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Existence and multiplicity of solutions for equations involving nonhomogeneous operators of $p(x)$-Laplace type in $\mathbb{R}^N$

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Abstract

We are concerned with the following elliptic equations with variable exponents: $-\operatorname{div}(\varphi(x,\nabla u)) + |u|^{p(x)-2}u = \lambda f(x,u)$ in $\mathbb{R}^N$, where the function $\varphi(x,v)$ is of type $|v|^{p(x)-2}v$ with continuous function $p:\mathbb{R}^N \to (1,\infty)$ and $f:\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. The purpose of this paper is to show the existence of at least one solution, and under suitable assumptions, infinitely many solutions for the problem above by using mountain pass theorem and fountain theorem.

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Keywords: $p(x)$-Laplace type; variable exponent Lebesgue-Sobolev spaces; weak solution; mountain pass theorem; fountain theorem

1 Introduction

The differential equations and variational problems with $p(x)$-growth conditions have been much interest in recent years since they can model physical phenomena which arise in the study of elastic mechanics, electro-rheological fluid dynamics and image processing, *etc.* We refer the readers to [1–4] and references therein.

In this paper, we establish some results about the existence and multiplicity of nontrivial weak solution to nonlinear elliptic equations of the $p(x)$-Laplacian type,

$$-\operatorname{div}\bigl(\varphi(x,\nabla u)\bigr) + |u|^{p(x)-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{B}$$

where the function $\varphi(x,v)$ is of type $|v|^{p(x)-2}v$ with continuous function $p:\mathbb{R}^N \to (1,\infty)$ and $f:\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. The essential interest in studying problem (B) starts from the presence of the $p(x)$-Laplace type operator $\operatorname{div}(\varphi(x,\nabla u))$, which is small perturbation of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. The study for the $p(x)$-Laplacian problems has been extensively considered by several authors in various ways; see for example [5–8] and references therein. Fan and Zhang [6] established the existence of solutions for the $p(x)$-Laplacian Dirichlet problems on bounded domains by using the variational method. For the case of the entire domain $\mathbb{R}^N$, the existence and multiplicity results of solutions for the $p(x)$-Laplacian equations have been discussed in [5]. Concerning the $p(x)$-Laplace type operator, Mihăilescu and Rădulescu in [3] investigated a multiplicity result for quasilinear nonhomogeneous problems with Dirichlet boundary conditions by adequate variational methods and a variant of mountain pass theorem which are crucial tools for finding solutions to elliptic problems. In particular, in order to obtain the existence of solutions for equations like (B) in [3], they assume that the functional Φ induced by φ is uniform convex, that is, there exists a constant $k > 0$ such that

$$\Phi\left(x, \frac{\xi+\eta}{2}\right) \le \frac{1}{2}\Phi(x,\xi) + \frac{1}{2}\Phi(x,\eta) - k|\xi-\eta|^{p(x)},$$

for all $x \in \overline{\Omega}$ and $\xi, \eta \in \mathbb{R}^N$, where Ω is a bounded domain in $\mathbb{R}^N$. When $p(x) \equiv p$ and $1 < p < 2$, it is well known that this condition is not applicable for the p-Laplacian problems because the function $\Phi(x,t) = t^p$ is not uniformly convex for $t > 0$. Recently the authors in [9] have studied the existence of infinitely many solutions for a class of quasilinear elliptic problems involving the $p(x)$-Laplace type operator with nonlinear boundary conditions without using the uniform convexity of Φ.

The aim of this paper is to show the existence of at least one nontrivial solution and infinitely many nontrivial solutions for problem (B) without the assumption of the uniform convexity of the functional Φ as in [9]; see also [10]. We give our main results in a more general setting than those of [5, 6] because (B) is a problem which involves the usual $p(x)$-Laplacian operator. Especially, our proof as regards the existence of infinitely many nontrivial solutions for (B) is different from those of [5, 6, 9].

This paper is organized as follows. In Section 2, we state some basic results for the variable exponent Lebesgue-Sobolev spaces. In Section 3, under certain conditions on φ and f, we establish several existence results of nontrivial weak solutions for problem (B) by employing as the main tools the variational principle.

2 Preliminaries

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^N)$ and the variable exponent Lebesgue-Sobolev spaces $W^{1,p(\cdot)}(\mathbb{R}^N)$ which will be treated in the next sections. For a deeper treatment on these spaces, we refer to [11–13].

Set

$$C_+(\mathbb{R}^N) = \left\{h \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} h(x) > 1\right\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we define

$$h_+ = \sup_{x\in\mathbb{R}^N} h(x) \quad \text{and} \quad h_- = \inf_{x\in\mathbb{R}^N} h(x).$$

For any $p \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\mathbb{R}^N) := \left\{u : u \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} |u(x)|^{p(x)}\,dx < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The dual space of $L^{p(\cdot)}(\mathbb{R}^N)$ is $L^{p'(\cdot)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(\cdot)}(\mathbb{R}^N) = \{u \in L^{p(\cdot)}(\mathbb{R}^N) : |\nabla u| \in L^{p(\cdot)}(\mathbb{R}^N)\},$$

where the norm is

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}. \tag{2.1}$$

It has the following equivalent norm:

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Lemma 2.1 ([11]) *The space $L^{p(\cdot)}(\mathbb{R}^N)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\mathbb{R}^N)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $v \in L^{p'(\cdot)}(\mathbb{R}^N)$, we have*

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \le 2\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{p'(\cdot)}(\mathbb{R}^N)}.$$

Lemma 2.2 *If $1/p(x) + 1/q(x) + 1/r(x) = 1$, then for any $u \in L^{p(\cdot)}(\mathbb{R}^N)$, $v \in L^{q(\cdot)}(\mathbb{R}^N)$, and $w \in L^{r(\cdot)}(\mathbb{R}^N)$,*

$$\begin{aligned}\left|\int_{\mathbb{R}^N} uvw\,dx\right| &\le \left(\frac{1}{p_-} + \frac{1}{q_-} + \frac{1}{r_-}\right)\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{q(\cdot)}(\mathbb{R}^N)}\|w\|_{L^{r(\cdot)}(\mathbb{R}^N)} \\ &\le 3\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{q(\cdot)}(\mathbb{R}^N)}\|w\|_{L^{r(\cdot)}(\mathbb{R}^N)}.\end{aligned}$$

Lemma 2.3 ([11]) *Denote*

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}\,dx, \quad \textit{for all } u \in L^{p(\cdot)}(\mathbb{R}^N).$$

