Yuan and Huang *Boundary Value Problems* (2015) 2015:36
DOI 10.1186/s13661-015-0295-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On existence and multiplicity of solutions for Kirchhoff-type equations with a nonsmooth potential

Ziqing Yuan and Lihong Huang*

*Correspondence:
lhhuang@hnu.edu.cn
College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P.R. China

Abstract

This paper is concerned with the following Kirchhoff-type problems with a nonsmooth potential: $-(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u \in \partial j(x,u)$ for a.a. $x \in \Omega$, $u = 0$ on $\partial\Omega$. Using the nonsmooth mountain pass theorem, the nonsmooth local linking theorem, and the nonsmooth fountain theorem, we establish the existence and multiplicity of solutions for the problem. All this is based on the nonsmooth critical point. Some recent results in the literature are generalized and improved.

Keywords: nonsmooth critical point; locally Lipschitz; Kirchhoff-type equation; multiple solutions

1 Introduction

In recent years, various Kirchhoff-type problems have been widely discussed by lots of authors. The Kirchhoff mode is an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, which takes into account the changes in length of the string produced by transverse vibrations. Some interesting studies of the Kirchhoff equations can be found in [1–7] and references therein. Especially, there exist lots of papers focused on studying the following Kirchhoff-type equations:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where f is a continuous function. For example, Perera and Zhang [6] derived nontrivial solutions for problem (1.1) with the help of the Yang index and critical groups. In [8], Chen *et al.*, by employing fibering map methods and the Nehari manifold, discussed problem (1.1) with concave and convex nonlinearities and obtained the existence of multiple positive solutions. Recently, Liang *et al.* in [9] firstly studied the bifurcation phenomena of problem (1.1) with the right-hand side of the first equation replaced by $\nu f(x,u)$ by using the topological degree and variational methods.

As is well known, many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities. Among these problems, we have the seepage surface problem [10], the obstacle problem [11], and the Elenbaas equation [12] and so on. Based on these results, the theory of nonsmooth variational analysis has been developed rapidly. For a comprehensive understanding, we refer to the monographs of [13–16].

Inspired by the above results, a natural question arises: what will happen when the potential function f is discontinuous in problem (1.1)? This is the main point of interest in our paper to study. For this purpose, we consider the following Kirchhoff-type problems with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u \in \partial j(x,u) & \text{for a.a. } x\in\Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subseteq\mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$ ($N=1,2,3$), $a,b>0$. By $\partial j(x,u)$ we denote the generalized subdifferential of $u\mapsto j(x,u)$.

Remark 1.1 If we let $a=1$ and $b=0$, then problem (1.2) turns into

$$\begin{cases} -\Delta u \in \partial j(x,u) & \text{for a.a. } x\in\Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) is a well-known semilinear elliptic equation with a nonsmooth potential and there exist many results focused on discussing problem (1.3); see [17–22] and references therein.

To the best of our knowledge, there exist few results on studying the Kirchhoff-type problems with nonsmooth potentials. We will face at least two difficulties in treating problem (1.2). Firstly, the presence of discontinuities probably leads to no solution of problem (1.2) in general. Therefore, in order to overcome this difficulty, our approach is based on the nonsmooth critical point theorem for locally Lipschitz functions due to Chang [11]. Specifically, we consider such a function f, which is locally essentially bounded measurable and we fill the discontinuity gaps of f, replacing f by an interval $\partial j(x,u)=[f^-(x,u),f^+(x,u)]$, where

$$f^-(x,u)=\lim_{\delta\to0^+}\operatorname*{ess\,inf}_{|t-u|<\delta} f(x,t), \qquad f^+(x,u)=\lim_{\delta\to0^+}\operatorname*{ess\,sup}_{|t-u|<\delta} f(x,t).$$

Secondly, it is well known that the classic (AR)-condition (see (J_4)) guarantees that every $(PS)_c$ sequence is bounded. However, in our Theorems 1.4 and 1.5, we abandon the classic (AR)-condition and have to find new conditions (see (J_7)-(J_9)) to ensure that every $(PS)_c$ sequence is bounded.

In our article, we need the following assumptions:

(J_1) $j:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function, and $j(\cdot,u)$ is measurable for all $u\in\mathbb{R}$ and $j(x,\cdot)$ is locally Lipschitz for almost all $x\in\Omega$;

(J_2) for almost all $x\in\Omega$, all $u\in\mathbb{R}$ and all $\omega\in\partial j(x,u)$, we have

$$|\omega|\le c\bigl(|u|^{p-1}+1\bigr) \quad \text{for some } 4<p<2^*=\begin{cases} 6, & \text{if } N=3, \\ \infty, & \text{if } N=1,2 \end{cases}$$

for some $c>0$;

(J_3) for almost all $x \in \Omega$, we have

$$\limsup_{|u|\to+\infty} \frac{j(x,u(x))}{|u(x)|^4} \le \alpha_1,$$

where $\alpha_1 \le \frac{b}{4c_4^4}$, and c_4 satisfies $\int_\Omega |u|^4\,\mathrm{d}x \le c_4^4(\int_\Omega |\nabla u|^2\,\mathrm{d}x)^2$;

(J_4) there exist $q > 4$ and $M_1 > 0$ such that for almost all $x \in \Omega$, all $|u| \ge M_1$ and all $\omega \in \partial j(x,u)$, we have $0 < qj(x,u) \le \omega u$;

(J_5) there exists $\delta > 0$, such that $\frac{a}{2}\lambda_k u^2 + \frac{b}{4c_4^4}\lambda_k^2 u^4 \le j(x,u) \le \frac{a}{2}\lambda_{k+1}u^2 + \frac{b}{8c_4^4}u^4$, for a.a. $x \in \Omega$ and all $|u| \le \delta$, $k \in \mathbb{N}$ (λ_k denotes the variational characterization (see (2.1) and (2.2)));

(J_6) $j(x,u) = j(x,-u)$ $\forall x \in \Omega$, $u \in \mathbb{R}$;

(J_7) there exists $M_2 > 0$ such that for a.a. $x \in \Omega$, all $|u| \ge M_2$ and all $\omega \in \partial j(x,u)$, we have $4j(x,u) \le u\omega$;

(J_8) $\lim_{|u|\to+\infty} \frac{j(x,u)}{u^4} \to +\infty$ uniformly for almost all $x \in \Omega$;

(J_9) $\lim_{|u|\to+\infty}(\omega u - 4j(x,u)) \to +\infty$ as $|u| \to +\infty$, and there exist $\sigma > 1 + \frac{2}{2^*-2}$ and a positive constant l such that $|\omega|^\sigma \le l(\omega u - 4j(x,u))|u|^\sigma$ for $|u|$ large and for a.a. $x \in \Omega$ and $\omega \in \partial j(x,u)$;

(J_{10}) $\lim_{x\to 0} \frac{j(x,u)}{|u|^2} \le \frac{\lambda_1}{2}a$ uniformly for a.a. $x \in \Omega$.

