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Multiple solutions for the p-Laplacian problem with supercritical exponent

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Abstract

In this paper, we explore the existence of multiple solutions to the following p-Laplacian type of equation with supercritical Sobolev-exponent:

$$\begin{cases} -\Delta_p u + |u|^{r-2}u = \gamma |u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smoothly bounded open domain in $\mathbb{R}^n$ $(n > p \geq 2)$, $r > p^*$, $p^* \triangleq \frac{np}{n-p}$ is a critical Sobolev exponent. We prove that if $1 < s < p < n$ and $\gamma \in \mathbb{R}^+$, the above equation possesses infinitely many weak solutions. Furthermore, if $1 < s = p < n$ and $\lambda_m < \gamma \leq \lambda_{m+1}$, there exists at least m-pair nontrivial solutions, where λ_m is the m-eigenvalue value defined in (2.2).

Keywords: p-Laplacian; supercritical exponent; functional; weak-convergence

1 Introduction

This paper is mainly concerned with the existence of solutions to the following p-Laplacian equation with supercritical exponent:

$$\begin{cases} -\Delta_p u + |u|^{r-2}u = \gamma |u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a smoothly bounded open domain in $\mathbb{R}^n$ and $\gamma > 0$, $1 < s \leq p < n$, $r > p^* \triangleq \frac{np}{n-p}$ is the critical Sobolev exponent.

When $p = 2$, (1.1) can be reduced to

$$\begin{cases} -\Delta u + |u|^{r-2}u = \gamma |u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $r, s \leq 2^*$, these types of equations have been widely studied. See [1–7] and references therein.

If $p \neq 2$, the p-Laplacian case has been studied extensively as well; see, *e.g.*, [8–12] for subcritical exponent, [13, 14] for critical case and the references therein. Recently, Avci in [15] proved the existence of nontrivial solutions for p-Laplacian equation via variational approach and the monotone operator method. The main difficulty for the case when $p \neq 2$ is that $W_0^{1,p}(\Omega)$ is not a Hilbert space and

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega)$$

in general does not imply that

$$|\nabla u_{n_k}|^{p-2}\frac{\partial u_{n_k}}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{in } L^{\frac{p}{p-1}}(\Omega).$$

So the concentration-compactness principle introduced by Lions in [16, 17] was applied to deal with this problem.

It seems that great progress has been made for the subcritical and critical exponents. Naturally, the question whether there exist solutions for the supercritical case or not is interesting for us. However, the problem for $r > p^* \triangleq \frac{np}{n-p}$ is more challenging since it should be shown that the Palais-Smale condition is still valid in this case.

Our main results are shown as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded open domain and $r > p^*$. If $1 < s < p < n$ and $\gamma \in \mathbb{R}^+$, problem* (1.1) *possesses infinitely many weak solutions in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$. If $1 < s = p < n$ and $\lambda_m < \gamma \leq \lambda_{m+1}$, there exists at least m-pair nontrivial solutions to problem* (1.1), *where λ_m is the m-eigenvalue value defined in* (2.2).

To the best of our knowledge, Theorem 1.1 is new and extends a similar result in [13, 14] for the p-Laplacian type problem. Clearly, solutions to (1.1) correspond to the critical points of the energy functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{r}\int_\Omega |u|^r\,dx - \frac{\gamma}{s}\int_\Omega |u|^s\,dx, \quad u \in W_0^{1,p}(\Omega) \cap L^r(\Omega). \tag{1.2}$$

One notes that $I(u)$ is an even C^1 functional which is bounded from below on the space $W_0^{1,p}(\Omega) \cap L^r(\Omega)$. If $I(u)$ satisfies the $(PS)_c$ condition, by using Ljusternik-Schnirelman's theory for Z_2 invariant functional (see [18]), we get a sequence of critical points $\{u_n\}$ of $I(u)$, which implies that $I(u)$ has multiple weak solutions.

The main difficulty in the proof is to verify that $I(u)$ satisfies the $(PS)_c$ condition. As $W_0^{1,p}(\Omega) \cap L^r(\Omega)$ is not a Hilbert space, even if we have a bounded $(PS)_c$ sequence $\{u_n\}$ for $I(u)$ and $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$, it is not clear whether there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying

$$|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}(\Omega),$$
$$|u_{n_k}|^r \rightharpoonup |u|^r \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

To overcome these difficulties, we use the concentration-compactness principle as in [14], which deals with a p-Laplacian type equation with critical exponent. As both a p-Laplacian operator and a supercritical exponent appear in this problem, more subtle analysis is needed, which is shown in the following theorem.