Then

(1) *$\rho(u) > 1$ $(= 1; < 1)$ if and only if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, respectively;*
(2) *if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$, then $\|u\|^{p_-}_{L^{p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(\cdot)}(\mathbb{R}^N)}$;*
(3) *if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1$, then $\|u\|^{p_+}_{L^{p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(\cdot)}(\mathbb{R}^N)}$.*

Remark 2.4 ([5]) Denote

$$\rho(u) = \int_{\mathbb{R}^N} \left(|u|^{p(x)} + |\nabla u|^{p(x)}\right) dx, \quad \text{for all } u \in X.$$

Then

(1) $\rho(u) > 1$ $(= 1; < 1)$ if and only if $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, respectively;
(2) if $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} > 1$, then $\|u\|^{p_-}_{W^{1,p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_+}_{W^{1,p(\cdot)}(\mathbb{R}^N)}$;
(3) if $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} < 1$, then $\|u\|^{p_+}_{W^{1,p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_-}_{W^{1,p(\cdot)}(\mathbb{R}^N)}$.

Lemma 2.5 ([13]) *Let $q \in L^\infty(\mathbb{R}^N)$ be such that $1 \le p(x)q(x) \le \infty$, for almost all $x \in \mathbb{R}^N$. If $u \in L^{q(\cdot)}(\mathbb{R}^N)$ with $u \neq 0$, then*

(1) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} > 1$, *then* $\|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le \|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)}$;
(2) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} < 1$, *then* $\|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le \|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)}$.

Lemma 2.6 ([12, 14]) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and let $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$. If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

for all $x \in \Omega$, then we have

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

Lemma 2.7 ([14]) *Suppose that $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with $1 < p_- \le p_+ < N$. Let $q \in L^\infty(\mathbb{R}^N)$ and $p(x) \le q(x) \le p^*(x)$, for almost all $x \in \mathbb{R}^N$. Then there is a continuous embedding $W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$.*

In what follows, let $p \in C_+(\mathbb{R}^N)$ be Lipschitz continuous with $1 < p_- \le p_+ < N$. We denote by the space $X := W^{1,p(\cdot)}(\mathbb{R}^N)$, and X^* be a dual space of X. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^* and Euclidean scalar product on $\mathbb{R}^N$, respectively.

3 Existence of solutions

In this section, we shall give the proof of the existence of nontrivial weak solutions for problem (B), by applying the mountain pass theorem, fountain theorem, and the basic properties of the spaces $L^{p(\cdot)}(\mathbb{R}^N)$ and $W^{1,p(\cdot)}(\mathbb{R}^N)$.

Definition 3.1 We say that $u \in X$ is a weak solution of problem (B) if

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uv \, dx = \lambda \int_{\mathbb{R}^N} f(x, u) v \, dx,$$

for all $v \in X$.

We assume that $\varphi(x, v) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to v of the mapping $\Phi_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The equalities

$$\Phi_0(x, 0) = 0 \quad \text{and} \quad \Phi_0(x, v) = \Phi_0(x, -v)$$

hold, for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J2) $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions: $\varphi(\cdot, v)$ is measurable, for all $v \in \mathbb{R}^N$, and $\varphi(x, \cdot)$ is continuous, for almost all $x \in \mathbb{R}^N$.

(J3) There are a function $a \in L^{p'(\cdot)}(\mathbb{R}^N)$ and a nonnegative constant b such that

$$\left|\varphi(x,v)\right| \le a(x) + b|v|^{p(x)-1},$$

for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J4) $\Phi_0(x,\cdot)$ is strictly convex in $\mathbb{R}^N$, for all $x \in \mathbb{R}^N$.

(J5) The relation

$$d|v|^{p(x)} \le \varphi(x,v)\cdot v \le p_+\Phi_0(x,v)$$

holds, for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where d is a positive constant.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)}|u|^{p(x)}\,dx.$$

The analog of the following lemma can be found in [3]. However, we will give the proof of those because our growth condition is slightly different from that of [3].

Lemma 3.2 *Assume that* (J1)-(J3) *and* (J5) *hold. Then the functional* Φ *is well defined on* X, $\Phi \in C^1(X,\mathbb{R})$ *and its Fréchet derivative is given by*

$$\left\langle \Phi'(u),v\right\rangle = \int_{\mathbb{R}^N} \varphi(x,\nabla u)\cdot\nabla v\,dx + \int_{\mathbb{R}^N} |u|^{p(x)-2}uv\,dx. \tag{3.1}$$

Proof A simple calculation as in [3] implies that the functional Φ is well defined on X. For a fixed $x \in \mathbb{R}^N$, it is clear that $\Phi_0 \in C^1(\mathbb{R}^N,\mathbb{R})$. Let $u,v \in X$, then given $x \in \mathbb{R}^N$ and $0 < |t| < 1$, by the classical mean value theorem, there exist $\theta_1,\theta_2 \in \mathbb{R}$ with $0 < |\theta_1| < |t|$ and $0 < |\theta_2| < |t|$ such that

$$\left|\frac{\Phi_0(x,\nabla u + t\nabla v) - \Phi_0(x,\nabla u)}{t}\right| = \left|\varphi(x,\nabla u + \theta_1\nabla v)\cdot\nabla v\right|$$

and

$$\left|\frac{1}{p(x)}\frac{|u+tv|^{p(x)} - |u|^{p(x)}}{t}\right| = \left||u+\theta_2 v|^{p(x)-2}(u+\theta_2 v)v\right|.$$

Since

$$\begin{aligned}\left|\varphi(x,\nabla u + \theta_1\nabla v)\cdot\nabla v\right| &\le \left|a(x) + b|\nabla u + \theta_1\nabla v|^{p(x)-1}\right||\nabla v| \\ &\le \left|a(x) + b\big(|\nabla u| + |\nabla v|\big)^{p(x)-1}\right||\nabla v|,\end{aligned}$$

it is easy to obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \left|a(x) + b\big(|\nabla u| + |\nabla v|\big)^{p(x)-1}\right||\nabla v|\,dx \\ &\quad\le 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^N)} + b\int_{\mathbb{R}^N}\left|\big(|\nabla u| + |\nabla v|\big)^{p(x)-1}|\nabla v|\right|dx\end{aligned}$$

