = f_1(u, v), & (x, t) \in \Omega \times (0, T), \\ v_{tt} - \Delta v + \int_0^t g_2(t - \tau)\Delta v(\tau) \, d\tau + |v_t|^{m-1} v_t = f_2(u, v), & (x, t) \in \Omega \times (0, T), \\ u = v = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega. \end{cases} \tag{1.7}$$

Local existence, global existence, uniqueness, and blow-up in finite time were obtained when f_1, f_2, g_1, g_2, and the initial values satisfy some conditions.

Messaoudi and Said-Houari [17] dealt with the problem (1.7) and proved a global nonexistence of solutions for a large class of initial data for which the initial energy takes positive values. Also, Said-Houari *et al.* [18] discussed (1.7) and proved a general decay result.

Liu [19] considered the following initial-boundary value problem for a coupled system of quasilinear viscoelastic equations:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - c_1 \Delta u_{tt} + \int_0^t g_1(t - \tau)\Delta u(\tau) \, d\tau + f(u, v) = 0, & (x, t) \in \Omega \times (0, \infty), \\ |v_t|^\rho v_{tt} - \Delta v - c_2 \Delta v_{tt} + \int_0^t g_2(t - \tau)\Delta v(\tau) \, d\tau + k(u, v) = 0, & (x, t) \in \Omega \times (0, \infty), \\ u = v = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega. \end{cases} \tag{1.8}$$

The author used perturbed energy method to show that dissipations given by the viscoelastic terms are strong enough to ensure the decay of the corresponding energy function.

Our purpose in this paper is to give the global nonexistence of solutions for coupled viscoelastic equations with damping and source terms by using the energy method.

The present work is organized as follows. In Section 2, we give some notations and material needed for this work. Section 3 contains the main result and the proof of the global nonexistence result.

2 Preliminary

In this section, we give some notations and material needed for the proof of our result.

We shall write $\|\cdot\|_p$ to denote the usual $L^p(\Omega)$ norm $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|$ to denote the usual $L^2(\Omega)$ norm $\|\cdot\|_{L^2(\Omega)}$.

First, we make the following assumptions on the C^1-nonnegative and nonincreasing relaxation functions g and h:

$$1 - \int_0^\infty g(s)\,ds = l > 0, \qquad 1 - \int_0^\infty h(s)\,ds = k > 0. \tag{2.1}$$

One can easily verify for $(\xi, \eta) \in R^2$,

$$\xi f_1(\xi, \eta) + \eta f_2(\xi, \eta) = 2(p+2)F(\xi, \eta),$$

$$F(\xi, \eta) = \frac{1}{2(p+2)}\big[a|\xi + \eta|^{2(p+2)}(\xi + \eta) + 2b|\xi\eta|^{p+2}\big]$$

and

$$f_1(\xi, \eta) = \frac{\partial F(\xi, \eta)}{\partial \xi}, \qquad f_2(\xi, \eta) = \frac{\partial F(\xi, \eta)}{\partial \eta}.$$

Define

$$(\phi \circ \psi)(t) := \int_0^t \phi(t - \tau) \int_\Omega |\psi(t) - \psi(\tau)|^2 \, dx \, d\tau$$

and

$$E(t) := \frac{1}{j+2}\big(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\big) + \frac{1}{2}\big(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\big)$$

$$+ \frac{1}{2}\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u\|^2 + \frac{1}{2}\bigg(1 - \int_0^t h(s)\,ds\bigg)\|\nabla v\|^2$$

$$- \int_\Omega F(u, v)\,dx + \frac{1}{2}(g \circ \nabla u + h \circ \nabla v) + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v\|_\beta^\beta. \tag{2.2}$$

3 Global nonexistence result

In this section, we give the global nonexistence result and its proof.