Theorem 1.2 *Let the assumptions of Theorem* 1.1 *hold, then the functional I satisfies the Palais-Smale condition, i.e., if $I(u_n) \to c$ and $I'(u_n) \to 0$, then u_n has a convergent subsequence.*

Our notations are standard: set $X \triangleq W_0^{1,p}(\Omega) \cap L^r(\Omega)$ with topology (see following Lemma 2.1)

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p} + \left(\int_\Omega |u|^r \, dx\right)^{1/r};$$

$\|\cdot\|_s$ denotes the norm of $L^s(\Omega)$; '$\to$' and '$\rightharpoonup$' represent strong and weak convergence in related function spaces, respectively; c, c_i, $i = 1, 2, \ldots$, denote constants and may be different in different places; $\mathfrak{D}(\mathbb{R}^n) = \{u \in C^\infty(\Omega) : \operatorname{supp} u \text{ is a compact subset of } \mathbb{R}^n\}$ and $\mathcal{M}(\mathbb{R}^n)$ is a Radon measure space; L- $\operatorname{mes}(A)$ denotes the Lebesgue measure of A.

This article is organized as follows. In Section 2 we present some preliminary results, and in Section 3 we prove our main theorems.

2 Preliminary results

We begin with the definition of weak solution for functional I.

Definition 2.1 $u \in X$ is said to be a weak solution for nonlinear elliptic equation (1.1) if for any $v \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, the following equality holds:

$$\left\langle -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right), v\right\rangle + \left\langle |u|^{r-2} u, v\right\rangle = \gamma \left\langle |u|^{s-2} u, v\right\rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is dual between X and $X^* \triangleq (W_0^{1,p}(\Omega) \cap L^r(\Omega))^* \subset W^{-1,\frac{p}{p-1}}(\Omega) \oplus L^{\frac{r}{r-1}}(\Omega)$.

In order to demonstrate that Definition 2.1 is valid, we need the following lemma. Since the proof of the lemma is quite basic, we chose to omit it here.

Lemma 2.1 *Let X be defined as above, then X is a reflexive Banach space with the duality $X^* \subset W^{-1,\frac{p}{p-1}}(\Omega) \oplus L^{\frac{r}{r-1}}(\Omega)$.*

Now here comes the definition of eigenvalue λ_m which is generated in the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to [8, 9], we set

$$\lambda_m = \inf_{A \in \Sigma_k} \left\{ \sup_{u \in A} \int_\Omega |\nabla u|^p \, dx : \int_\Omega |u|^p \, dx = 1, A \subset W_0^{1,p}(\Omega) \backslash \{0\} \right\}, \tag{2.2}$$

where Σ_k is the collection of symmetric subsets A of $W_0^{1,p}(\Omega)$ such that $\Gamma(A) \geq m$ and the set $\{v \in A : \|v\|_p = (\int_\Omega |v|^p \, dx)^{1/p} = 1\}$ is compact. It is well known that $\{\lambda_m\}$ is a nondecreasing divergent sequence, and λ_m is an eigenvalue of $-\Delta_p$ for every $m \geq 1$. And the Z_2-index is defined by $\Gamma(A) = \min\{m \in Z_+ |$ odd continuous map $\varphi : A \to \mathbb{R}^m \backslash \{0\}\}$.

Now, we denote $B(x_0, a) = \{x \in \mathbb{R}^n | |x - x_0| \leq a\}$, and let $\varphi \in \mathfrak{D}(\mathbb{R}^n)$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ for $x \in B(0, \frac{1}{2})$ and $\operatorname{supp} \varphi \subset B(0,1)$. Also, for $\epsilon > 0$, we write $\varphi_\epsilon = \varphi(\frac{x}{\epsilon})$ for $x \in \mathbb{R}^n$, then we have the following.

Proposition 2.1 (see [14]) *For any $x_j \in \mathbb{R}^n$, $u \in L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{np}{n-p}$, we have*

$$\int_{\mathbb{R}^n} \bigl|u(x)\nabla\varphi_\epsilon(x - x_j)\bigr|^p \, dx \leq \left(\int_{\mathbb{R}^n} |\nabla\varphi|^{\frac{p^* p}{p^*-p}} \, dx\right)^{\frac{p^*-p}{p^*}} \left(\int_{B(x_j,\epsilon)} |u|^{p^*} \, dx\right)^{\frac{p}{p^*}}.$$

Let us recall that the energy functional of problem (1.1) is

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{r}\int_\Omega |u|^r \, dx - \frac{\gamma}{s}\int_\Omega |u|^s \, dx, \quad u \in W_0^{1,p}(\Omega) \cap L^r(\Omega).$$

For $c \in \mathbb{R}$, we say that a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a $(PS)_c$ sequence of $I(u)$ if

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } X^*,$$

where I' is the Fréchet derivative of I.

3 Proofs

3.1 Proof of Theorem 1.2

Now, we first present the following lemmas which will be used in proving Theorem 1.2.