Our main results are the following:

Theorem 1.1 *If hypotheses* (J_1), (J_2), (J_4), (J_{10}), *and* $j(x,0) = 0$ *for a.a.* $x \in \Omega$ *are satisfied, then problem* (1.2) *has at least one nontrivial solution.*

Theorem 1.2 *If hypotheses* (J_1), (J_2), (J_3), (J_5), *and* $j(x,0) = 0$ *for a.a.* $x \in \Omega$ *are satisfied, then problem* (1.2) *has at least two nontrivial solutions.*

Motivated by [23], we obtain the existence of infinitely solutions for problem (1.2).

Theorem 1.3 *If hypotheses* (J_1), (J_2), (J_4), *and* (J_6) *are satisfied, then problem* (1.2) *has infinitely many large energy solutions.*

Theorem 1.4 *If hypotheses* (J_7) *and* (J_8) *are used in place of* (J_4), *then the conclusion of Theorem* 1.3 *holds.*

Theorem 1.5 *If hypothesis* (J_9) *is used in place of* (J_7), *then the conclusion of Theorem* 1.4 *holds.*

Remark 1.2 (i) It is not difficult to see that there exist many functions, which, respectively, satisfy Theorem 1.1 and Theorem 1.2. For example, for simplicity, we drop the x-dependence

$$j_1(u) = \begin{cases} \frac{3}{16}c|u|^{\frac{16}{3}} & \text{if } |u| \ge 1, \\ \frac{3}{16}c|u|^2 & \text{if } |u| < 1, \end{cases}$$

where $0 < c \leq \frac{8}{3}\lambda_1 a$.

$$j_2(u) = \begin{cases} \theta_1 |u|^{\frac{5}{2}} & \text{if } |u| < 1, \\ \theta_1 |u|^3 & \text{if } |u| \geq 1, \end{cases}$$

for some $\theta_1 > 0$. Then it is easy to check that $j_1(u)$ satisfies Theorem 1.1 and $j_2(u)$ satisfies Theorem 1.2.

(ii) Since we do not assume $j(x,u) > 0$ in hypothesis (J_7), the assumption (J_4) cannot imply (J_7). So Theorem 1.3 and Theorem 1.4 are two different theorems. Furthermore, there exist functions $j(x,u)$, which satisfy all hypotheses of Theorem 1.3 and Theorem 1.4, while they do not satisfy hypothesis (J_9). For example

$$j_3(u) = \begin{cases} \frac{2}{11}|u|^{\frac{11}{2}} & \text{if } |u| \geq 1, \\ \frac{2}{11}|u|^2 & \text{if } |u| < 1. \end{cases}$$

Then $j_3(u)$ satisfies all conditions of Theorem 1.3 and Theorem 1.4, while it does not satisfy Theorem 1.5.

(iii) There exist lots of functions which satisfy all assumptions of Theorem 1.5, while they do not satisfy (J_4) and (J_7), for example, for small $\varepsilon > 0$, let

$$j_4(u) = \begin{cases} |u|^{4+\varepsilon} + 2\varepsilon |u|^4 \sin^2(\frac{|u|^\varepsilon}{\varepsilon}) & \text{if } |u| \geq 1, \\ [1 + 2\varepsilon \sin^2(\frac{1}{\varepsilon})]|u| & \text{if } |u| < 1. \end{cases}$$

Then j_4 does not satisfy (J_4) and (J_7). This means that Theorem 1.5 is different from Theorem 1.3 and Theorem 1.4.

This paper is divided into three sections. In Section 2, we recall some basic definitions and propositions which will be used in the sequel. In Section 3, we give the proof of the main results.

2 Preliminaries

In this section we state some definitions and lemmas, which will be used throughout this paper. First of all, we give some definitions: $(X, \|\cdot\|)$ will denote a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. While $u_n \to u$ (respectively, $u_n \rightharpoonup u$) in X means the sequence $\{u_n\}$ converges strongly (respectively, weakly) in X. As usual, 2^* denotes the critical Sobolev exponent, *i.e.*, $2^* = \frac{2N}{N-2}$ if $2 < N$, and $2^* = +\infty$ if $2 \geq N$. We denote by $|\cdot|_p$ the usual L^p-norm. The n-dimensional Lebesgue measure of a set $E \in \mathbb{R}^n$ is denoted by $|E|$. Since Ω is a bounded domain, $X \hookrightarrow L^r(\Omega)$ continuously for $r \in [1, 2^*]$, compactly for $r \in [1, 2^*)$, and there exists $c_r > 0$, such that

$$|u|_r \leq c_r \|u\|, \quad \forall u \in X.$$

Definition 2.1 A function $I : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $\nu, \eta \in U$

$$\bigl|I(\nu) - I(\omega)\bigr| \leq L\|\nu - \eta\|.$$

Definition 2.2 Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional, $u, v \in X$: the generalized derivative of I in u along the direction v,

$$I^0(u;v) = \limsup_{\omega \to u, \tau \to 0^+} \frac{I(\omega + \tau v) - I(\omega)}{\tau}.$$

It is easy to see that the function $v \mapsto I^0(u;v)$ is sublinear, continuous and so is the support function of a nonempty, convex, and ω^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, v\rangle_X \le I^0(u;v) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \{I'(u)\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

A point $x \in X$ is a critical point of I, if $0 \in \partial I(u)$. It is easy to see that, if $u \in X$ is a local minimum of I, then $0 \in \partial I(u)$. For more on locally Lipschitz functionals and their subdifferential calculus, we refer the reader to Clarke [24].

Definition 2.3 If $I : X \to \mathbb{R}$ is a locally Lipchitz function, then we say that I satisfies the nonsmooth C-condition, if the following holds:

Every sequence $\{u_n\} \subset X$, such that

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_X) m^I(u_n) \to 0,$$

where $m^I(u_n) = \inf_{u_n^* \in \partial I(x,u_n)} \|u_n^*\|_{X^*}$, has a strongly convergent subsequence.

In the following, we introduce the eigenvalues of the negative Laplacian with a Dirichlet boundary condition. By $\{u_n\}_{n\ge1}$ we denote the corresponding eigenfunctions. We know that $\{u_n\}_{n\ge1} \subset C_0^1(\overline{\Omega})$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. Also $\lambda_n \to +\infty$ as $n \to +\infty$. λ_1 is isolated and simple, and $u_1 \in C_0^1(\overline{\Omega})$ is the only eigenfunction with constant sign. Furthermore, we derive the following variational characterization of $\{\lambda_n\}_{n\ge1}$:

$$\lambda_1 = \min\left\{\frac{|\nabla u|_2^2}{|u|_2^2} : u \in H_0^1(\Omega), u \ne 0\right\} = \frac{|\nabla u_1|_2^2}{|u_1|_2^2}. \tag{2.1}$$

For $n \ge 2$, we have

$$\begin{aligned}\lambda_n &= \min\left\{\frac{|\nabla u|_2^2}{|u|_2^2} : u \in H_0^1(\Omega), u \perp \{u_1, \dots, u_{n-1}\}, u \ne 0\right\} \\ &= \max\left\{\frac{|\nabla u|_2^2}{|u|_2^2} : u \in H_0^1(\Omega), u \in \operatorname{span}\{u_k\}_{k=1}^n, u \ne 0\right\} \\ &= \frac{|\nabla u_n|_2^2}{|u_n|_2^2}.\end{aligned} \tag{2.2}$$

The following properties can be found in [25].