Theorem 3.1 *Assume that*

$$\max\{j+1, m+1, r+1\} < 2(p+2), \qquad E(0) < 0,$$

and there exists constant γ such that

$$\max\{\alpha, \beta\} < \gamma < 2(p+2)$$

and

$$\min\{l, k\} > \frac{1/(2\gamma)}{(\gamma/2) - 1 + 1/(2\gamma)}.$$

Then any solution of (1.1) *cannot exist for all time.*

Proof By multiplying the first and second equation of (1.1) by u_t and v_t, respectively, integrating over Ω and summing them up, we obtain

$$\frac{d}{dt}E(t) = -\left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1}\right) + \frac{1}{2}\left(g' \circ \nabla u + h' \circ \nabla v\right)$$
$$- \frac{1}{2}\left(g(t)\|\nabla u\|^2 + h(t)\|\nabla v\|^2\right).$$

Define

$$H(t) := -E(t),$$

then

$$0 < H(0) \le H(t) \le \int_\Omega F(u, v)\, dx \le \frac{C_1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right). \qquad (3.1)$$

Throughout, C and C_i represent generic positive constants.

Define

$$\Theta(t) := H^{1-\delta}(t) + \frac{\varepsilon}{j+1}\int_\Omega \left(|u_t|^j u_t u + |v_t|^j v_t v\right) dx - \varepsilon \int_\Omega (\Delta u u_t + \Delta v v_t)\, dx,$$

where ε is a positive constant to be determined, and

$$0 < \delta$$
$$\le \min\left\{\frac{2(p+2) - (j+2)}{2(j+2)(p+2)}, \frac{2(p+2) - (m+1)}{2m(p+2)}, \frac{2(p+2) - (r+1)}{2r(p+2)}, \frac{\alpha - 2}{2\alpha}, \frac{\beta - 2}{2\beta}\right\}. \qquad (3.2)$$

By differentiation, we obtain

$$\Theta'(t) = (1 - \delta)H^{-\delta}(t)H'(t) + \varepsilon \int_\Omega \left[\left(|u_t|^j u_{tt} u + |v_t|^j v_{tt} v\right) + \frac{1}{j+1}\left(|u_t|^{j+2} + |v_t|^{j+2}\right)\right] dx$$
$$+ \varepsilon\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \varepsilon \int_\Omega (u \Delta u_{tt} + v \Delta v_{tt})\, dx$$
$$= (1 - \delta)H^{-\delta}(t)H'(t) + \frac{\varepsilon}{j+1}\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right)$$
$$- \varepsilon \int_\Omega \left(|u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v\right) dx + \varepsilon\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right)$$
$$- \varepsilon\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - \varepsilon\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\beta^\beta\right) + 2\varepsilon(p+2)\int_\Omega F(u, v)\, dx$$
$$+ \varepsilon\left(\int_0^t g(s)\, ds\right)\|\nabla u\|^2 + \varepsilon\left(\int_0^t h(s)\, ds\right)\|\nabla v\|^2$$

$$+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) \big[\nabla u(s) - \nabla u(t) \big] \, dx \, ds$$

$$+ \varepsilon \int_0^t h(t-s) \int_\Omega \nabla v(t) \big[\nabla v(s) - \nabla v(t) \big] \, dx \, ds.$$

Using Hölder's inequality and Young's inequality, we get

$$\int_0^t g(t-s) \int_\Omega \nabla u(t) \big[\nabla u(s) - \nabla u(t) \big] \, dx \, ds$$

$$\leq \int_0^t g(t-s) \big\| \nabla u(t) \big\| \big\| \nabla u(s) - \nabla u(t) \big\| \, ds$$

$$\leq \lambda (g \circ \nabla u) + \frac{1}{4\lambda} \left(\int_0^t g(s) \, ds \right) \| \nabla u \|^2,$$

$$\int_0^t h(t-s) \int_\Omega \nabla v(t) \big[\nabla v(s) - \nabla v(t) \big] \, dx \, ds$$

$$\leq \int_0^t h(t-s) \big\| \nabla v(t) \big\| \big\| \nabla v(s) - \nabla v(t) \big\| \, ds$$

$$\leq \lambda (h \circ \nabla v) + \frac{1}{4\lambda} \left(\int_0^t h(s) \, ds \right) \| \nabla v \|^2,$$