Lemma 3.1 *If $u_n \rightharpoonup u$ in X, then*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \qquad u_n \rightharpoonup u \quad \text{in } L^r(\Omega), \text{as } n \to \infty; \tag{3.1}$$

$$u_n \to u \quad \text{in } L^t(\Omega), \forall 2 \leq t < r, \text{as } n \to \infty; \tag{3.2}$$

$$|u_n|^{r-2}u_n \rightharpoonup |u|^{r-2}u \quad \text{in } L^{\frac{r}{r-1}}(\Omega), \text{as } n \to \infty. \tag{3.3}$$

Proof First, to prove (3.1), by applying Lemma 2.1, one sees that for any $f \in X^*$, there exist $f_1 \in W^{-1,\frac{p}{p-1}}(\Omega)$ and $f_2 \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$\langle f, u_n\rangle_{X^*,X} = \langle f_1, u_n\rangle_{W^{-1,\frac{p}{p-1}}(\Omega), W_0^{1,p}(\Omega)} + \langle f_2, u_n\rangle_{L^{\frac{r}{r-1}}(\Omega), L^r(\Omega)}. \tag{3.4}$$

Now, choosing $f_2 = 0$ in (3.4) (noting that $W^{-1,\frac{p}{p-1}}(\Omega) \times \{0\} \subset X^*$) and combining with $u_n \rightharpoonup u$ in X, one deduces that for any $f_1 \in W^{-1,\frac{p}{p-1}}(\Omega)$,

$$\begin{aligned}
\langle f, u_n\rangle_{X^*,X} &= \langle f_1, u_n\rangle_{W^{-1,\frac{p}{p-1}}(\Omega), W_0^{1,p}(\Omega)} + \langle 0, u_n\rangle_{L^{r/(r-1)}(\Omega), L^r(\Omega)} \\
&= \langle f_1, u_n\rangle_{W^{-1,\frac{p}{p-1}}(\Omega), W_0^{1,p}(\Omega)} \\
&\to \langle f_1, u\rangle_{W^{-1,\frac{p}{p-1}}(\Omega), W_0^{1,p}(\Omega)},
\end{aligned}$$

which implies that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Similarly, choosing $f_1 = 0$ in (3.4) (also noting that $\{0\} \times L^{\frac{r}{r-1}}(\Omega) \subset X^*$), we can get $u_n \rightharpoonup u$ in $L^r(\Omega)$ and this finishes the proof of (3.1).

Second, convergence (3.2) is a direct conclusion from

$$\|u-v\|_{L^t(\Omega)} \le \|u-v\|_{L^2(\Omega)}^{\theta}\|u-v\|_{L^r(\Omega)}^{1-\theta}$$

and $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ together with (3.1).

Finally, we prove assertion (3.3). It follows from the mean value theorem and the Hölder inequality that for $v \in W_0^{1,p}(\Omega) \cap L^{r+\alpha}(\Omega)$ with $\alpha > 0$,

$$\begin{aligned}
&\langle |u_n|^{r-2}u_n - |u|^{r-2}u, v\rangle \\
&\quad = \int_\Omega \bigl[|u_n|^{r-2}u_n - |u|^{r-2}u\bigr] v\,dx \\
&\quad \le \int_\Omega r\bigl|\theta u_n + (1-\theta)u\bigr|^{r-2}|u_n-u||v|\,dx \\
&\quad \le r\biggl[\int_\Omega \bigl\{\bigl|\theta u_n + (1-\theta)u\bigr|^{r-2}\bigr\}^{\frac{r}{r-2}}\,dx\biggr]^{\frac{r-2}{r}} \biggl(\int_\Omega \bigl[|u_n-u||v|\bigr]^{\frac{r}{2}}\,dx\biggr)^{\frac{2}{r}} \\
&\quad \le C\biggl(\biggl[\int_\Omega \bigl\{|u_n-u|^{\frac{r}{2}}\bigr\}^{\frac{2t}{r}}\,dx\biggr]^{\frac{r}{2t}}\biggr)^{\frac{2}{r}} \biggl(\biggl[\int_\Omega \bigl\{|v|^{\frac{r}{2}}\bigr\}^{\frac{2t}{2t-r}}\,dx\biggr]^{\frac{2t-r}{2t}}\biggr)^{\frac{2}{r}} \quad \text{with } t < r < 2t \\
&\quad \le C\biggl[\int_\Omega |u_n-u|^t\,dx\biggr]^{\frac{1}{t}} \biggl[\int_\Omega |v|^{r+\alpha}\,dx\biggr]^{\frac{2t-r}{rt}} \quad \text{with } \alpha = \frac{r(r-t)}{2t-r}.
\end{aligned} \tag{3.5}$$