$$
\begin{aligned}
&\le 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^N)} + 2b\big\|\big(|\nabla u| + |\nabla v|\big)^{p(x)-1}\big\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^N)} \\
&\le 2C\left[\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)} + b\left\{1 + \left(\int_{\mathbb{R}^N}\big(|\nabla u| + |\nabla v|\big)^{p(x)}\,dx\right)^{\frac{1}{(p')_-}}\right\}\right]\|v\|_X \\
&\le 2C\left[\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)} + b\left\{1 + 2^{\frac{p_+-1}{(p')_-}}\left(\int_{\mathbb{R}^N}\big(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\big)\,dx\right)^{\frac{1}{(p')_-}}\right\}\right]\|v\|_X
\end{aligned}
$$

for a positive constant C. Hence $|a+b(|\nabla u|+|\nabla v|)^{p(\cdot)-1}||\nabla v| \in L^1(\mathbb{R}^N)$. Since $t \to 0$ implies $\theta_1 \to 0$ and $\theta_2 \to 0$, it follows from the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\big(\Phi(u+tv) - \Phi(u)\big) &= \lim_{t\to 0}\int_{\mathbb{R}^N}\frac{\Phi_0(x,\nabla u + t\nabla v) - \Phi_0(x,\nabla u)}{t}\,dx \\
&\quad + \lim_{t\to 0}\int_{\mathbb{R}^N}\frac{1}{p(x)}\,\frac{|u+tv|^{p(x)} - |u|^{p(x)}}{t}\,dx \\
&= \int_{\mathbb{R}^N}\lim_{\theta_1\to 0}\varphi(x,\nabla u + \theta_1\nabla v)\cdot\nabla v\,dx \\
&\quad + \int_{\mathbb{R}^N}\lim_{\theta_2\to 0}\big(|u+\theta_2 v|^{p(x)-2}(u+\theta_2 v)v\big)\,dx \\
&= \int_{\mathbb{R}^N}\varphi(x,\nabla u)\cdot\nabla v\,dx + \int_{\mathbb{R}^N}|u|^{p(x)-2}uv\,dx \\
&= \big\langle\Phi'(u), v\big\rangle.
\end{aligned}
$$

Let $\Lambda_1 : X \to L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)$ and $\Lambda_2 : X \to L^{p'(\cdot)}(\mathbb{R}^N)$ be an operators defined by

$$
\Lambda_1(u)(x) := \varphi\big(x, \nabla u(x)\big) \quad\text{and}\quad \Lambda_2(u)(x) := \big|u(x)\big|^{p(x)-2}u(x).
$$

Then the operators Λ_1 and Λ_2 are continuous on X. In fact, for any $u \in X$, let $u_n \to u$ in X as $n \to \infty$. Then there exist a subsequence $\{u_{n_k}\}$ and functions v, w_j in $L^{p(\cdot)}(\mathbb{R}^N)$ for $j = i,\dots,N$ such that $u_{n_k}(x) \to u(x)$ as $k \to \infty$, and $|u_{n_k}(x)| \le v(x)$ and $|(\partial u_{n_k}/\partial x_j)(x)| \le w_j(x)$, for all $k \in \mathbb{N}$ and for almost all $x \in \mathbb{R}^N$. Without loss of generality, we assume that $\|\Lambda_1(u_{n_k}) - \Lambda_1(u)\|_{L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)} < 1$ and $\|\Lambda_2(u_{n_k}) - \Lambda_2(u)\|_{L^{p'(\cdot)}(\mathbb{R}^N)} < 1$. Then we have

$$
\big\|\Lambda_1(u_{n_k}) - \Lambda_1(u)\big\|^{(p')_+}_{L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)} \le \int_{\mathbb{R}^N}\big|\varphi(x,\nabla u_{n_k}) - \varphi(x,\nabla u)\big|^{p'(x)}\,dx
$$

and

$$
\big\|\Lambda_2(u_{n_k}) - \Lambda_2(u)\big\|^{(p')_+}_{L^{p'(\cdot)}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N}\big|\big|u_{n_k}(x)\big|^{p(x)-2}u_{n_k}(x) - \big|u(x)\big|^{p(x)-2}u(x)\big|^{p'(x)}\,dx.
$$

Hence (J3) implies that the integrands at the right-hand sides in the above estimates are dominated by integrable functions. Since the function φ satisfies (J2) and $u_{n_k} \to u$ in X as $k \to \infty$, we obtain $\varphi(x,\nabla u_{n_k}(x)) \to \varphi(x,\nabla u(x))$ and $|u_{n_k}(x)|^{p(x)-2}u_{n_k}(x) \to |u(x)|^{p(x)-2}u(x)$ as $k \to \infty$, for almost all $x \in \mathbb{R}^N$. Therefore, the Lebesgue dominated convergence theorem tells us that $\Lambda_1(u_{n_k}) \to \Lambda_1(u)$ in $L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)$ and $\Lambda_2(u_{n_k}) \to \Lambda_2(u)$ in $L^{p'(\cdot)}(\mathbb{R}^N)$ as

$k \to \infty$. Thus, Λ_1 and Λ_2 are continuous on X. From the Hölder inequality, we have

$$\begin{aligned}
&\left|\left\langle \Phi'(u_n) - \Phi'(u), v\right\rangle\right| \\
&\quad = \left|\int_{\mathbb{R}^N} \left(\varphi(x, \nabla u_n) - \varphi(x, \nabla u)\right) \cdot \nabla v\, dx + \int_{\mathbb{R}^N} \left(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u\right) v\, dx\right| \\
&\quad \le 2\left\| \varphi(x, \nabla u_n) - \varphi(x, \nabla u)\right\|_{L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)} \|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^N)} \\
&\qquad + 2\left\| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u\right\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{p(\cdot)}(\mathbb{R}^N)},
\end{aligned}$$

for all $v \in X$, and thus

$$\begin{aligned}
\left\|\Phi'(u_n) - \Phi'(u)\right\|_{X^*} &= \sup_{\|v\|_X \le 1} \left|\left\langle \Phi'(u_n) - \Phi'(u), v\right\rangle\right| \\
&\le 2\Big\{ \left\| \varphi(x, \nabla u_n) - \varphi(x, \nabla u)\right\|_{L^{p'(\cdot)}(\mathbb{R}^N;\mathbb{R}^N)} \\
&\quad + \left\| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u\right\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \Big\} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Consequently, the operator Φ' is continuous on X. □

Now we will show that the operator Φ' is a mapping of type (S_+), which plays a key role in obtaining our main results. To do this, we first prove the following useful result.

Lemma 3.3 *Assume that* (J1)-(J5) *hold. If the sequence* $\{v_n\}$ *in* $\mathbb{R}^N$ *such that*

$$\left\langle \varphi(x, v_n) - \varphi(x, v), v_n - v\right\rangle \to 0$$

as $n \to \infty$, *for* $v \in \mathbb{R}^N$ *and for almost all* $x \in \mathbb{R}^N$, *then* $v_n \to v$ *in* $\mathbb{R}^N$ *as* $n \to \infty$.