(p_1) $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$.
(p_2) $|\nabla u|_2^2 \geq \lambda_1 |u|_2^2$ for all $u \in H_0^1(\Omega)$.
(p_3) There exists an eigenfunction u_1 corresponding to λ_1 such that $u_1 \in \operatorname{int}(C_0^1(\bar{\Omega}))$ as well as $|u_1|_{L^2(\Omega)} = 1$.

Next, we list the nonsmooth mountain pass theorem.

Theorem 2.1 (Nonsmooth mountain pass theorem [16]) *If there exist $u_1 \in X$ and $r > 0$, such that $\|u_1\|_X > r$,*

$$\max\{I(0), I(u_1)\} \leq \inf_{\|u\|=r} I(u)$$

and I satisfies the nonsmooth C-condition with

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1];X) : \gamma(0) = 0, \gamma(1) = u_1\}$, then $c \geq \inf_{\|u\|=r} I(u)$ and c is a critical value of I. Moreover, if $c = \inf\{I(u) : \|u\| = r\}$, then there exists a critical point u_0 of I with $I(u_0) = c$ and $\|u_0\| = r$ (i.e., $K_c^I \cap \partial B_r \neq \emptyset$).

Recently, Kandilakis *et al.* [26] proved a multiplicity result under the so-called local linking conditions. The result is a nonsmooth version of result due to Brézis and Nirenberg [27].

Theorem 2.2 *If X is a reflexive Banach space, $X = Y \oplus V$ with dim $Y < +\infty$, $I : X \to \mathbb{R}$ is a locally Lipschitz function which is bounded below and satisfies the nonsmooth C-condition*; *we have* $I(0) = 0$, $\inf_X I < 0$ *and there exists* $r > 0$, *such that*

$$\begin{cases} I(u) \leq 0 & \text{if } u \in Y, \|u\| \leq r, \\ I(u) \geq 0 & \text{if } u \in V, \|u\| \leq r \end{cases} \quad \textit{(local linking condition)},$$

then I has at least two nontrivial critical points.

In the following, we introduce a nonsmooth version fountain theorem which was proved by Dai [23]. The smooth fountain theorem was established by Bartsh in [28, 29].

Definition 2.4 Assume that the compact group G acts diagonally on V^k

$$g(v_1, \ldots, v_k) = (gv_1, \ldots, gv_k),$$

where V is a finite dimensional space. The action of G is admissible if every continuous equivariant map $\partial U \to V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in V^k, $k \geq 2$, has a zero.

Example 2.1 The antipodal action $G = \mathbb{Z}$ on $V = \mathbb{R}$ is admissible.

We consider the following situation:

(A_1) The compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{m\in\mathbb{N}} X_m}$, the space X_m are invariant and there exists a finite dimensional space V such that, for every $m \in \mathbb{N}$, $X_m \simeq V$ and the action of G on V is admissible.

In the theorem, we will use the following notations:

$$\begin{aligned} &Y_k = \bigoplus_{m=0}^{k} X_m, \qquad Z_k = \overline{\bigoplus_{m=k}^{\infty} X_m}, \\ &B_k = \{u \in Y_k : \|u\| \le \rho_k\}, \qquad N_k = \{u \in Z_k : \|u\| = r_k\}, \end{aligned} \tag{2.3}$$

where $\rho_k > r_k > 0$.

The following lemma is very important when we use the fountain theorem to prove infinite solutions for problem (1.2).

Lemma 2.1 (see [29]) *If* $1 \le p < 2^*$, *then we have*

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} |u|_p \to 0, \quad k \to \infty.$$

Theorem 2.3 *Under assumption* (A_1), *let* $I : X \to \mathbb{R}$ *be an invariant locally Lipshitz functional. If for every* $k \in \mathbb{N}$, *there exists* $\rho_k > r_k > 0$ *such that*

(A_2) $a_k = \max_{u\in Y_k, \|u\|=\rho_k} I(u) \le 0$;
(A_3) $b_k = \inf_{u\in Z_k, \|u\|=r_k} I(u) \to \infty$, $k \to \infty$;
(A_4) *I satisfies the nonsmooth $(PS)_c$ condition for every $c > 0$,*

then I has an unbounded sequence of critical values.

In this paper we let $X = H_0^1(\Omega)$ be the Sobolev space equipped with the norm $\|u\| = |\nabla u|_2$.

We say that u is a weak solution to problem (1.2), if $u \in X$ and

$$\left(a + b\|u\|^2\right) \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega \omega v \, \mathrm{d}x = 0,$$

for all $v \in X$ and $\omega \in \partial j(x, u)$ a.e. on Ω.

Seeking a weak solution of problem (1.2) is equivalent to finding a critical point of the energy function $I : X \to \mathbb{R}$ for problem (1.2), defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j\left(x, u(x)\right) \mathrm{d}x, \quad \forall u \in X. \tag{2.4}$$

I is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see [24], p.83).

3 Proof of the main results

In order to give the proofs of our main results, we firstly prove the following lemma.

Lemma 3.1 *If* (J_1) *and* (J_2) *hold, assume that* $\{u_n\}_{n\ge1} \subseteq X$ *is a bounded sequence with* $m^I(u_n) \to 0$, *then* $\{u_n\}_{n\ge1} \subseteq X$ *has a convergent sequence.*

Proof Since $\{u_n\} \subset X$ is bounded and the embedding $X \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, 2^*)$, passing to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \to u \quad \text{in } L^r(\Omega), \qquad u_n \to u(x) \quad \text{for a.a. } x \in \Omega, \\ &\bigl|u_n(x)\bigr| \le k(x) \quad \text{for a.a. } x \in \Omega \text{ and all } n \ge 1, \text{ with } k \in L^r(\Omega)_+. \end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned} &\langle u_n^* - u^*, u_n - u\rangle \\ &\quad = \left(a + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)\int_\Omega \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x \\ &\qquad - \left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x - \int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x \\ &\quad = \left(a + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)\int_\Omega \bigl|\nabla(u_n - u)\bigr|^2\,\mathrm{d}x - b\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right) \\ &\qquad \cdot \int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x - \int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x \\ &\quad \ge \min\{a, 1\}\|u_n - u\|^2 - b\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)\int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x \\ &\qquad - \int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x, \end{aligned}$$

where $u^* \in \partial I(u)$, $u_n^* \in \partial I(u_n)$, $\omega \in \partial j(x, u)$ and $\omega_n \in \partial j(x, u_n)$ for almost all $x \in \Omega$, then we obtain

$$\begin{aligned} \min\{a, 1\}\|u_n - u\|^2 \le{}& \langle u_n^* - u^*, u_n - u\rangle + b\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right) \\ & \cdot \int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x + \int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