Due to (3.2), the right-hand side of inequality (3.5) $\to 0$ as $n \to \infty$. Now since $W_0^{1,p}(\Omega) \cap L^{r+\alpha}(\Omega)$ is dense in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$, for any $\epsilon > 0$, $v \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, there exists a $v_\epsilon \in W_0^{1,p}(\Omega) \cap L^{r+\alpha}(\Omega)$ such that $\|v_\epsilon - v\|_X \le \frac{\epsilon}{2M}$, and for v_ϵ, from (3.5), there exists $N \in \mathbb{N}$ such that for $n \ge N$, we have

$$\langle |u_n|^{r-2}u_n - |u|^{r-2}u, v_\epsilon\rangle < \frac{\epsilon}{2}.$$

Therefore, together again with (3.5), there exists an integer N such that for $n > N$,

$$\langle |u_n|^{r-2}u_n - |u|^{r-2}u, v\rangle = \langle |u_n|^{r-2}u_n - |u|^{r-2}u, v - v_\epsilon\rangle + \langle |u_n|^{r-2}u_n - |u|^{r-2}u, v_\epsilon\rangle \le \epsilon.$$

This implies that $|u_n|^{r-2}u_n \rightharpoonup |u|^{r-2}u$ in $L^{\frac{r}{r-1}}(\Omega)$. □

The next step is to consider the property of the functional I.

Lemma 3.2 *The functional I is C^1 and coercive on X.*

Proof Note that the assumptions $1 < s \le p < n$ and $r > p^*$. One can conclude that $\lim_{\|u\|_X \to \infty} I(u) \to \infty$, which implies that the functional I is coercive. Moreover, the standard argument yields that I is C^1 functional on X with

$$I'(u) \triangleq -\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) + |u|^{r-2}u - \gamma|u|^{s-2}u. \tag{3.6}$$

□

Before considering the $(PS)_c$ sequence of $I(u)$, we need the following crucial lemma.

Lemma 3.3 *Let the assumptions of Lemma* 3.2 *hold. If $I(u_n) \to c$ and $I'(u_n) \to 0$, then there exists the subsequence of u_n denoted by u_{n_k} such that*

$$-\operatorname{div}\big(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\big) \rightharpoonup -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \quad \text{in } W^{-1,\frac{p}{p-1}}(\Omega). \tag{3.7}$$

Clearly, this is equivalent to proving that

$$|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}(\Omega). \tag{3.8}$$

Remark 3.1 Obviously, $|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$ by simple calculations. And from the reflexivity of $L^{\frac{p}{p-1}}(\Omega)$, the sequence $|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}$ will weakly converge in $L^{\frac{p}{p-1}}(\Omega)$ to a function χ. To establish (3.8), we need to show that $\chi = |\nabla u|^{p-2}\nabla u$. And the standard argument leads to show

$$\nabla u_{n_k}(x) \to \nabla u(x), \quad \text{a.e. } x \in \Omega. \tag{3.9}$$

It should be noted that this is different from the critical or subcritical case.

Proof of Lemma 3.3 The proof is divided into two steps.

Step 1: Due to Lemma 3.2, it follows from $I(u_n) \to c$ that the sequence $\{u_n\}$ is bounded in X. Hence, there exists a subsequence(also denoted by u_n) such that $u_n \rightharpoonup u$ in X. So, from Lemma 3.1, we conclude that u_n satisfies the following conditions:

$$\begin{cases} u_n \rightharpoonup u \text{ in } X; \\ u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightharpoonup u \text{ in } L^r(\Omega); \\ u_n \to u \text{ in } L^t(\Omega) \text{ with } 2 \le t < r; \\ u_n \to u \text{ a.e. in } \Omega; \\ |u_n|^{r-2}u_n \rightharpoonup |u|^{r-2}u \text{ in } L^{\frac{r}{r-1}}(\Omega). \end{cases} \tag{3.10}$$

Now we extend the functions u_n, u to $\mathbb{R}^n$ with zero. For convenience, those extensions are also denoted by u_n and u. Without loss of the generality, we assume that there exist nonnegative Radon measures μ, ϑ, ν such that (see [19–21])

$$|\nabla u_n|^p \rightharpoonup \mu; \qquad |u_n|^{p^*} \rightharpoonup \vartheta; \qquad |u_n|^r \rightharpoonup \nu \quad \text{in } \mathcal{M}\big(\mathbb{R}^n\big). \tag{3.11}$$

Here $a_n \rightharpoonup a$ in $\mathcal{M}(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} h\,da_n \to \int_{\mathbb{R}^n} h\,da \quad \text{for each } h \in C_0\big(\mathbb{R}^n\big), \text{as } n \to \infty,$$

where $C_0(\mathbb{R}^n)$ denotes the space of continuous, real-valued functions on $\mathbb{R}^n$ with compact support.

