Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

# Lower bounds for the blow-up time of the nonlinear non-local reaction diffusion problems in $\mathbb{R}^N$ $(N \geq 3)$

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**Abstract**
This paper deals with the blow-up of the solution to a non-local reaction diffusion problem in $\mathbb{R}^N$ for $N \geq 3$ under nonlinear boundary conditions. Utilizing the technique of a differential inequality, lower bounds for the blow-up time are derived when the blow-up does occur under some suitable assumptions.
**MSC:** 35K20; 35K55; 35K65

**Keywords:** blow-up; lower bounds; non-local reaction diffusion problem

## 1 Introduction

There is a vast literature on the question of blow-up of solutions to nonlinear parabolic equations and systems. Readers can refer to the books of Straughan [1] and Quittner and Souple [2], as well as to the survey paper of Bandle and Brunner [3]. For more recent work, one can refer to [4–12].

In practical situations, one would like to know among other things whether the solution blows up. In this paper, we consider the blow-up for the solution of the following nonlinear non-local reaction diffusion problems, which have been studied by Song in [4]:

$$\frac{\partial u}{\partial t} = \triangle u + \int_\Omega u^p\,dx - ku^q \quad \text{in } \Omega \times (0, t^*), \tag{1.1}$$

$$u = 0 \quad \text{in } \partial\Omega \times (0, t^*), \tag{1.2}$$

$$u(x, 0) = f(x) \geq 0 \quad \text{in } \Omega, \tag{1.3}$$

where $\triangle$ is the Laplace operator, $\partial\Omega$ the boundary of $\Omega$ and $t^*$ the possible blow-up time, $p, q > 1$. In [4, 13–16], the authors have studied the question of blow-up for the solution of parabolic problems by imposing two different nonlinear boundary conditions: homogeneous Dirichlet boundary conditions or homogeneous Neumann boundary conditions. They determine, for solutions that blow up, a lower bound for the blow-up time $t^*$ in a bounded domain $\Omega \subset \mathbb{R}^N$ for $N = 3$. Besides, some authors have also started to consider the blow-up phenomena of those problems under Robin boundary conditions (see [17–19]). However, for the case $N \geq 3$, the Sobolev inequality, which is important for the result obtained in [4], is no longer applicable. Recently, some papers begin to pay attentions to the study of the blow-up phenomena of solution to an equation in $\Omega \subset \mathbb{R}^N$, for $N \geq 3$ (see [20–22]).

In the present paper, for convenience, we set $p = s + 1$, $s > 0$ and rewrite (1.1) as follows:

$$\frac{\partial u}{\partial t} = \triangle u + \int_\Omega u^{s+1}\,dx - ku^q \quad \text{in } \Omega \times \bigl(0, t^*\bigr). \tag{1.4}$$

As indicated in [23], it is well known that if $p \leq q$ the solution will not blow up in finite time. Also it is well known that if the initial data are small enough the solution will actually decay exponentially as $t \longrightarrow \infty$ (see *e.g.* [1, 24]). Since we are interested in a lower bound for $t$, in the case of blow-up, we are only concerned with the case $q < p$.

We see by the parabolic maximum principles [25, 26] that $u$ is nonnegative in $x$ for $t \in [0, t^*)$.

In Section 2, we derive the lower bound for the blow-up time of the system (1.1)-(1.3) in $\mathbb{R}^N$. The obtained results extend the corresponding conclusions in the literature to $\mathbb{R}^N$ for any $N \geq 3$.

## 2 A lower bound for the blow-up time

In this section we seek the lower bound for the blow-up time $t^*$ and establish the following theorem.