From (3.1) and the boundedness of $\{u_n\}$ in X, we have

$$b\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega \nabla|u_n|^2\,\mathrm{d}x\right) \cdot \int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x \to 0, \tag{3.3}$$

as $n \to +\infty$. Furthermore, from (J_2) and the Hölder inequality

$$\begin{aligned} \int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x &\le \int_\Omega |\omega_n - \omega||u_n - u|\,\mathrm{d}x \\ &\le \int_\Omega c_1\bigl(|u_n|^{p-1} + |u|^{p-1} + 1\bigr)|u_n - u|\,\mathrm{d}x \\ &\le c_1|\Omega|^{\frac{1}{2}}|u_n - u|_2 + c_1|u_n - u|_p\bigl(|u_n|_p^{p-1} + |u|_p^{p-1}\bigr) \\ &\le c_1|\Omega|^{\frac{1}{2}}|u_n - u|_2 + c_2|u_n - u|_p, \end{aligned}$$

where c_1, c_2 are some positive constants. Since $|u_n - u|_2 \to 0$ and $|u_n - u|_p \to 0$ as $n \to +\infty$, we infer that

$$\int_\Omega (\omega_n - \omega)(u_n - u)\,\mathrm{d}x \to 0, \tag{3.4}$$

as $n \to +\infty$. Note that

$$\langle u_n^* - u^*, u_n - u\rangle \to 0 \quad \text{as } n \to +\infty. \tag{3.5}$$

Hence, from (3.2)-(3.5), we deduce that $\|u_n - u\| \to 0$. This means that $\{u_n\}_{n\geq 1} \subseteq X$ has a convergent sequence. □

Remark If we use $(1 + \|u_n\|)m^I(u_n) \to 0$ in place of $m^I(u_n) \to 0$, the proposition remains true.

In the following, we will use the nonsmooth mountain pass theorem to prove Theorem 1.1.

Proof of Theorem 1.1

Claim 1. I satisfies the nonsmooth C-condition.

Let $\{u_n\}_{n\geq 1} \subseteq X$ such that

$$\bigl|I(u_n)\bigr| \leq M_4 \quad \text{for all } n \geq 1 \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)m^I(u_n) \to 0 \quad \text{as } n \to \infty, \tag{3.6}$$

where $M_4 > 0$. From Lemma 3.1, we only need to prove that $\{u_n\}_{n\geq 1} \subseteq X$ is a bounded sequence. It follows from (3.6) that

$$-\langle u_n^*, u_n\rangle = -\left(a + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + \int \omega_n u_n\,\mathrm{d}x \leq \varepsilon_n, \tag{3.7}$$

and

$$\frac{qa}{2}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + \frac{qb}{4}\|u_n\|^4 - \int_\Omega qj\bigl(x, u_n(x)\bigr)\,\mathrm{d}x \leq qM_4, \tag{3.8}$$

where $\varepsilon_n \to 0$, $u_n^* \in \partial I(u_n)$, $\omega_n \in \partial j(x, u_n)$ a.a. on Ω. Adding (3.7) and (3.8), we obtain

$$a\left(\frac{q}{2} - 1\right)\|u_n\|^2 + b\left(\frac{q}{4} - 1\right)\|u_n\|^4 + \int_\Omega \bigl(\omega_n u_n - qj\bigl(x, u_n(x)\bigr)\bigr)\,\mathrm{d}x \leq \varepsilon_n + qM_4. \tag{3.9}$$

By virtue of (J_4), we have

$$a\left(\frac{q}{2} - 1\right)\|u_n\|^2 + b\left(\frac{q}{4} - 1\right)\|u_n\|^4 + \int_{|u_n|<M_1} \bigl(\omega_n u_n - qj\bigl(x, u_n(x)\bigr)\bigr)\,\mathrm{d}x \leq \varepsilon_n + qM_4. \tag{3.10}$$

Since $a, b > 0$, $q > 4$, from (3.10) and (J_2), we deduce that

$$a\left(\frac{q}{2} - 1\right)\|u_n\|^2 + b\left(\frac{q}{4} - 1\right)\|u_n\|^4 \leq M_5, \tag{3.11}$$

for some $M_5 > 0$ and all $n \geq 1$, then from (3.11), we deduce that

$$\{u_n\}_{n\geq 1} \subseteq X \quad \text{is bounded.}$$

Hence, from Lemma 3.1, we find that I satisfies the nonsmooth C-condition.

By virtue of (J_2) and (J_{10}), there exists $c_1 > 0$ such that

$$j(x,u) \leq \frac{a}{2}\lambda_1|u|^2 + c_1|u|^p$$

for almost all $x \in \Omega$, and all $u \in \mathbb{R}$, then we obtain

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x,u)\,\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 - \frac{a\lambda_1}{2}|u|_2^2 + \frac{b}{4}\|u\|^4 - c_1\int_\Omega |u|^p\,\mathrm{d}x \\ &\geq \frac{b}{4}\|u\|^4 - c_1c_p^p\|u\|^p, \end{aligned}$$

where c_p^p satisfies $|u|_p^p \leq c_p^p\|u\|^p$. Since $p > 4$, set $r_0 = (\frac{b}{4c_1c_p^p})^{p-4}$, then for all $0 < r < r_0$ we have

$$\inf\bigl\{I(u) : \|u\| = r\bigr\} > 0. \tag{3.12}$$

Claim 2. There exists $u_0 \in X$ with $\|u_0\| > r > 0$ such that $I(u_0) < 0$.

Let N_0 be the Lebesgue-null set outside which hypotheses (J_1), (J_2), and (J_4) hold. Let $x \in \Omega \setminus N_0$ and $u \in \mathbb{R}$ with $|u| \geq M_1$. We set

$$h(x,\tau) = j(x,\tau u), \qquad \tau \geq 1.$$

It is obvious that $h(x,\cdot)$ is locally Lipschitz and from the nonsmooth chain rule (see Clark [24], p.45), we obtain

$$\partial h(x,\tau) \subseteq \partial_u(x,\tau u)u,$$

thus

$$\tau\partial h(x,\tau) \subseteq \partial_u(x,\tau u)\tau u.$$

By virtue of hypothesis (J_4), we obtain

$$\tau h'(x,\tau) \geq qh(x,\tau)$$

for all $x \in \Omega \setminus N_0$ and a.a. $\tau \geq 1$. Consequently

$$\frac{q}{\tau} \leq \frac{h'(x,\tau)}{h(x,\tau)}$$

for all $x \in \Omega \setminus N_0$ and a.a. $\tau \geq 1$. Integrating from 1 to $\tau_0 > 1$, we derive

$$\ln \tau_0^q \leq \ln \frac{h(x,\tau_0)}{h(x,1)} \quad \Rightarrow \quad \tau_0^q h(x,1) \leq h(x,\tau_0).$$