According to the concentration-compactness principle established by Berestycki and Lions in [22, 23], there exists an index set J and $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, $\{\vartheta_j\}_j \subset [0,+\infty)$, $\{\mu_j\}_{j\in J} \subset [0,+\infty)$ such that

(i) $\vartheta = |u|^{p^*} + \sum_{j\in J}\vartheta_j\delta_{x_j}$,
(ii) $\mu \ge |\nabla u|^p + \sum_{j\in J}\mu_j\delta_{x_j}$

and $\vartheta_j^{p/p^*} \le \mu_j/S$, $\forall j \in J$.

Here δ_{x_j} is a Dirac measure on x_j and $S = \inf_{\substack{u\in W^{1,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx}{(\int_{\mathbb{R}^N} |u|^{p^*}\,dx)^{p/p^*}}$ is the optimal constant in the Sobolev inequality. Since $p^* < r$, by (3.10), we derive that $\vartheta_j = 0$.

Similarly, for the Radon measure μ defined in (3.11), we have the following.

Claim 1 $\mu_j = 0$.

Proof Since u_n is extended with zero to $\mathbb{R}^n$, for any $j \in J$, $x_j \in \overline{\Omega}$. Now, given ϵ (small enough), for the sequence $\{\varphi_\epsilon(x - x_j)u_n\}$ with fixed x_j, one can conclude that

$$\int_\Omega \left|\varphi_\epsilon(x - x_j)u_n\right|^r dx \le \int_\Omega |u_n|^r\,dx, \tag{3.12}$$

$$\int_\Omega \left|\varphi_\epsilon(x - x_j)u_n\right|^p dx \le \int_\Omega |u_n|^p\,dx, \tag{3.13}$$

$$\begin{aligned}
&\left|\int_\Omega \left|\nabla\bigl(\varphi_\epsilon(x - x_j)u_n\bigr)\right|^p dx\right| \\
&\quad\le 2^{p-1}\left[\int_\Omega \left(\left|\nabla\varphi_\epsilon(x - x_j)\right|^p |u_n|^p + \left|\varphi_\epsilon(x - x_j)\right|^p |\nabla u_n|^p\right)dx\right] \\
&\quad\le 2^{p-1}\left[\left(\int_\Omega |u_n|^{p^*}\,dx\right)^{p/p^*} \int_{\mathbb{R}^n}\left(|\nabla\varphi|^{p^*p/(p^*-p)}\,dx\right)^{(p^*-p)/p^*} + \int_\Omega |\nabla u_n|^p\,dx\right] \\
&\quad\le C\left(\left(\int_\Omega |u_n|^{p^*}\,dx\right)^{p/p^*} + \int_\Omega |\nabla u_n|^p\,dx\right) \le C_1\int_\Omega |\nabla u_n|^p\,dx.
\end{aligned} \tag{3.14}$$

To obtain (3.14), we have used Proposition 2.1. Here C, C_1 are constants independent of u, ϵ, j. This implies that $\{\varphi_\epsilon(x-x_j)u_n\}$ is bounded (independent on ϵ and j) in $W_0^{1,p}(\Omega)\cap L^r(\Omega)$.

At the same time, (3.6) says that

$$-\operatorname{div}\bigl(|\nabla u_n|^{p-2}\nabla u_n\bigr) + |u_n|^{r-2}u_n = \gamma|u_n|^{s-2}u_n - I'(u_n). \tag{3.15}$$

Now, multiplying (3.15) by $\varphi_\epsilon(x - x_j)u_n$ and integrating by parts with $I'(u_n) \to 0$, we have as $n \to \infty$,

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\bigl(\varphi_\epsilon(x - x_j)u_n\bigr)\,dx + \int_\Omega \varphi_\epsilon(x - x_j)|u_n|^r\,dx \\
&\quad= \int_\Omega \varphi_\epsilon(x - x_j)u_n\bigl(\gamma|u_n|^{s-2}u_n\bigr)\,dx + o(1).
\end{aligned} \tag{3.16}$$

In other words,

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^p\varphi_\epsilon(x - x_j)\,dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot u_n \cdot \nabla\varphi_\epsilon(x - x_j)\,dx + \int_\Omega \varphi_\epsilon(x - x_j)|u_n|^r\,dx \\
&\quad= \int_\Omega \varphi_\epsilon(x - x_j)u_n\bigl(\gamma|u_n|^{s-2}u_n\bigr)\,dx + o(1).
\end{aligned} \tag{3.17}$$

By applying the Hölder inequality and the Young inequality for the term

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot u_n \cdot \nabla\varphi_\epsilon(x - x_j)\,dx,$$

we see that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot u_n \cdot \nabla\varphi_\epsilon(x-x_j)\,dx\right| \le \delta\int_\Omega |\nabla u_n|^p\,dx + C_\delta\int_\Omega \left|u_n\nabla\varphi_\epsilon(x-x_j)\right|^p\,dx.$$