for $0 < \lambda < \frac{\gamma}{2}$, thus

$$\begin{aligned}
\Theta'(t) \geq\ & (1-\delta) H^{-\delta}(t) H'(t) + \varepsilon \left(\frac{1}{j+1} + \frac{\gamma}{j+2} \right) \left(\| u_t \|_{j+2}^{j+2} + \| v_t \|_{j+2}^{j+2} \right) \\
& - \varepsilon \int_\Omega \left(|u_t|^{m-1} u_t u + |v_t|^{r-1} v_t v \right) dx + \varepsilon \left(1 + \frac{\gamma}{2} \right) \left(\| \nabla u_t \|^2 + \| \nabla v_t \|^2 \right) \\
& + \gamma \varepsilon H(t) + \varepsilon \left(2(p+2) - \gamma \right) \int_\Omega F(u,v) \, dx \\
& + \varepsilon \left[\left(\frac{\gamma}{2} - 1 \right) - \left(\frac{\gamma}{2} - 1 + \frac{1}{4\lambda} \right) \int_0^\infty g(s) \, ds \right] \| \nabla u \|^2 \\
& + \varepsilon \left[\left(\frac{\gamma}{2} - 1 \right) - \left(\frac{\gamma}{2} - 1 + \frac{1}{4\lambda} \right) \int_0^\infty h(s) \, ds \right] \| \nabla v \|^2 \\
& + \varepsilon \left(\frac{\gamma}{2} - \lambda \right) (g \circ \nabla u + h \circ \nabla v) + \varepsilon \left(\frac{\gamma}{\alpha} - 1 \right) \| \nabla u \|_\alpha^\alpha + \varepsilon \left(\frac{\gamma}{\beta} - 1 \right) \| \nabla v \|_\beta^\beta. \quad (3.3)
\end{aligned}$$

By Young's inequality, we get

$$\begin{aligned}
\int_\Omega |u_t|^{m-1} u_t u \, dx &\leq \frac{\delta_1^{m+1}}{m+1} \| u \|_{m+1}^{m+1} + \frac{m \delta_1^{-\frac{m+1}{m}}}{m+1} \| u_t \|_{m+1}^{m+1} \\
&\leq \frac{\delta_1^{m+1}}{m+1} \| u \|_{m+1}^{m+1} + \frac{m \delta_1^{-\frac{m+1}{m}}}{m+1} H'(t)
\end{aligned} \quad (3.4)$$

and

$$\int_\Omega |v_t|^{r-1} v_t v \, dx \leq \frac{\delta_2^{r+1}}{r+1} \| v \|_{r+1}^{r+1} + \frac{r \delta_2^{-\frac{r+1}{r}}}{r+1} \| v_t \|_{r+1}^{r+1} \leq \frac{\delta_2^{r+1}}{r+1} \| v \|_{r+1}^{r+1} + \frac{r \delta_2^{-\frac{r+1}{r}}}{r+1} H'(t). \quad (3.5)$$

Set

$$\delta_1^{-\frac{m+1}{m}} = M_1 H^{-\delta}, \qquad \delta_2^{-\frac{r+1}{r}} = M_2 H^{-\delta},$$

in which M_1 and M_2 will be specified later, and γ obeys

$$\min\left\{\frac{\gamma}{\alpha} - 1, \frac{\gamma}{\beta} - 1\right\} > 0$$

and

$$1 + \frac{\gamma}{2} > 0.$$

Then due to (3.2), we get

$$\delta_1^{m+1} = M_1^{-m} H^{m\delta} \leq \frac{M_1^{-m} C_1^{m\delta}}{(2(p+2))^{m\delta}} \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{m\delta} \tag{3.6}$$

and

$$\delta_2^{r+1} = M_2^{-r} H^{r\delta} \leq \frac{M_2^{-r} C_1^{r\delta}}{(2(p+2))^{r\delta}} \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{r\delta}. \tag{3.7}$$

Since $L^{2(p+2)}(\Omega) \hookrightarrow L^{m+1}(\Omega)$, $L^{2(p+2)}(\Omega) \hookrightarrow L^{r+1}(\Omega)$, we have

$$\|u\|_{m+1}^{m+1} \leq C\|u\|_{2(p+2)}^{m+1}, \qquad \|v\|_{r+1}^{r+1} \leq C\|v\|_{2(p+2)}^{r+1},$$

thus

$$\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{m\delta} \|u\|_{m+1}^{m+1} \leq C_2 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{m\delta + \frac{m+1}{2(p+2)}} \tag{3.8}$$

and

$$\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{r\delta} \|v\|_{r+1}^{r+1} \leq C_3 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{r\delta + \frac{r+1}{2(p+2)}}. \tag{3.9}$$