Hence we have shown that for $x \in \Omega \setminus N_0$, $|u| \geq M_1 > 1$, and $\tau \geq 1$, we have

$$\tau^q j(x,u) \leq j(x,\tau u). \tag{3.13}$$

Then for all $u \geq M_1$, due to (3.13), we have

$$j(x,u) = j\left(x, \frac{u}{M_1} M_1\right) \geq \left(\frac{u}{M_1}\right)^q j(x,M_1). \tag{3.14}$$

For all $u \leq -M_1$, we obtain

$$j(x,u) = j\left(x, \frac{u}{-M_1}(-M_1)\right) \geq \left(\frac{|u|}{M_1}\right)^q j(x,-M_1). \tag{3.15}$$

From hypothesis (J_2) we can find $c_2 > 0$ such that

$$\bigl|j(x,u)\bigr| \leq c_2. \tag{3.16}$$

for all $x \in \Omega \setminus N_0$ and all $|u| \leq M_1$. Together with (3.14)-(3.16), we infer that

$$j(x,u) \geq c_3 |u|^q - c_2, \tag{3.17}$$

for a.a. $x \in \Omega$, all $u \in \mathbb{R}$ and some $c_2, c_3 > 0$. From (3.17), for $v \in X \setminus \{0\}$ and $t > 1$, we obtain

$$\begin{aligned} I(tv) &= \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \int_\Omega j(x,tv)\,\mathrm{d}x \\ &\leq \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - c_3 t^q \int_\Omega |v|^q\,\mathrm{d}x + c_2|\Omega|. \end{aligned}$$

Note that $q > 4$, which implies $I(tv) \to -\infty$ as $t \to +\infty$. So we can choose a t_0 large enough such that $I(t_0 v) < 0$, and set $u_0 = t_0 v$, then u_0 is the desired element.

Define

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1],X\bigr) : \gamma(0) = 0, \gamma(1) = u_0\bigr\}, \qquad c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I\bigl(\gamma(t)\bigr),$$

then $c \geq \inf_{\|u\|=r} I(u)$. From $I(0) = 0$, Claims 1, 2, (3.12), and the nonsmooth mountain pass theorem, we infer that there exists a point $u \in X$ such that

$$I(u) = c \geq \inf\bigl\{I(u) : \|u\| = r\bigr\} > 0, \tag{3.18}$$

$$0 \in \partial I(u). \tag{3.19}$$

From (3.18) it immediately follows that $u \neq 0$. By (3.19), on account of [30] (p.362) we thus have

$$\left(a + b\|u\|^2\right) \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega \omega v \, \mathrm{d}x \quad \forall v \in X, \omega \in \partial j(x,u) \text{ a.a. on } \Omega,$$

which evidently means

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \, \mathrm{d}x)\Delta u = \omega & \text{for a.a. } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\omega \in \partial j(x,u)$ a.a. on Ω. Hence the function $u \in X$ turns out to be a nontrivial solution for problem (1.2). □

Proof of Theorem 1.2 We consider the orthogonal decomposition $X = Y \oplus V$, where $Y = E_k = \bigoplus_{i=1}^k E(\lambda_i)$, $E(\lambda_i)$ be the eigenvalue space ($i = 1, 2, \ldots$) and $V = E_k^\perp$.

Claim 1. I is coercive.

From (J_2) and (J_3), we have

$$j(x, u(x)) \leq \alpha_1 |u(x)|^4 + c_5 \tag{3.20}$$

for almost all $x \in \Omega$ and some $c_5 > 0$. Then

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x, u(x)) \, \mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \alpha_1 \int_\Omega |u(x)|^4 \, \mathrm{d}x - c_5|\Omega| \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \alpha_1 c_4^4 \|u\|^4 - c_5|\Omega| \\ &\geq \frac{a}{2}\|u\|^2 - c_5|\Omega| \quad \left(\text{see } (J_3)\right). \end{aligned}$$

This means that $I(u)$ is coercive and so it is bounded below and satisfies the nonsmooth $(PS)_c$.

Claim 2. I satisfies a local linking at 0 with respect to (Y, V).

For $u \in V$, (J_2) and (J_5) mean that

$$j(x,u) < \frac{a}{2}\lambda_{k+1}|u|^2 + \frac{b}{8c_4^4}|u|^4 + c_6|u|^p, \tag{3.21}$$

for some $c_6 > 0$ and a.a. $x \in \Omega$, $u \in \mathbb{R}$. Then

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x, u(x)) \, \mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\lambda_{k+1} \int_\Omega |u|^2 \, \mathrm{d}x - \frac{b}{8c_4^4} \int_\Omega |u|^4 \, \mathrm{d}x - c_6 \int_\Omega |u|^p \, \mathrm{d}x \\ &\geq \frac{b}{8}\|u\|^4 - c_6 c_p^p \|u\|^p, \end{aligned}$$

where c_p^p satisfies $\int_\Omega |u|^p \, dx \le c_p^p (\int_\Omega |\nabla u|^2 \, dx)^2$. Since $p > 4$, letting $r_1 = (\frac{b}{8c_6 c_p^p})^{p-4}$, for all $0 < r < r_1$, we have $I(u) \ge 0$. For $u \in Y$, from (J_5), there exists $r_2 > 0$ with $\|u\| \le r_2 = \frac{\delta}{\mu}$. When $|u| \le \mu \|u\| \le \delta$.

$$j(x,u) \ge \frac{a}{2}\lambda_k u^2 + \frac{b}{4c_4^4}\lambda_k^2 |u|^4.$$

Since $\dim Y_k = k < +\infty$, then

$$I(u) \le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\lambda_k |u|_2^2 - \frac{b}{4c_4^4}\lambda_k^2 |u|_4^4 \le 0.$$

Choosing $r = \min\{r_1, r_2\}$, we find that I satisfies a local linking at 0 with respect to (Y, V).

If $\inf_{u \in X} I(u) < 0 = I(0)$, from Theorem 2.1, we obtain two nontrivial critical points $\hat{u}_1, \hat{u}_2 \in X$ of I, and hence two nontrivial solutions of problem (1.2).

If $\inf_{u \in X} I(u) = 0$, then all $y \in Y \setminus \{0\}$ with $\|y\| \le r_2$ satisfy

$$I(y) = \inf_{y \in X} I(u)$$

and so all are the nontrivial critical points of I, and hence we find a continuum of nontrivial solutions of problem (1.2). In both cases, by standing regularity theory, these solutions belong in $C_0^1(\bar{\Omega})$. □

In the following, we choose an orthonormal basis (e_j) of X and we define $X_j = \mathbb{R}e_j$. We will use the nonsmooth fountain theorem with the antipodal action of $\mathbb{Z}_2$ to prove Theorem 1.3.

Proof of Theorem 1.3 From the proof of Theorem 1.1, we have already checked that $I(u)$ satisfies the nonsmooth $(PS)_c$. Note that I is an even functional. We only need to prove that for k large enough there exist $\rho_k > r_k > 0$ such that

(A_2) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \le 0$,
(A_3) $b_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \to \infty$, as $k \to +\infty$.