**Theorem 1** *Let* $u(x,t)$ *be the classical nonnegative solution of problem* (1.1)-(1.3) *in a bounded star-shaped domain* $\Omega \in \mathbb{R}^N$ $(N \geq 3)$ *and assume that* $q < p$. *Then the quantity*

$$\varphi(t) = \int_\Omega u^{ns}\,dx \tag{2.1}$$

*satisfies the differential inequality*

$$\frac{d\varphi}{dt} \leq k_1 \varphi^{1+\beta}, \tag{2.2}$$

*from which follows that the blow-up time* $t^*$ *is bounded from below*; *i.e.*, *we have*

$$t^* \geq \int_{\varphi(0)}^{\infty} \frac{1}{k_1 \eta^{1+\beta}}\,d\eta = \frac{1}{k_1 \beta}\varphi^{-\beta}(0), \tag{2.3}$$

*where* $k_1$, $\beta$ *are positive constants which will be defined later.*

*Proof* Firstly we compute

$$\begin{aligned}\varphi'(t) &= ns \int_\Omega u^{ns-1}\left[\triangle u + \int_\Omega u^{s+1}\,dx - ku^q\right] dx \\ &\leq -\frac{4(ns-1)}{ns}\int_\Omega \bigl|\nabla u^{\frac{ns}{2}}\bigr|^2\,dx + ns|\Omega|\int_\Omega u^{s(n+1)}\,dx - kns\int_\Omega u^{ns+q-1}\,dx.\end{aligned} \tag{2.4}$$

For convenience, we now set

$$v = u^s, \qquad \alpha = \frac{q-1}{s}. \tag{2.5}$$

Since $q < s + 1$, $\alpha < 1$. Thus, we obtain

$$\varphi'(t) \le -\frac{4(ns-1)}{ns}\int_\Omega \left|\nabla v^{\frac{n}{2}}\right|^2 dx + ns|\Omega|\int_\Omega v^{(n+1)}\,dx - kns\int_\Omega v^{n+\alpha}\,dx. \tag{2.6}$$

By the Hölder inequality, we have

$$\int_\Omega v^{n+1}\,dx \le \left(\int_\Omega v^{n+\alpha}\,dx\right)^{\frac{\gamma-1}{\gamma-\alpha}}\left(\int_\Omega v^{n+\gamma}\,dx\right)^{\frac{1-\alpha}{\gamma-\alpha}} \tag{2.7}$$

for positive constant $\gamma > 1$. By the inequality

$$a^r + b^{1-r} \le ra + (1-r)b, \quad a, b > 0, 0 < r < 1, \tag{2.8}$$

we have

$$\int_\Omega v^{n+1}\,dx \le \frac{\gamma-1}{\gamma-\alpha}\varepsilon_1\int_\Omega v^{n+\alpha}\,dx + \frac{1-\alpha}{\gamma-\alpha}\varepsilon_1^{-\frac{\gamma-1}{1-\alpha}}\int_\Omega v^{n+\gamma}\,dx, \tag{2.9}$$

where $\varepsilon_1$ is a positive constant. If we insert (2.9) into (2.6) and choose

$$\varepsilon_1 = \frac{k(\gamma-\alpha)}{|\Omega|(\gamma-1)},$$

then (2.6) yields

$$\varphi'(t) \le -\frac{4(ns-1)}{ns}\int_\Omega \left|\nabla v^{\frac{n}{2}}\right|^2 dx + ns|\Omega|\frac{1-\alpha}{\gamma-\alpha}\varepsilon_1^{-\frac{\gamma-1}{1-\alpha}}\int_\Omega v^{n+\gamma}\,dx. \tag{2.10}$$

By the Hölder inequality again, we have

$$\int_\Omega v^{n+\gamma}\,dx \le \left(\int_\Omega v^n\,dx\right)^{\frac{2n-\gamma(N-2)}{2n}}\left(\int_\Omega v^{\frac{n}{2}\cdot\frac{2N}{N-2}}\,dx\right)^{\frac{\gamma(N-2)}{2n}}, \tag{2.11}$$

where we have chosen $2n > \gamma N$. Now let $c_1$ be the best imbedding constant defined in [27]. Using the Sobolev inequality for $W_0^{1,2} \hookrightarrow L^{\frac{2N}{N-2}}$ for $N \ge 3$, we have

$$\int_\Omega v^{\frac{n}{2}\cdot\frac{2N}{N-2}}\,dx \le c_1^{\frac{2N}{N-2}}\left(\int_\Omega \left|\nabla v^{\frac{n}{2}}\right|^2 dx\right)^{\frac{N}{N-2}}.$$