Proof Let $\{v_{n_k}\}$ be a subsequence of the sequence $\{v_n\}$ in $\mathbb{R}^N$ satisfied

$$\left\langle \varphi(x, v_{n_k}) - \varphi(x, v), v_{n_k} - v\right\rangle \to 0 \tag{3.2}$$

as $k \to \infty$ for any $v \in \mathbb{R}^N$. Then there exists $M > 0$ such that, for almost all $x \in \mathbb{R}^N$,

$$\left\langle \varphi(x, v_{n_k}), v_{n_k}\right\rangle \le M + \left|\varphi(x, v_{n_k})\right||v| + \left|\varphi(x, v)\right||v_{n_k}| + \left|\varphi(x, v)\right||v|.$$

This together with assumptions (J3), (J5), and Young?s inequality imply that

$$\begin{aligned}
d|v_{n_k}|^{p(x)} &\le \left\langle \varphi(x, v_{n_k}), v_{n_k}\right\rangle \\
&\le M + \left|\varphi(x, v_{n_k})\right||v| + \left|\varphi(x, v)\right||v_{n_k}| + \left|\varphi(x, v)\right||v| \\
&\le M + \left(a(x) + b|v_{n_k}|^{p(x)-1}\right)|v| + \left|\varphi(x, v)\right||v_{n_k}| + \left|\varphi(x, v)\right||v| \\
&\le M + a(x)|v| + \frac{d}{3}|v_{n_k}|^{p(x)} + b^{p(x)}\left(\frac{3}{d}\right)^{\frac{p(x)}{p'(x)}} |v|^{p(x)} \\
&\quad + \frac{d}{3}|v_{n_k}|^{p(x)} + \left(\frac{3}{d}\right)^{\frac{p'(x)}{p(x)}} \left|\varphi(x, v)\right|^{p'(x)} + \left|\varphi(x, v)\right||v|,
\end{aligned}$$

for almost all $x \in \mathbb{R}^N$, and hence

$$\frac{d}{3}|v_{n_k}|^{p(x)} \leq M + a(x)|v| + b^{p(x)}\left(\frac{3}{d}\right)^{\frac{p(x)}{p'(x)}}|v|^{p(x)} + \left(\frac{3}{d}\right)^{\frac{p'(x)}{p(x)}}\left|\varphi(x,v)\right|^{p'(x)} + \left|\varphi(x,v)\right||v|,$$

for almost all $x \in \mathbb{R}^N$, where d is the positive constant from (J5). Since $d > 0$, the sequence $\{|v_{n_k}|\}$ is bounded, and then the sequence $\{v_{n_k}\}$ is bounded in $\mathbb{R}^N$. By passing to a subsequence, we can assume that $v_{n_k} \to \xi$ as $k \to \infty$, for some $\xi \in \mathbb{R}^N$. Then we obtain $\varphi(x, v_{n_k}) \to \varphi(x,\xi)$ as $k \to \infty$ and the relation (3.2) implies that

$$0 = \lim_{k\to\infty}\langle\varphi(x,v_{n_k}) - \varphi(x,v), v_{n_k} - v\rangle = \langle\varphi(x,\xi) - \varphi(x,v), \xi - v\rangle.$$

Since it follows from assumption (J4) and Proposition 25.10 in [15] that φ is monotone on X, this relation occurs only if $\xi = v$, that is, $v_n \to v$ in $\mathbb{R}^N$ as $n \to \infty$. Since these arguments hold for any subsequence of the sequence $\{v_n\}$, we conclude that $v_n \to v$ in $\mathbb{R}^N$ as $n \to \infty$. □

Next we give the following assertion, which is based on the idea of the proof in [16]; see [17] for the case of bounded domain in $\mathbb{R}^N$.

Lemma 3.4 *Assume that* (J1)-(J5) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle\Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* X *as* $n \to \infty$.

Proof Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and

$$\limsup_{n\to\infty}\langle\Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0. \tag{3.3}$$

It follows from $u_n \rightharpoonup u$ in X as $n \to \infty$ that $\langle\Phi'(u), u_n - u\rangle \to 0$ as $n \to \infty$. Since Φ is strictly convex by (J4), it is obvious that the operator Φ' is monotone, that is,

$$\langle\Phi'(u_n) - \Phi'(u), u_n - u\rangle \geq 0. \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\langle\varphi(x,\nabla u_n) - \varphi(x,\nabla u), \nabla u_n - \nabla u\rangle\,dx \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\,dx \\
&\quad = \lim_{n\to\infty}\langle\Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0.
\end{aligned}$$

Hence the sequences $\{\langle\varphi(x,\nabla u_n) - \varphi(x,\nabla u), \nabla u_n - \nabla u\rangle\}$ and $\{(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)\}$ converge to 0 in $L^1(\mathbb{R}^N;\mathbb{R}^N)$ and $L^1(\mathbb{R}^N)$ as $n \to \infty$, respectively. By Lemma 3.3, we have $\nabla u_n(x) \to \nabla u(x)$ in $\mathbb{R}^N$ and $u_n(x) \to u(x)$ in $\mathbb{R}$ as $n \to \infty$, for almost all $x \in \mathbb{R}^N$. Then (3.3) holds in the stronger form

$$\lim_{n\to\infty}\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle = 0. \tag{3.5}$$

It follows from the convexity of Φ that

$$\Phi(u)+\langle \Phi'(u_n),u_n-u\rangle \geq \Phi(u_n),$$

and hence we obtain $\Phi(u)\geq \limsup_{n\to\infty}\Phi(u_n)$ by (3.5). Since the functional Φ is strictly convex and C^1-functional on X, it follows that Φ is weakly lower semicontinuous on X. Then it is immediate that $\Phi(u)\leq \liminf_{n\to\infty}\Phi(u_n)$. Thus it implies

$$\Phi(u)=\lim_{n\to\infty}\Phi(u_n). \tag{3.6}$$

Consider the sequence $\{h_n\}$ in $L^1(\mathbb{R}^N)$ defined pointwise by

$$\begin{aligned} h_n(x) &= \frac{1}{2}\big\{\Phi_0(x,\nabla u_n)+\Phi_0(x,\nabla u)\big\}-\Phi_0\left(x,\frac{\nabla u_n-\nabla u}{2}\right) \\ &\quad+\frac{1}{2p(x)}\big(\left|u_n(x)\right|^{p(x)}+\left|u(x)\right|^{p(x)}\big)-\frac{1}{p(x)}\left|\frac{u_n(x)-u(x)}{2}\right|^{p(x)}. \end{aligned}$$