For $u \in Y_k$ (see (2.3)), from Claim 2 in Theorem 1, there exist $M_5 > 0$, $\alpha_2 > 0$, and $\alpha_3 > 0$ such that

$$j(x,u) \ge \alpha_2 |u|^q - \alpha_3 \tag{3.22}$$

for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$. From (3.22), we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x,u) \, dx \\ &\le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \alpha_2 |u|_q^q - \alpha_3 |\Omega|. \end{aligned} \tag{3.23}$$

Noting that $\dim Y_k < +\infty$, so all norms of Y_k are equivalent. Then, from (3.23), we can find $\rho_k > 0$ large enough such that

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \le 0. \tag{3.24}$$

For $u \in Z_k$, letting $\beta_k = \sup_{u\in Z_k,\|u\|=1} |u|_p$, $k = 1,2,\ldots$, from Lemma 2.1 and the mean value theorem, we have $\beta_k \to 0$ as $k \to \infty$. Set $r_k = (b^{-1}\alpha_4 p\beta_k^p)^{\frac{1}{4-p}}$ for $u \in Z_k$ with $\|u\| = r_k$. By virtue of (J_2), we derive

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x,u)\,\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \alpha_4\int_\Omega |u|^p\,\mathrm{d}x - c_8|\Omega| \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \alpha_4\beta_k^p\|u\|^p - c_8|\Omega| \\ &= \frac{a}{2}\left(b^{-1}\alpha_4 p\beta_k^p\right)^{\frac{2}{4-p}} + \left(\frac{1}{4}-\frac{1}{p}\right)\left(b^{-1}\right)^{\frac{p}{4-p}}\left(\alpha_4 p\beta_k^p\right)^{\frac{4}{4-p}} - c_8|\Omega| \end{aligned}$$

for some $\alpha_4, c_8 > 0$. Since $p > 4$ and $\beta_k \to 0$ as $k \to +\infty$, we obtain

$$b_k = \inf_{u\in Z_k,\|u\|=r_k} I(u) \to +\infty \quad \text{as } k \to +\infty. \tag{3.25}$$

So from (3.24), (3.25), and noting that $I(u)$ satisfies the nonsmooth $(PS)_c$, by the nonsmooth fountain theorem, we deduce Theorem 1.3. □

Proof of Theorem 1.4 From the proof of Theorem 1.3, we need to prove that any $(PS)_c$ sequence is bounded and the condition (A_2) is satisfied. For $u \in Y_k$, by virtue of (J_8) and (J_2), we know that for any $c_9 > 0$, there exist constants $M_6 > 0$, $|u| \geq M_6$, and $c_{10} > 0$ such that

$$j(x,u) \geq c_9|u|^4 - c_{10}$$

for a.a. $x \in \Omega$, all $u \in \mathbb{R}$. Then

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega j(x,u)\,\mathrm{d}x \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_9|u|_4^4 + c_{10}|\Omega|. \end{aligned}$$

Since all norms are equivalent on the finitely dimensional space Y_k, we can find some $\theta > 0$ such that

$$I(u) \leq \frac{a}{2}\|u\|^2 - \left(c_9\theta - \frac{b}{4}\right)\|u\|^4 + c_{10}|\Omega|. \tag{3.26}$$

Let $c_9 > \frac{b}{4\theta}$. Then, from (3.26), we can find $\rho_k > 0$ large enough such that

$$a_k = \max_{u\in Y_k,\|u\|=\rho_k} I(u) \leq 0. \tag{3.27}$$

So the condition (A_2) holds.

Next, we show that I satisfies the nonsmooth $(PS)_c$ on X. Let $\{u_n\}_{n\geq 1} \subseteq X$ such that

$$\left|I(u_n)\right| \leq M_7 \quad \text{for all } n \geq 1 \quad \text{and} \quad m^I(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.28}$$

Recalling that $u_n^* \in \partial I(u_n)$ a.a. on Ω, from (3.28), we obtain

$$-\langle u_n^*, u_n\rangle = -\left(a + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + \int_\Omega \omega_n u_n\,\mathrm{d}x \le \varepsilon_n \tag{3.29}$$

and

$$2a\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + b\|u_n\|^4 - \int_\Omega 4j(x,u_n)\,\mathrm{d}x \le 4M_7, \tag{3.30}$$

where $\omega_n \in \partial j(x,u_n)$ a.a. on Ω. Adding (3.29) and (3.30), we have

$$a\|u_n\|^2 + \int_\Omega \big(\omega_n u_n - 4j(x,u_n)\big)\,\mathrm{d}x \le \varepsilon_n + 4M_7.$$

Then in a similar way as used in the proof of Theorem 1.1, we can infer that $\{u_n\}_{n\ge1} \subseteq X$ is bounded in X. From Lemma 3.1, we find that I satisfies the nonsmooth $(PS)_c$. Hence we complete the proof of Theorem 1.4. □

Proof of Theorem 1.5 From the proofs of Theorem 1.3 and Theorem 1.4, it is necessary to show that every $(PS)_c$ sequence $\{u_n\}_{n\ge1} \subset X$ of I is bounded in X. Let $\{u_n\}_{n\ge1} \subseteq X$ be a sequence, such that

$$\big|I(u_n)\big| \le M_8 \quad \text{and} \quad m^I(u_n) \to 0.$$

Remember that $u_n^* \in \partial I(u_n)$ a.e. on Ω and $m^I(u_n) = \|u_n^*\|_{X^*}$ for $n \ge 1$. From Lemma 3.1, we only need to show that $\{x_n\}_{n\ge1} \subset X$ is bounded in X. Supposing that $\{u_n\}_{n\ge1} \subseteq X$ is not bounded in X, we may assume that $\|u_n\| \to +\infty$ as $n \to +\infty$, and we have

$$\begin{aligned}4M_8 + 1 + \|u_n\| &\ge 4I(u_n) - \langle u_n^*, u_n\rangle \\ &= a\|u_n\|^2 + \int_\Omega \big(\omega_n u_n - 4j(x,u_n)\big)\,\mathrm{d}x \\ &\ge \min\{1,a\}\|u_n\|^2 + \int_\Omega \big(\omega_n u_n - 4j(x,u_n)\big)\,\mathrm{d}x,\end{aligned} \tag{3.31}$$

where $\omega_n \in \partial j(x,u_n(x))$ a.a. on Ω. From (3.31), for n large enough, we obtain

$$\begin{aligned}4M_8 + 1 &\ge \min\{1,a\}\|u_n\|^2 - \|u_n\| + \int_\Omega \big(\omega_n u_n - 4j(x,u_n)\big)\,\mathrm{d}x \\ &\ge \int_\Omega \big(\omega_n u_n - 4j(x,u_n)\big)\,\mathrm{d}x.\end{aligned} \tag{3.32}$$