Using the Strauss lemma (see [22, 23]) to the last term in the above inequality, we get from (3.10) that

$$\lim_{n\to\infty}\int_\Omega \left|u_n\nabla\varphi_\epsilon(x-x_j)\right|^p\,dx = \int_\Omega \left|u\nabla\varphi_\epsilon(x-x_j)\right|^p\,dx.$$

Furthermore, Proposition 2.1 says that

$$\int_\Omega \left|u\nabla\varphi_\epsilon(x-x_j)\right|^p\,dx \le C_1\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{p/p^*},$$

where $C_1 = (\int_{\mathbb{R}^n} |\nabla\varphi|^{\frac{pp^*}{p^*-p}}\,dx)^{\frac{p^*-p}{p}}$ is a constant (independent on ϵ and δ).

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot u_n \cdot \nabla\varphi_\epsilon(x-x_j)\,dx\right| \\ &\quad\le \delta\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\,dx + C_\delta C_1\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{p/p^*}.\end{aligned} \tag{3.18}$$

Also combining the assumptions $1 < s \le p < n$ and $r > p^*$ with (3.10), one concludes from the Strauss lemma that

$$\lim_{n\to\infty}\int_\Omega \varphi_\epsilon(x-x_j)\bigl[\gamma|u_n|^{s-2}u_n\bigr]u_n\,dx = \gamma\int_\Omega \varphi_\epsilon(x-x_j)|u|^s\,dx. \tag{3.19}$$

By virtue of inequalities (3.17)-(3.19), we can get

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \varphi_\epsilon(x-x_j)|\nabla u_n|^p\,dx + \lim_{n\to\infty}\int_\Omega \varphi_\epsilon(x-x_j)|u_n|^r\,dx \\ &\quad\le \int_\Omega \varphi_\epsilon(x-x_j)\gamma|u|^s\,dx + \delta\varlimsup_{n\to\infty}\int_\Omega |\nabla u_n|^p\,dx + C_\delta C_1\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{p/p^*},\end{aligned}$$

namely, as $n\to\infty$,

$$\begin{aligned}&\int_{\mathbb{R}^n} \varphi_\epsilon(x-x_j)\,d\mu + \int_{\mathbb{R}^n} \varphi_\epsilon(x-x_j)\,d\nu \\ &\quad\le \gamma\int_\Omega \varphi_\epsilon(x-x_j)|u|^s\,dx + \delta C_2 + C_\delta\cdot C_1\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{p/p^*} \\ &\quad= \gamma\int_{B(x_j,\epsilon)} \varphi_\epsilon(x-x_j)|u|^s\,dx + \delta C_2 + C_\delta\cdot C_1\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{p/p^*}.\end{aligned} \tag{3.20}$$

Accordingly, by sending $\epsilon\to 0^+$ first and then $\delta\to 0^+$ on the right-hand side of (3.20), we deduce that

$$\mu_j = 0. \tag{3.21}$$

This implies that the index set J is just an empty set and the proof of Claim 1 is finished. □

Furthermore, for the Radon measure ν defined in (3.11), we have the following.

Claim 2 For any nonnegative function $f \in C_0(\mathbb{R}^n)$, it holds that

$$\int_{\mathbb{R}^n} f\,d\nu = \lim_{n\to\infty}\int_\Omega f|u_n|^r\,dx \geq \int_\Omega f|u|^r\,dx = \int_{\mathbb{R}^n} f|u|^r\,dx. \tag{3.22}$$

Proof Since u_n, u are extended with zero to $\mathbb{R}^n$ with (3.11), we obtain

$$\int_{\mathbb{R}^n} f\,d\nu = \lim_{n\to\infty}\int_{\mathbb{R}^n} f|u_n|^r\,dx = \lim_{n\to\infty}\int_\Omega f|u_n|^r\,dx.$$

Note that $u \in L^r(\Omega)$ (*cf.* Lemma 3.1) and a nonnegative function $f \in C_0(\mathbb{R}^n)$. By applying the Fatou lemma with $u_n \to u$ a.e. in Ω (or saying in $\mathbb{R}^n$), we deduce that

$$\lim_{n\to\infty}\int_\Omega f|u_n|^r\,dx \geq \int_\Omega f|u|^r\,dx = \int_{\mathbb{R}^n} f|u|^r\,dx.$$

So, combining with the above two formulas, we obtain (3.22). □

Step 2: Thanks to the above results, and in order to prove (3.9) in Remark 3.1, it suffices to prove that as $n \to \infty$,

$$\int_\Omega \bigl(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\bigr)(\nabla u_n - \nabla u)\,dx \to 0. \tag{3.23}$$

Now, we choose a R (big enough) such that $\Omega \subset \{x \in \mathbb{R}^n \,|\, |x| < R\}$ and set a nonnegative function $\psi \in \mathfrak{D}(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset B(0,3R)$ and $\psi(x) \equiv 1$ as $x \in B(0,2R)$. By applying the same argument as in (3.12)-(3.14), one deduces that $\{\psi u_n\}$ is bounded in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$.