From (J1) and (J4), it is clear that the sequence $h_n\geq 0$. Since $\Phi_0(x,\cdot)$ is continuous, for almost all $x\in\mathbb{R}^N$, we obtain $h_n(x)\to\Phi_0(x,\nabla u)+(1/p(x))|u(x)|$ as $n\to\infty$, for almost all $x\in\mathbb{R}^N$. Hence, by the Fatou lemma and (3.6), we have

$$\begin{aligned} \Phi(u) &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} h_n(x)\,dx \\ &= \Phi(u)-\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\Phi_0\left(x,\frac{\nabla u_n-\nabla u}{2}\right)+\frac{1}{p(x)}\left|\frac{u_n-u}{2}\right|^{p(x)}\right)dx. \end{aligned}$$

Thus we get

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\Phi_0\left(x,\frac{\nabla u_n-\nabla u}{2}\right)+\frac{1}{p(x)}\left|\frac{u_n-u}{2}\right|^{p(x)}\right)dx\leq 0,$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\Phi_0\left(x,\frac{\nabla u_n-\nabla u}{2}\right)+\frac{1}{p(x)}\left|\frac{u_n-u}{2}\right|^{p(x)}\right)dx = 0.$$

Then by assumption (J5), $\lim_{n\to\infty}\|u_n-u\|_X=0$, we conclude that $u_n\to u$ in X as $n\to\infty$. □

Until now, we considered some properties for the integral operator corresponding to the divergence part in problem (B). To deal with our main results in this section, we need the following assumptions for f. Denoting $F(x,t)=\int_0^t f(x,s)\,ds$, we assume that

(H1) $p,q\in C_+(\mathbb{R}^N)$, $p(x)<N$, and $1<p_-\leq p_+<q_-\leq q_+<p^*(x)$, for all $x\in\mathbb{R}^N$.

(H2) $m \in L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for some $r \in C_+(\mathbb{R}^N)$ with $q(x) < r(x) < p^*(x)$ and $\operatorname{meas}\{x \in \mathbb{R}^N : m(x) > 0\} > 0$, for all $x \in \mathbb{R}^N$.

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot, t)$ is measurable, for all $t \in \mathbb{R}$, and $f(x, \cdot)$ is continuous, for almost all $x \in \mathbb{R}^N$.

(F2) f satisfies the following growth condition: For all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\left|f(x,t)\right| \le \left|m(x)\right| |t|^{q(x)-1},$$

where q and m are given in (H1) and (H2), respectively.

(F3) There exists a positive constant θ such that $\theta > p_+$ and

$$0 < \theta F(x,t) \le f(x,t)t, \quad \text{for all } t \in \mathbb{R} \setminus \{0\} \text{ and } x \in \mathbb{R}^N.$$

(F4) $f(x,t) = o(|t|^{p_+-1})$, as $|t| \to 0$ uniformly, for all $x \in \mathbb{R}^N$.

Then it follows from assumption (F2) that

(F2′) $|F(x,t)| \le \frac{|m(x)|}{q(x)} |t|^{q(x)}$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Define the functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Then it is easy to check that $\Psi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\,dx \tag{3.7}$$

for any $u, v \in X$. Next we define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Then it follows that the functional $I_\lambda \in C^1(X, \mathbb{R}^N)$ and its Fréchet derivative is

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v\,dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uv\,dx - \lambda \int_{\mathbb{R}^N} f(x,u)v\,dx$$

for any $u, v \in X$.

Lemma 3.5 *Assume that* (H1)-(H2) *and* (F2) *hold. Then* Ψ *and* Ψ' *are weakly strongly continuous on* X.

Proof Proceeding the argument analogous to Lemma 3.2 of [5], it implies that the functionals Ψ and Ψ' are weakly strongly continuous on X. □

With the aid of Lemma 3.5, we prove that the energy functional I_λ satisfies the Palais-Smale condition ((PS)-condition for short). This plays a key role in obtaining the existence of a nontrivial weak solution for the given problem.

Lemma 3.6 *Assume that* (J1)-(J5), (H1)-(H2), *and* (F1)-(F3) *hold. Then* I_λ *satisfies the* (*PS*)-*condition, for all* $\lambda > 0$.

Proof Note that Ψ' is of the type (S_+), since Ψ' is weakly strongly continuous. Let $\{u_n\}$ be a (PS)-sequence in X, *i.e.*, $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Since I'_λ is of type (S_+) and X is reflexive, it suffices to verify that the sequence $\{u_n\}$ is bounded in X. Suppose that $\|u_n\|_X \to \infty$, in the subsequence sense. By assumption (J5), we deduce that

$$\begin{aligned}
I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle &= \int_{\mathbb{R}^N}\left(\Phi_0(x,\nabla u_n) - \frac{1}{\theta}\varphi(x,\nabla u_n)\cdot\nabla u_n\right)dx \\
&\quad + \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|u|^{p(x)} - \frac{1}{\theta}|u|^{p(x)}\right)dx \\
&\quad + \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\geq \left(1-\frac{p_+}{\theta}\right)\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla u_n)\,dx + \int_{\mathbb{R}^N}\frac{1}{p(x)}|u|^{p(x)}\,dx\right) \\
&\quad + \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx,
\end{aligned}$$

where θ is a positive constant from (F3). By condition (F3), we have

$$\left(1-\frac{p_+}{\theta}\right)\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla u_n)\,dx + \int_{\mathbb{R}^N}\frac{1}{p(x)}|u|^{p(x)}\,dx\right) \leq I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle.$$

For n large enough, we may assume that $\|u_n\|_X > 1$. Then it follows from (J5) and Remark 2.4(2) that

$$\left(1-\frac{p_+}{\theta}\right)\frac{1}{p_+}\|u_n\|_X^{p_-} \leq I_\lambda(u_n) + \frac{1}{\theta}\left\|I'_\lambda(u_n)\right\|_{X^*}\|u_n\|_X.$$

Since $\theta > p_+$ and $p_- > 1$, this is a contradiction. □

We are now prepared to prove our main results for the existence of at least one solution and infinitely many solutions for problem (B), following the basic idea in [10]. The following consequence can be established by applying the mountain pass theorem with Lemmas 3.4 and 3.6.

Theorem 3.7 *Assume that* (J1)-(J5), (H1)-(H2) *and* (F1)-(F4) *hold. Then problem* (B) *has a nontrivial weak solution, for all* $\lambda > 0$.