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \ge 1.$$

Then $\|y_n\| = 1$. Note that

$$\frac{\langle u_n^*, u_n\rangle}{\|u_n\|^4} = \frac{a\|u_n\|^2}{\|u_n\|^4} + \frac{b\|u_n\|^4}{\|u_n\|^4} - \frac{\int_\Omega \omega_n u_n\,\mathrm{d}x}{\|u_n\|^4},$$

where $\omega_n \in \partial j(x, u_n(x))$ a.e. on Ω. Since $\langle u_n^*, u_n\rangle \le \|u_n^*\|_{X^*}\|u_n\|$ and $\|u_n^*\|_{X^*} \to 0$, we have

$$\limsup_{n\to+\infty} \frac{\int_\Omega \omega_n u_n \,\mathrm{d}x}{\|u_n\|^4} = b,$$

and

$$\liminf_{n\to+\infty} \frac{\int_\Omega \omega_n u_n \,\mathrm{d}x}{\|u_n\|^4} = b,$$

for $\omega_n \in \partial j(x, u_n(x))$ a.a. on Ω, then we obtain

$$\lim_{n\to+\infty} \frac{\int_\Omega \omega_n u_n \,\mathrm{d}x}{\|u_n\|^4} = b. \tag{3.33}$$

In the following, we will prove

$$\lim_{n\to+\infty} \frac{\int_\Omega \omega_n u_n \,\mathrm{d}x}{\|u_n\|^4} = 0, \tag{3.34}$$

where $\omega_n \in \partial j(x, u_n(x))$ a.a. on Ω. For convenience, we set $H(x,u) = \omega_n u_n - 4j(x,u_n)$ for $\omega_n \in \partial j(x, u_n(x))$ a.a. $x \in \Omega$. $h(\rho) = \inf\{H(x,u) : x \in \Omega, |u| \ge \rho\}$, $\Lambda_n(\alpha,\beta) = \{x \in \Omega : \alpha \le |u_n(x)| < \beta\}$ and $E_\alpha^\beta = \inf\{\frac{H(x,u)}{|u|^2} : x \in \Omega, \alpha \le |u| < \beta\}$. Then from (J_8) and (J_9), we have $h(\rho) \to +\infty$ as $\rho \to +\infty$ and for large $\alpha > 0$, $h(\alpha) > 0$, $E_\alpha^\beta > 0$, and

$$H(x,u_n) \ge E_\alpha^\beta |u_n|^2, \quad \forall x \in \Lambda_n(\alpha,\beta).$$

By virtue of (J_6) and (3.32), for large n and α with $\alpha < \beta$, we have

$$\begin{aligned}
4M_8 + 1 &\ge \int_{\Lambda_n(0,\alpha)} H(x,u_n)\,\mathrm{d}x + \int_{\Lambda_n(\alpha,\beta)} H(x,u_n)\,\mathrm{d}x + \int_{\Lambda_n(\beta,+\infty)} H(x,u_n)\,\mathrm{d}x \\
&\ge \int_{\Lambda_n(0,\alpha)} H(x,u_n)\,\mathrm{d}x + E_\alpha^\beta \int_{\Lambda_n(\alpha,\beta)} |u_n|^2\,\mathrm{d}x + h(\beta)\bigl|\Lambda_n(\beta,+\infty)\bigr|.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{4M_8+1}{\|u_n\|^2} \ge{}& \frac{1}{\|u_n\|^2}\int_{\Lambda_n(0,\alpha)} H(x,u_n)\,\mathrm{d}x \\
&+ \frac{E_\alpha^\beta}{\|u_n\|^2}\int_{\Lambda_n(\alpha,\beta)} |u_n|^2\,\mathrm{d}x + h(\beta)\frac{|\Lambda_n(\beta,+\infty)|}{\|u_n\|^2}.
\end{aligned} \tag{3.35}$$

From (J_2), we know that $\frac{1}{\|u_n\|^2}\int_{\Lambda_n(0,\alpha)} H(x,u_n)\,\mathrm{d}x$ is bounded. Hence from (3.35) and noting that $\lim_{\beta\to+\infty} h(\beta) \to +\infty$, we derive that

$$\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\alpha,\beta)} H(x,u_n)\,\mathrm{d}x \quad \text{and} \quad \frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)} H(x,u_n)\,\mathrm{d}x$$

are bounded and $\lim_{\beta\to+\infty}\frac{|\Lambda_n(\beta,+\infty)|}{\|u_n\|^2}=0$ uniformly in n. Without loss of generality, we assume that $2^*<+\infty$, therefore from the Hölder inequality, for any $r\in[2,2^*]$

$$\begin{aligned}\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)}|y_n|^r\,\mathrm{d}x&\le\frac{1}{\|u_n\|^{\frac{2r}{2^*}}}\left(\int_{\Lambda_n(\beta,+\infty)}\left(|y_n|^r\right)^{\frac{2^*}{r}}\,\mathrm{d}x\right)^{\frac{r}{2^*}}\left|\frac{|\Lambda_n(\beta,+\infty)|}{\|u_n\|^2}\right|^{\frac{2^*-r}{2^*}}\\&\le\frac{c_{2^*}^r}{\|u_n\|^{\frac{2r}{2^*}}}\left|\frac{|\Lambda_n(\beta,+\infty)|}{\|u_n\|^2}\right|^{\frac{2^*-r}{2^*}}\to0\end{aligned}$$

as $\beta\to+\infty$ uniformly in n.

Since $\|u_n\|\to+\infty$, we can find a positive integral number N_0 such that $\|u_n\|\ge1$ if $n>N_0$. Setting $r=\frac{2\sigma}{\sigma-1}$, and noting that $\sigma>1+\frac{2}{2^*-2}$, we obtain $r\in(2,2^*]$ and $\sigma=\frac{r}{r-2}$. By virtue of condition (J_9), we have

$$\begin{aligned}\left|\int_{\Lambda_n(\beta,+\infty)}\frac{\omega_nu_n}{\|u_n\|^4}\,\mathrm{d}x\right|&\le\int_{\Lambda_n(\beta,+\infty)}\frac{|\omega_n|}{\|u_n\||u_n|}\cdot\frac{y_n^2}{\|u_n\|}\,\mathrm{d}x\\&\le\left[\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)}\left(\frac{|\omega_n|}{|u_n|}\right)^\sigma\mathrm{d}x\right]^{\frac{1}{\sigma}}\left[\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)}|y_n|^r\,\mathrm{d}x\right]^{\frac{2}{r}}\\&\le\left[\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)}lH(x,u_n)\,\mathrm{d}x\right]^{\frac{1}{\sigma}}\left[\frac{1}{\|u_n\|^2}\int_{\Lambda_n(\beta,+\infty)}|y_n|^r\,\mathrm{d}x\right]^{\frac{2}{r}}\\&\to0\end{aligned}\tag{3.36}$$

as $\beta\to+\infty$ uniformly in n and $\omega_n\in\partial j(x,u_n)$ a.a. on Ω. Furthermore, from (J_2), we have