Using (3.2) and the following inequality:

$$z^\nu \leq z + 1 \leq \left(1 + \frac{1}{a}\right)(z + a), \quad \forall z \geq 0, 0 < \nu \leq 1,$$

we obtain, for $t \geq 0$,

$$\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{m\delta + \frac{m+1}{2(p+2)}} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + H(0)\right)$$

$$\leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + H(t)\right) \tag{3.10}$$

and

$$\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right)^{r\delta + \frac{r+1}{2(p+2)}} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + H(t)\right), \tag{3.11}$$

for $d = 1 + \frac{1}{H(0)}$.

By (3.3)-(3.11) we have

$$
\begin{aligned}
\Theta'(t) \geq{} & (1 - \delta - M\varepsilon)H^{-\delta}(t)H'(t) + \varepsilon\left(\frac{1}{j+1} + \frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right) \\
& + (\gamma - K_1)\varepsilon H(t) + \varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\beta^\beta\right) + \varepsilon K_3\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
& + \varepsilon\left(\frac{\gamma}{2} - \lambda\right)(g \circ \nabla u + h \circ \nabla v) + \varepsilon\left(1 + \frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
& + \varepsilon\left(\frac{(2(p+2) - \gamma)C_1}{2(p+2)} - K_1\right)\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \frac{M_1 m}{m+1} + \frac{M_2 r}{r+1}, \\
K_1 &= \frac{M_1^{-m} C_1^{m\delta} C_2 d}{(m+1)(2(p+2))^{m\delta}} + \frac{M_2^{-r} C_1^{r\delta} C_3 d}{(r+1)(2(p+2))^{r\delta}}, \\
K_2 &= \min\left\{\frac{\gamma}{\alpha} - 1, \frac{\gamma}{\beta} - 1\right\}
\end{aligned}
$$

and

$$
K_3 = \left(\frac{\gamma}{2} - 1\right) - \left(\frac{\gamma}{2} - 1 + \frac{1}{4\lambda}\right)\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right).
$$

Choose M_1, M_2 large enough such that

$$
\gamma - K_1 > 0, \qquad \frac{(2(p+2) - \gamma)C_1}{2(p+2)} - K_1 > 0.
$$

For fixed M_1 and M_2, we choose $\varepsilon > 0$ small enough such that

$$
1 - \delta - M\varepsilon \geq 0.
$$

Then we can find positive constants a_1 and a_2 such that

$$
\begin{aligned}
\Theta'(t) \geq{} & (1 - \delta - M\varepsilon)H^{-\delta}(t)H'(t) + \varepsilon\left(\frac{1}{j+1} + \frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right) \\
& + \varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\beta^\beta\right) + \varepsilon\left(1 + \frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
& + \varepsilon a_1 H(t) + \varepsilon a_2\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right) \geq 0.
\end{aligned}
$$

Because of $H'(t) \geq 0$, there exists constant $\Lambda > 0$ such that

$$
\begin{aligned}
\Theta'(t) \geq{} & \Lambda\big(H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\beta^\beta \\
& + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\big) \geq 0.
\end{aligned}
\tag{3.12}
$$

On the other hand, we can choose ε even smaller so that

$$
\Theta(0) = H^{1-\delta}(0) + \varepsilon\int_\Omega (u_0 u_1 + v_0 v_1)\,dx > 0.
\tag{3.13}
$$

Furthermore, we have

$$\Theta(t) \geq \Theta(0), \quad t \geq 0. \tag{3.14}$$

Next we estimate $\Theta^{\frac{1}{1-\delta}}(t)$.