Therefore, (2.11) may be rewritten as

$$\int_\Omega v^{n+\gamma}\,dx \le c_1^{\frac{\gamma N}{2n}}\left(\int_\Omega v^n\,dx\right)^{\frac{2n-\gamma(N-2)}{2n}}\left(\int_\Omega \left|\nabla v^{\frac{n}{2}}\right|^2 dx\right)^{\frac{N\gamma}{2n}}. \tag{2.12}$$

Using (2.8) again, we have

$$\begin{aligned}\int_\Omega v^{n+\gamma}\,dx \le{}& \frac{2n-N\gamma}{2n}c_1^{\frac{\gamma N}{2n}}\varepsilon_2^{-\frac{N}{2n-\gamma(N-2)}}\left(\int_\Omega v^n\,dx\right)^{\frac{2n-\gamma(N-2)}{2n}} \\ &+ \frac{N\gamma}{2n}c_1^{\frac{\gamma N}{2n-N\gamma}}\varepsilon_2\int_\Omega \left|\nabla v^{\frac{n}{2}}\right|^2 dx,\end{aligned} \tag{2.13}$$

where $\varepsilon_2$ is a positive constant to be chosen as follows:

$$\varepsilon_2 = \frac{8(ns-1)(\gamma-\alpha)}{N\gamma|\Omega|ns^2(1-\alpha)}\varepsilon_1^{\frac{\gamma-1}{1-\alpha}}c_1^{-\frac{\gamma N}{2n}}, \tag{2.14}$$

and inserting (2.13) back into (2.10), we have

$$\varphi'(t) \le k_1\varphi^{\frac{2n-\gamma(N-2)}{2n-N\gamma}}, \tag{2.15}$$

where

$$k_1 = ns|\Omega|\frac{1-\alpha}{\gamma-\alpha}\varepsilon_1^{-\frac{\gamma-1}{1-\alpha}}\frac{2n-N\gamma}{2n}c_1^{\frac{\gamma N}{2n}}\varepsilon_2^{-\frac{N}{2n-\gamma(N-2)}}. \tag{2.16}$$

If we set

$$\beta = \frac{2\gamma}{2n-N\gamma} > 0, \tag{2.17}$$

then (2.15) can be written as

$$\varphi'(t) \le k_1\varphi^{1+\beta}, \tag{2.18}$$

or

$$\frac{d\varphi}{k_1\varphi^{1+\beta}} \le 1. \tag{2.19}$$

Upon integration we have for $t < t^*$,

$$t^* \ge \int_{\varphi(0)}^{\infty}\frac{1}{k_1\eta^{1+\beta}}\,d\eta = \frac{1}{k_1\beta}\varphi^{-\beta}(0), \tag{2.20}$$

where $\varphi(0) = \varphi(t) = \int_\Omega u_0^{ns}\,dx$. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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**Acknowledgements**
The authors would like to express their gratitude to the anonymous referees for helpful and very careful reading on this paper. This research was supported by the Natural Science Foundation of Hunan Province (No. 14JJ4044; No. 15JJ2063; No. 13JJ3085) and the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (Grant 2013LYM0112).

Received: 14 August 2014 Accepted: 8 December 2014 Published online: 20 December 2014

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doi:10.1186/s13661-014-0265-5
**Cite this article as:** Tang et al.: **Lower bounds for the blow-up time of the nonlinear non-local reaction diffusion problems in $\mathbb{R}^N$ ($N \geq 3$).** *Boundary Value Problems* 2014 **2014**:265.