$$\left|\int_{\Lambda_n(0,\alpha)}\frac{\omega_nu_n}{\|u_n\|^4}\,\mathrm{d}x\right|\le\int_{\Lambda_n(0,\alpha)}\frac{c(\alpha)|y_n|}{\|u_n\|^3}\,\mathrm{d}x\to0\tag{3.37}$$

as $n\to+\infty$, $\omega_n\in\partial j(x,u_n)$ a.a. on Ω. $c(\alpha)$ is a positive constant, and

$$\frac{1}{\|u_n\|^4}\left|\int_{\Lambda_n(\alpha,\beta)}\omega_nu_n\,\mathrm{d}x\right|\le\int_{\Lambda_n(\alpha,\beta)}\frac{c(\alpha,\beta)|y_n|}{\|u_n\|^3}\,\mathrm{d}x\to0\tag{3.38}$$

as $n\to+\infty$, $\omega_n\in\partial j(x,u_n)$ a.a. on Ω. Then from hypothesis (J_6) and (3.36)-(3.38), we obtain

$$\begin{aligned}b=\lim_{n\to+\infty}\frac{\int_\Omega\omega_nu_n\,\mathrm{d}x}{\|u_n\|^4}&\le\lim_{n\to+\infty}\left|\int_{\Lambda_n(0,\alpha)}\frac{\omega_nu_n}{\|u_n\|^4}\,\mathrm{d}x\right|+\lim_{n\to+\infty}\frac{1}{\|u_n\|^4}\left|\int_{\Lambda_n(\alpha,\beta)}\omega_nu_n\,\mathrm{d}x\right|\\&\quad+\lim_{n\to+\infty}\left|\int_{\Lambda_n(\beta,+\infty)}\frac{\omega_nu_n}{\|u_n\|^4}\,\mathrm{d}x\right|=0,\end{aligned}$$

i.e., $b=0$; a contradiction to the fact $b>0$. Hence $\{u_n\}_{n\ge1}\subseteq X$ is a bounded sequence. From Lemma 3.1, we find that $\{u_n\}_{n\ge1}$ satisfies $(PS)_c$. Hence we complete the proof of Theorem 1.5. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors wish to thank the referees for their corrections and remarks. This research is partly supported by the NSFC (Nos. 11371127).

Received: 24 October 2014 Accepted: 28 January 2015 Published online: 20 February 2015

References

1. Lei, C, Liao, J, Tang, C: Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents. J. Math. Anal. Appl. **421**, 521-538 (2015)
2. Chen, J: Multiple positive solutions to a class of Kirchhoff equation on $\mathbb{R}^3$ with indefinite nonlinearity. Nonlinear Anal. **96**, 134-145 (2014)
3. Park, S: General decay of a transmission problem for Kirchhoff type wave equations with boundary memory condition. Acta Math. Sci. **34B**(5), 1395-1403 (2014)
4. Mao, A, Zhang, Z: Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition. Nonlinear Anal. **70**, 1275-1287 (2009)
5. Figueiredo, G, Rodrigo, C, Júnior, J: Study of a nonlinear Kirchhoff equation with non-homogeneous material. J. Math. Anal. Appl. **416**, 597-608 (2014)
6. Perera, K, Zhang, Z: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246-255 (2006)
7. Liu, Z, Guo, S: Existence and concentration of positive ground states for a Kirchhoff equation involving critical Sobolev exponent. Z. Angrew. Math. Phys. (2014). doi:10.1007/s00033-014-0431-8
8. Chen, C, Kuo, Y, Wu, T: The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions. J. Differ. Equ. **250**, 1876-1908 (2011)
9. Liang, Z, Li, F, Shi, J: Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **31**, 155-167 (2014)
10. Chang, K: On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms. Sci. Sin. **21**, 139-158 (1978)
11. Chang, K: Variational methods for nondifferentiable functions and their applications to partial differential equations. J. Math. Anal. Appl. **80**, 102-129 (1981)
12. Chang, K: The obstacle problem and partial differential equations with discontinuous nonlinearities. Commun. Pure Appl. Math. **33**, 117-146 (1980)
13. Motreanu, D, Rădulescu, V: Variational and Non-Variational Methods in Nonlinear Analysis and Boundary Value Problems. Kluwer Academic, Boston (2003)
14. Naniewicz, Z, Pangiotopoulos, P: Mathematical Theory of Hemivariational Inequalities and Applications. Dekker, New York (1995)
15. Motreanu, D, Pangiotopoulos, P: Minimax Theorems and Qualitative Properties of the Solutions of Hemivaritational Inequalities. Kluwer Academic, Dordrecht (1999)
16. Gasiński, L, Papageorgiou, N: Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems. Chapman and Hall/CRC Press, Boca Raton (2005)
17. Iannizzotto, A: Three solutions for a partial differential inclusion via nonsmooth critical point theory. Set-Valued Anal. **19**, 311-327 (2011)
18. Filippakis, M, Gasiński, L, Papageorgiou, N: Multiple positive solutions for eigenvalue problems of hemivariational inequalities. Positivity **10**, 495-515 (2006)
19. Motreanu, D, Papageorgiou, N: Existence and multiplicity of solutions for Neumann problems. J. Differ. Equ. **232**, 1-35 (2007)
20. Kyritsi, S, Papageorgiou, N: Solvability of semilinear hemivariational inequalities at resonance using generalized Landesman-Lazer conditions. Monatshefte Math. **142**, 227-241 (2004)
21. Gasiński, L, Motreanu, D, Papageorgiou, N: Multiplicity of nontrivial solutions for elliptic equations with nonsmooth potential and resonance at higher eigenvalues. Proc. Indian Acad. Sci. Math. Sci. **116**, 233-255 (2006)
22. Denkowski, Z, Gasiński, L, Papageorgiou, N: Nontrivial solutions for resonant hemivariational inequalities. J. Glob. Optim. **34**, 317-337 (2006)
23. Dai, G: Nonsmooth version of fountain theorem and its application to a Dirichlet-type differential inclusion problem. Nonlinear Anal. **72**, 1454-1461 (2010)
24. Clarke, F: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
25. Lê, A: Eigenvalue problems for the p-Laplacian. Nonlinear Anal. **64**, 1057-1099 (2006)
26. Kandilakis, D, Kourogenis, N, Papageorgiou, N: Two nontrivial critical points for nonsmooth functions via local linking and applications. J. Glob. Optim. **34**, 317-337 (2006)
27. Brézis, H, Nirenberg, L: Remarks on finding critical points. Commun. Pure Appl. Math. **44**, 939-963 (1991)
28. Bartsh, T: Infinitely many solutions of a symmetric Dirichlet problem. Nonlinear Anal. TMA **20**, 1205-1216 (1993)
29. Willem, M: Minimax Theorems. Birkhäuser, Boston (1996)
30. Denkowski, Z, Migorski, S, Papagergiou, N: An Introduction to Nonlinear Analysis: Theory. Kluwer, New York (2003)