We have

$$\Theta^{\frac{1}{1-\delta}}(t)$$

$$= \left(H^{1-\delta}(t) + \frac{\varepsilon}{j+1} \int_\Omega \left(|u_t|^j u_t u + |v_t|^j v_t v \right) dx - \varepsilon \int_\Omega (\Delta u u_t + \Delta v v_t) \, dx \right)^{\frac{1}{1-\delta}}$$

$$\leq C \left(H(t) + \left| \frac{\varepsilon}{j+1} \int_\Omega \left(|u_t|^j u_t u + |v_t|^j v_t v \right) dx - \varepsilon \int_\Omega (\Delta u u_t + \Delta v v_t) \, dx \right|^{\frac{1}{1-\delta}} \right). \tag{3.15}$$

By Hölder's inequality, the Sobolev embedding theorem, and Young's inequality, we get

$$\left| \frac{\varepsilon}{j+1} \int_\Omega \left(|u_t|^j u_t u + |v_t|^j v_t v \right) dx - \varepsilon \int_\Omega (\Delta u u_t + \Delta v v_t) \, dx \right|^{\frac{1}{1-\delta}}$$

$$\leq C \left(\|u\|^{\frac{1}{1-\delta}} \|u_t\|^{\frac{j+1}{1-\delta}} + \|v\|^{\frac{1}{1-\delta}} \|v_t\|^{\frac{j+1}{1-\delta}} + \|\nabla u\|^{\frac{1}{1-\delta}} \|\nabla u_t\|^{\frac{1}{1-\delta}} + \|\nabla v\|^{\frac{1}{1-\delta}} \|\nabla v_t\|^{\frac{1}{1-\delta}} \right)$$

$$\leq C \left(\|u\|_{2(p+2)}^{\frac{s}{1-\delta}} + \|u_t\|_{j+2}^{\frac{(j+1)\tau}{1-\delta}} + \|v\|_{2(p+2)}^{\frac{s}{1-\delta}} + \|v_t\|_{j+2}^{\frac{(j+1)\tau}{1-\delta}} + \|\nabla u\|_\alpha^{\frac{\lambda}{1-\delta}} \right.$$

$$\left. + \|\nabla u_t\|^{\frac{\mu}{1-\delta}} + \|\nabla v\|_\beta^{\frac{\lambda}{1-\delta}} + \|\nabla v_t\|^{\frac{\mu}{1-\delta}} \right),$$

where

$$\frac{1}{s} + \frac{1}{\tau} = 1,$$

$$\frac{1}{\lambda} + \frac{1}{\mu} = 1.$$

Set

$$\frac{(j+1)\tau}{1-\delta} = j+2, \qquad \mu = 2(1-\delta),$$

thus

$$\frac{s}{1-\delta} = \frac{j+2}{1-(j+2)\delta}, \qquad \frac{\lambda}{1-\delta} = \frac{2}{1-2\delta}.$$

Furthermore, we have

$$\|u\|_{2(p+2)}^{\frac{j+2}{1-(j+2)\delta}} \leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + H(t) \right),$$

$$\|v\|_{2(p+2)}^{\frac{j+2}{1-(j+2)\delta}} \leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + H(t) \right),$$

$$\|\nabla u\|_\alpha^{\frac{2}{1-2\delta}} \leq d \left(\|\nabla u\|_\alpha^\alpha + H(t) \right),$$

$$\|\nabla v\|_\beta^{\frac{2}{1-2\delta}} \leq d \left(\|\nabla v\|_\beta^\beta + H(t) \right)$$

and

$$\left| \frac{\varepsilon}{j+1} \int_{\Omega} \left(|u_t|^j u_t u + |v_t|^j v_t v \right) dx - \varepsilon \int_{\Omega} (\Delta u u_t + \Delta v v_t) \, dx \right|^{\frac{1}{1-\delta}}$$

$$\leq C \big(H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta} + \|\nabla u_t\|^2 + \|\nabla v_t\|^2$$

$$+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \big).$$

Thus we obtain

$$\Theta^{\frac{1}{1-\delta}}(t) \leq C \big(H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta}$$

$$+ \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \big). \tag{3.16}$$

A combination of (3.12) and (3.16) yields

$$\Theta'(t) \geq \zeta \, \Theta^{\frac{1}{1-\delta}}(t),$$

where ζ is some positive constant. A simple integration gives

$$\Theta^{\delta/(1-\delta)} \geq \frac{1}{\Theta^{-\delta/(1-\delta)}(0) - \frac{\zeta \delta t}{1-\delta}}.$$

Thus $H(t)$ blows up in a finite time, and the estimation of the blow-up time T^* obeys

$$T^* \leq \frac{1-\delta}{\zeta \delta \Theta^{\delta/(1-\delta)}(0)}.$$

This completes the proof of Theorem 3.1. $\qquad\qquad\square$

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