Multiplying (3.15) by ψu_n and integrating on Ω, we have as $n \to \infty$,

$$\begin{aligned}
&\int_\Omega \psi|\nabla u_n|^p\,dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot u_n\cdot\nabla\psi\,dx + \int_\Omega \psi|u_n|^r\,dx \\
&\quad = \int_\Omega \psi\bigl[\gamma|u_n|^{s-2}u_n\bigr]u_n\,dx + o(1).
\end{aligned} \tag{3.24}$$

Due to the assumptions $1 < s \leq p < n$ and $r > p^*$, and conclusion (3.10) and the Strauss lemma, we obtain

$$\lim_{n\to\infty}\int_\Omega \psi\bigl[\gamma|u_n|^{s-2}u_n\bigr]u_n\,dx = \gamma\int_\Omega \psi u^s\,dx. \tag{3.25}$$

Therefore, combining (3.11), (3.24), (3.25) with Claim 2 and the fact that $\nabla\psi(x) = 0$ for $x \in \overline{\Omega}$, one concludes from the left-hand side in (3.24) that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \psi|\nabla u_n|^p\,dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot u_n\cdot\nabla\psi\,dx + \int_\Omega \psi|u_n|^r\,dx \\
&\quad = \lim_{n\to\infty}\biggl\{\int_\Omega \psi|\nabla u_n|^p\,dx + \int_\Omega \psi|u_n|^r\,dx\biggr\}
\end{aligned}$$

$$= \lim_{n\to\infty} \int_{\Omega} \psi |\nabla u_n|^p \, dx + \int_{\mathbb{R}^n} \psi \, d\nu$$
$$= \gamma \int_{\Omega} \psi |u|^s \, dx. \tag{3.26}$$

Similarly, multiplying (3.15) by ψu and integrating over Ω yields that as $n \to \infty$,

$$\int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx + \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n u \nabla \psi \, dx + \int_{\Omega} \psi |u_n|^{r-2} u_n u \, dx$$
$$= \int_{\Omega} \psi \big[\gamma |u_n|^{s-2} u_n\big] u \, dx + o(1). \tag{3.27}$$

Therefore, by (3.10) (or Lemma 3.1) and Claim 2 together with the Strauss lemma and $\nabla \psi(x) = 0$ for $x \in \overline{\Omega}$, we conclude from the left-hand side in (3.27) that

$$\lim_{n\to\infty} \int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx + \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n u \nabla \psi \, dx + \int_{\Omega} \psi |u_n|^{r-2} u_n u \, dx$$
$$= \lim_{n\to\infty} \left\{ \int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx + \int_{\Omega} \psi |u_n|^{r-2} u_n u \, dx \right\}$$
$$= \lim_{n\to\infty} \int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx + \int_{\Omega} \psi |u|^r \, dx$$
$$= \lim_{n\to\infty} \int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx + \int_{\mathbb{R}^n} \psi |u|^r \, dx$$
$$= \gamma \int_{\Omega} \psi |u|^s \, dx. \tag{3.28}$$

Here the term $\lim_{n\to\infty} \int_{\Omega} \psi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \, dx$ exists because $\{|\nabla u_n|^{p-2} \nabla u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$, and by choosing the subsequence, $\{|\nabla u_n|^{p-2} \nabla u_n\}$ is weakly convergent in $L^{\frac{p}{p-1}}(\Omega)$.

On the other hand, from the difference of (3.26) and (3.28) and inequality (3.22) in Claim 2 and the fact that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, we deduce that as $n \to \infty$,

$$\lim_{n\to\infty} \int_{\Omega} \psi \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\big)(\nabla u_n - \nabla u) \, dx$$
$$= \lim_{n\to\infty} \int_{\Omega} \big(\psi |\nabla u_n|^p - \psi |\nabla u_n|^{p-2} \nabla u_n \nabla u - \psi |\nabla u|^{p-2} \nabla u \nabla u_n + \psi |\nabla u|^{p-2} \nabla u \nabla u\big) \, dx$$
$$= \lim_{n\to\infty} \int_{\Omega} \big(\psi |\nabla u_n|^p - \psi |\nabla u_n|^{p-2} \nabla u_n \nabla u\big) \, dx$$
$$- \lim_{n\to\infty} \int_{\Omega} \big(\psi |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u)\big) \, dx$$
$$= \lim_{n\to\infty} \int_{\Omega} \big(\psi |\nabla u_n|^p - \psi |\nabla u_n|^{p-2} \nabla u_n \nabla u\big) \, dx$$
$$= \left(\gamma \int_{\Omega} \psi |u|^s \, dx - \int_{\mathbb{R}^n} \psi \, d\nu\right) - \left(\gamma \int_{\Omega} \psi |u|^s \, dx - \int_{\mathbb{R}^n} \psi |u|^r \, dx\right)$$
$$= \int_{\mathbb{R}^n} \psi |u|^r \, dx - \int_{\mathbb{R}^n} \psi \, d\nu$$
$$\leq 0. \tag{3.29}$$