Proof Note that $I_\lambda(0) = 0$. Since I_λ satisfies the (PS)-condition, it is enough to show the geometric conditions in the mountain pass theorem, *i.e.*,

(1) there is a positive constant R such that

$$\inf_{\|u\|_X = R} I_\lambda(u) > 0;$$

(2) there exists an element v in X satisfying

$$I_\lambda(tv) \to -\infty \quad \text{as } t \to \infty.$$

Let us prove the condition (1). By Lemma 2.7, there exists a positive constant d_1 such that $\|u\|_{L^{p_+}(\mathbb{R}^N)} \leq d_1\|u\|_X$. Let $\varepsilon > 0$ be small enough such that $\lambda\varepsilon d_1^{p_+} \leq \min\{d,1\}/(2p_+)$ for the positive constant d from (J5). By assumptions (F2) and (F4), for any $\varepsilon > 0$, there exists a positive constant denoted by $C(\varepsilon)$ such that

$$\left|F(x,t)\right| \leq \varepsilon|t|^{p_+} + C(\varepsilon)\left|m(x)\right||t|^{q(x)},$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Assume that $\|u\|_X < 1$. Then it follows from (J5) and Lemmas 2.1, 2.5(2), 2.7, and Remark 2.4 that

$$\begin{aligned}
I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)}|u|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq \frac{\min\{d,1\}}{p_+}\left(\int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N} |u|^{p(x)}\,dx\right) \\
&\quad - \lambda\int_{\mathbb{R}^N}\left(\varepsilon|u|^{p_+} + C(\varepsilon)\left|m(x)\right||u|^{q(x)}\right)dx \\
&\geq \frac{\min\{d,1\}}{p_+}\|u\|_X^{p_+} - \lambda\varepsilon d_1^{p_+}\|u\|_X^{p_+} - 2\lambda C(\varepsilon)\|m\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\|u\|_{L^{r(\cdot)}(\mathbb{R}^N)}^{q_-} \\
&\geq \frac{\min\{d,1\}}{p_+}\|u\|_X^{p_+} - \lambda\varepsilon d_1^{p_+}\|u\|_X^{p_+} - 2\lambda C(\varepsilon)C_1\|u\|_X^{q_-}
\end{aligned}$$

for a positive constant C_1. Then it follows that

$$I_\lambda(u) \geq \frac{\min\{d,1\}}{2p_+}\|u\|_X^{p_+} - C(\lambda,\varepsilon)C_1\|u\|_X^{q_-}.$$

Since $q_- > p_+$, there exist $R > 0$ small enough and $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ when $\|u\|_X = R$.

Next we show the condition (2). Meanwhile, observe that (J4) implies that, for all $s \geq 1$, $x \in \mathbb{R}^N$, and $\xi \in \mathbb{R}^N$,

$$\Phi_0(x,s\xi) \leq s^{p_+}\Phi_0(x,\xi). \tag{3.8}$$

Indeed, let us define $g(k) = \Phi_0(x,k\xi)$. Then we have

$$g'(k) = \varphi(x,k\xi)\xi = \frac{1}{k}\varphi(x,k\xi)\cdot k\xi \leq \frac{p_+}{k}\Phi_0(x,k\xi) = \frac{p_+}{k}g(k).$$

It implies that

$$\frac{g'(k)}{g(k)} \leq \frac{p_+}{k}.$$

Integrating this inequality over $(1,s)$, we have

$$\ln g(s) - \ln g(1) \leq p_+ \ln s$$

and so

$$\frac{g(s)}{g(1)} \leq s^{p_+}.$$

Hence we find that (3.8) holds. In a similar way, we find that condition (F3) implies

$$F(x,s\eta) \geq s^{\theta} F(x,\eta), \tag{3.9}$$

for all $\eta \in \mathbb{R}$, $x \in \mathbb{R}^N$, and $s \geq 1$.

Take $v \in X \setminus \{0\}$. Then it follows from (3.8) and (3.9) that

$$\begin{aligned} I_\lambda(tv) &= \int_{\mathbb{R}^N} \Phi_0(x, t\nabla v)\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)}|tv|^{p(x)}\,dx - \lambda \int_{\mathbb{R}^N} F(x,tv)\,dx \\ &\leq t^{p_+}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla v)\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)}|v|^{p(x)}\,dx\right) - \lambda t^{\theta} \int_{\mathbb{R}^N} F(x,v)\,dx, \end{aligned}$$

where $t \geq 1$. Since $\theta > p_+$, we see that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. Therefore I_λ satisfies the geometry of the mountain pass theorem. □

Now, adding the oddity on f and using the fountain theorem in Theorem 3.6 in [18], we shall demonstrate infinitely many pairs of weak solutions for problem (B). To employ the fountain theorem, we consider the following situation. This lemma holds for a reflexive and separable Banach space.

Lemma 3.8 ([6]) *Let W be a reflexive and separable Banach space. Then there are $\{e_n\} \subseteq W$ and $\{f_n^*\} \subseteq W^*$ such that*

$$W = \overline{\operatorname{span}\{e_n : n = 1,2,\ldots\}}, \qquad W^* = \overline{\operatorname{span}\{f_n^* : n = 1,2,\ldots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $W_n = \operatorname{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^{k} W_n$, and $Z_k = \overline{\bigoplus_{n=k}^{\infty} W_n}$.

Theorem 3.9 *Assume that* (J1)-(J5), (H1)-(H2), *and* (F1)-(F4) *hold. If $f(x,-t) = -f(x,t)$ holds, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, then I_λ has a sequence of critical points $\{\pm u_n\}$ in X such that $I_\lambda(\pm u_n) \to \infty$ as $n \to \infty$.*

Proof Obviously, I_λ is an even functional and satisfies (PS)-condition. It is enough to show that there exist $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ as $n \to \infty$;

(2) $a_k := \max\{I_\lambda(u) : u \in Y_k, \|u\|_X = \rho_k\} \leq 0$,

for k large enough.

Denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_X = 1} \left(\int_{\mathbb{R}^N} \frac{1}{r(x)}|u|^{r(x)}\,dx\right).$$

Then $\alpha_k \to 0$ as $k \to \infty$. In fact, suppose that it is false. Then there exist $\varepsilon_0 > 0$ and the sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_X = 1, \qquad \int_{\mathbb{R}^N} \frac{1}{r(x)} |u_k|^{r(x)}\, dx \geq \varepsilon_0,$$

for all $k \geq k_0$. Since the sequence $\{u_k\}$ is bounded in X, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X as $n \to \infty$ and

$$\langle f_j^*, u \rangle = \lim_{k\to\infty} \langle f_j^*, u_k \rangle = 0$$

for $j = 1, 2, \dots$. Hence we get $u = 0$. But we have

$$\varepsilon_0 \leq \lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{1}{r(x)} |u_k|^{r(x)}\, dx = \int_{\mathbb{R}^N} \frac{1}{r(x)} |u|^{r(x)}\, dx = 0,$$

which provides a contradiction.

For any $u \in Z_k$, it follows from (F2′), (J5) and Lemmas 2.1 and 2.7, and Remark 2.4 that

$$\begin{aligned} I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)}\, dx - \lambda \int_{\mathbb{R}^N} F(x,u)\, dx \\ &\geq \frac{\min\{d,1\}}{p_+} \left(\int_{\mathbb{R}^N} |\nabla u|^{p(x)}\, dx + \int_{\mathbb{R}^N} |u|^{p(x)}\, dx \right) - \frac{\lambda}{q_-} \int_{\mathbb{R}^N} |m(x)| |u|^{q(x)}\, dx \\ &\geq \frac{\min\{d,1\}}{p_+} \|u\|_X^{p_-} - \frac{2\lambda}{q_-} \|m\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} \|u\|_{L^{r(\cdot)}(\mathbb{R}^N)}^{q_+} \\ &\geq \frac{\min\{d,1\}}{p_+} \|u\|_X^{p_-} - \frac{2\lambda}{q_-} C \|u\|_{L^{r(\cdot)}(\mathbb{R}^N)}^{q_+} \\ &\geq \frac{\min\{d,1\}}{p_+} \|u\|_X^{p_-} - \frac{2\lambda}{q_-} \alpha_k^{q_+} C_2 \|u\|_X^{q_+}, \end{aligned} \tag{3.10}$$

where $C = \|m\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}$ and C_2 is a positive constant. Choose $\delta_k = (2\lambda q_+ C_2 \alpha_k^{q_+} / (q_- \min\{d,1\}))^{\frac{1}{p_- - q_+}}$. Then $\delta_k \to \infty$ as $k \to \infty$ since $p_- < q_+$ and $\alpha_k \to 0$ as $k \to \infty$. Hence, if $u \in Z_k$ and $\|u\|_X = \delta_k$, we deduce that

$$I_\lambda(u) \geq \min\{d,1\} \left(\frac{1}{p_+} - \frac{1}{q_+} \right) \delta_k^{p_-} - C_3 \to \infty \quad \text{as } k \to \infty,$$

for a positive constant C_3, which implies (1).

To show (2), from (F4) we see that, for $\varepsilon = 1$, there exists $\delta > 0$ such that

$$f(x,s) \leq |s|^{p_+ - 1}, \tag{3.11}$$

for all $x \in \mathbb{R}^N$ and for all $|s| < \delta$. Then we know that there exists $\omega \in L^\infty(\mathbb{R}^N)$ such that $\omega(x) > 0$, for almost all $x \in \mathbb{R}^N$, and

$$F(x,s) \geq \omega(x)|s|^\theta, \tag{3.12}$$

for almost all $x \in \mathbb{R}^N$ and all $|s| \geq \delta$. In fact, by (F3), we have, for all $t \geq \delta$,

$$\frac{\theta}{t} \leq \frac{f(x,t)}{F(x,t)} = \frac{\frac{d}{dt}F(x,t)}{F(x,t)}.$$

Then it follows that

$$\int_\delta^s \frac{\theta}{t}\,dt \leq \int_\delta^s \frac{\frac{d}{dt}F(x,t)}{F(x,t)}\,dt,$$

and thus

$$\ln\left(\frac{s}{\delta}\right)^\theta \leq \ln\frac{F(x,s)}{F(x,\delta)}.$$

Hence we get

$$F(x,s) \geq \frac{s^\theta}{\delta^\theta}F(x,\delta).$$

Similarly, we obtain

$$F(x,s) \geq \frac{|s|}{\delta}F(x,-\delta),$$

for all $s \leq -\delta$. Thus, $F(x,s) \geq \omega(x)|s|^\theta$, for almost all $x \in \mathbb{R}^N$ and all $|s| \geq \delta$, where $\omega(x) = \min\{F(x,\delta)/\delta^\theta, F(x,-\delta)/\delta^\theta\}$. Also assumptions (F2) and (F3) imply that $\omega \in L^\infty(\mathbb{R}^N)$ and $\omega > 0$.

Assume that $\|u\|_X > 1$. For any $u \in Y_k$, by (J5), (F3), (3.11), (3.12), Lemmas 2.1, 2.3, 2.7, and Remark 2.4, we have

$$\begin{aligned}
I_\lambda(u) &= \int_{\mathbb{R}^N}\Phi_0(x,\nabla u)\,dx + \int_{\mathbb{R}^N}\frac{1}{p(x)}|u|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N}F(x,u)\,dx \\
&\leq \int_{\mathbb{R}^N}\big|a(x)\big||\nabla u|\,dx + \frac{b}{p_-}\int_{\mathbb{R}^N}|\nabla u|^{p(x)}\,dx + \frac{1}{p_-}\int_{\mathbb{R}^N}|u|^{p(x)}\,dx \\
&\quad - \lambda\left(\int_{\{x\in\mathbb{R}^N:|u(x)|\geq\delta\}}\omega(x)|u|^\theta\,dx - \int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}}\frac{1}{p_+}|u|^{p_+}\,dx\right) \\
&\leq 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + \frac{\max\{b,1\}}{p_-}\|u\|_X^{p_+} + \frac{\lambda}{p_+}\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}}|u|^{p_+}\,dx \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}}\theta\min\left\{\frac{F(x,\delta)}{\delta^\theta},\frac{F(x,-\delta)}{\delta^\theta}\right\}\frac{|u|^\theta}{\theta}\,dx - \lambda\int_{\mathbb{R}^N}\omega(x)|u|^\theta\,dx \\
&\leq 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_X + \left(\frac{\max\{b,1\}}{p_-} + \frac{\lambda}{p_+}\right)\|u\|_X^{p_+} \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}}\frac{\min\{f(x,\delta)\delta, f(x,-\delta)(-\delta)\}}{\delta^\theta}\frac{|u|^{\theta-p_+}}{\theta}|u|^{p_+}\,dx \\
&\quad - \lambda\int_{\mathbb{R}^N}\omega(x)|u|^\theta\,dx \\
&\leq 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_X + \left(\frac{\max\{b,1\}}{p_-} + \frac{\lambda}{p_+}\right)\|u\|_X^{p_+}
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} \frac{\delta^{p_+-1}}{\delta^{\theta-1}} \frac{\delta^{\theta-p_+}}{\theta} |u|^{p_+}\,dx - \lambda \int_{\mathbb{R}^N} \omega(x)|u|^\theta\,dx \\
&\le 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_X + \left(\frac{\max\{b,1\}}{p_-} + \frac{\lambda}{p_+}\right)\|u\|_X^{p_+} + \frac{\lambda}{\theta}\int_{\mathbb{R}^N} |u|^{p_+}\,dx \\
&- \lambda \int_{\mathbb{R}^N} \omega(x)|u|^\theta\,dx. \qquad (3.13)
\end{aligned}
$$

By Hölder?s inequality and Lemma2.7, we deduce that

$$
\int_{\mathbb{R}^N} \omega(x)|u|^\theta\,dx \le \|\omega\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^\theta\,dx \le C_4\|u\|_X^\theta
$$

for a positive constant C_4. Notice that in the finite dimensional subspace X_1, the norm $\|\cdot\|_{L^\theta(\mathbb{R}^N)}$ is equivalent to the norm $\|\cdot\|_X$. Therefore, it follows from (3.13) that

$$
I_\lambda(u) \le 2\|a\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_X + \left(\frac{\max\{b,1\}}{p_-} + \frac{\lambda}{p_+} + \frac{\lambda}{\theta}\right)\|u\|_X^{p_+} - \lambda C_5\|u\|_X^\theta
$$

for a positive constant C_5. Since $\theta > p_+$, we obtain

$$
I_\lambda(u) \to -\infty \quad \text{as } \|u\|_X \to \infty
$$

and thus we can choose $\rho_k > \delta_k > 0$. This completes the proof. □

The following consequence is the other way to show the existence of infinitely many pairs of weak solutions for the given problem (B) without assumption (F4).

Theorem 3.10 *Assume that* (J1)-(J5), (H1)-(H2), *and* (F1)-(F3) *hold. In addition, suppose that there exist* $\gamma_0 \in L^1(\mathbb{R}^N)$ *and* $\gamma_1 \in L^\infty(\mathbb{R}^N)$ *with* $\gamma_1(x) > 0$, *for almost all* $x \in \mathbb{R}^N$, *such that*

$$
F(x,s) \ge \gamma_1(x)|s|^\theta - \gamma_0(x), \qquad (3.14)
$$

for almost all $x \in \mathbb{R}^N$ *and for all* $s \in \mathbb{R}$ *where* $\theta > p_+$. *If* $f(x,-t) = -f(x,t)$ *holds, for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, *then* I_λ *has a sequence of critical points* $\{\pm u_n\}$ *in* X *such that* $I_\lambda(\pm u_n) \to \infty$ *as* $n \to \infty$.

Proof Obviously, I_λ is an even functional and satisfies (PS)-condition. It is enough to show that there exist $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ as $n \to \infty$;
(2) $a_k := \max\{I_\lambda(u) : u \in Y_k, \|u\|_X = \rho_k\} \le 0$,

for k large enough. The same argument in Theorem 3.9 implies (1).

To show (2), for $k = 1, 2, \ldots$, write

$$
\sigma_k = \inf_{u\in Y_k, \|u\|_X=1} \left(\int_{\mathbb{R}^N} \gamma_1(x)|u|^\theta\,dx\right).
$$

It is easy to see that $\sigma_k > 0$. For $u \in Y_k$ with $\|u\|_X = 1$ and $t > 1$, by (3.14) we have

$$\begin{aligned} I_\lambda(tu) &= \int_{\mathbb{R}^N} \Phi_0(x, t\nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)}|tu|^{p(x)}\,dx - \lambda \int_{\mathbb{R}^N} F(x, tu)\,dx \\ &\le C_6 t^{p_+} - \lambda \sigma_k t^\theta + \lambda \int_{\mathbb{R}^N} \gamma_0(x)\,dx \end{aligned} \tag{3.15}$$

for a positive constant C_6. Since $\theta > p_+$, it follows from (3.15) that

$$I_\lambda(tu) \to -\infty \quad \text{as } t \to \infty$$

and thus we can choose $\rho_k > \delta_k > 0$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Diening, L, Harjulehto, P, Hästö, P, Růžička, M: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Berlin (2011)
2. Edmunds, DE, Rákosník, J: Density of smooth functions in $W^{k,p(x)}(\Omega)$. Proc. R. Soc. Lond. Ser. A **437**, 229-236 (1992)
3. Mihăilescu, M, Rădulescu, V: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **462**, 2625-2641 (2006)
4. Růžička, M: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
5. Fan, X, Han, X: Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$. Nonlinear Anal. **59**, 173-188 (2004)
6. Fan, X, Zhang, QH: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843-1852 (2003)
7. Kim, IH, Kim, Y-H: Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents. Manuscr. Math. (in press)
8. Mihăilescu, M, Rădulescu, V: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. Proc. Am. Math. Soc. **135**, 2929-2937 (2007)
9. Boureanu, M-M, Preda, F: Infinitely many solutions for elliptic problems with variable exponent and nonlinear boundary conditions. Nonlinear Differ. Equ. Appl. **19**, 235-251 (2012)
10. Lee, JS, Kim, Y-H: Existence and multiplicity of solutions for nonlinear elliptic equations of p-Laplace type in $\mathbb{R}^N$. (submitted)
11. Fan, X, Zhao, D: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424-446 (2001)
12. Kim, IH, Kim, Y-H: Positivity of the infimum eigenvalue for equations of $p(x)$-Laplace type in $\mathbb{R}^N$. Bound. Value Probl. **2013**, 214 (2013)
13. Szulkin, A, Willem, M: Eigenvalue problem with indefinite weight. Stud. Math. **135**, 191-201 (1995)
14. Fan, X: Sobolev embeddings for unbounded domain with variable exponent having values across N. Math. Inequal. Appl. **13**, 123-134 (2010)
15. Zeidler, E: Nonlinear Functional Analysis and Its Applications. II/B. Springer, New York (1990)
16. Colasuonno, F, Pucci, P, Varga, C: Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. **75**, 4496-4512 (2012)
17. Le, VK: On a sub-supersolution method for variational inequalities with leray-lions operators in variable exponent spaces. Nonlinear Anal. **71**, 3305-3321 (2009)
18. Willem, M: Minimax Theorems. Birkhäuser, Basel (1996)