Moreover, one notes that

$$\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u) \geq 0.$$

It follows from $0 \leq \psi \leq 1$ and $\psi \equiv 1$ as $x \in \overline{\Omega}$ with (3.29) that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty}\int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,dx \\ &\leq \lim_{n\to\infty}\int_\Omega \psi\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,dx \\ &\leq 0. \end{aligned} \tag{3.30}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,dx = 0. \tag{3.31}$$

Then standard arguments lead to

$$\nabla u_n(x) \to \nabla u(x), \quad \text{a.e. } x \in \Omega.$$

Applying the fact that $\{|\nabla u_n|^{p-2}\nabla u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$ and choosing the subsequence, one obtains that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}(\Omega). \tag{3.32}$$

The proof of Lemma 3.3 is completely finished. □

Finally, we are in a position to prove our Theorem 1.2.

Completion of the proof of Theorem 1.2.

Due to Lemma 3.1 and Lemma 3.3, for any $h \in X$, we have as $n \to \infty$,

$$\begin{cases} \langle -\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n), h\rangle \to \langle -\operatorname{div}(|\nabla u|^{p-2}\nabla u), h\rangle, \\ \langle |u_n|^{r-2}u_n, h\rangle \to \langle |u|^{r-2}u, h\rangle, \\ \langle |u_n|^{s-2}u_n, h\rangle \to \langle |u|^{s-2}u, h\rangle \end{cases} \tag{3.33}$$

i.e.,

$$\begin{aligned} \big\langle I'(u_n), h\big\rangle &= \big\langle -\operatorname{div}\big(|\nabla u_n|^{p-2}\nabla u_n\big) + |u_n|^{r-2}u_n - \gamma|u_n|^{s-2}u_n, h\big\rangle \\ &\to \big\langle -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{r-2}u - \gamma|u|^{s-2}u, h\big\rangle. \end{aligned}$$

Moreover, with the assumption $I'(u_n) \to 0$, we have

$$\big\langle -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{r-2}u - \gamma|u|^{s-2}u, h\big\rangle = 0, \quad \forall h \in X,$$

which implies that $I'(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{r-2}u - \gamma|u|^{s-2}u = 0$ in X^*.

And then, by a simple calculation, one can get

$$\begin{aligned}&\langle I'(u_n) - I'(u), u_n - u\rangle + \langle \gamma|u_n|^{s-2}u_n - \gamma|u|^{s-2}u, u_n - u\rangle \\ &\quad = \langle -\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) - (-\operatorname{div}(|\nabla u|^{p-2}\nabla u)), u_n - u\rangle + \langle |u_n|^{r-2}u_n - |u|^{r-2}u, u_n - u\rangle \\ &\quad = \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle + \langle |u_n|^{r-2}u_n - |u|^{r-2}u, u_n - u\rangle \\ &\quad \geq \frac{1}{2^p}\int_\Omega |\nabla u_n - \nabla u|^p\,dx + \frac{1}{2^r}\int_\Omega |u_n - u|^r\,dx. \end{aligned} \tag{3.34}$$

Hence, from $I'(u_n) \to 0$, $I'(u) = 0$ in X^*, $\gamma|u_n|^{s-2}u_n \to \gamma|u|^{s-2}u$ in $L^{\frac{r}{r-1}}$ (due to (3.2)) and inequality (3.34), one deduces that $u_n \to u$ in X. This indicates that the functional I satisfies the Palais-Smale condition and the proof of Theorem 1.2 is completely finished.

3.2 Proof of Theorem 1.1

From the assumption $1 < s \leq p < n$ and $r > p^*$, the functional I defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{r}\int_\Omega |u|^r\,dx - \frac{\gamma}{s}\int_\Omega |u|^s\,dx, \quad u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$$

is clearly even.

In order to prove Theorem 1.1, we need the following lemma about the existence of solutions (see [24]).

Lemma 3.4 *Let $\mathcal{A}$ be the all close and symmetric set of X with respect to the zero point and Γ be the Z_2 index on $\mathcal{A}$. Suppose that the functional $\mathcal{J} \in C^1(X, \mathbb{R}^1)$ is even and satisfies the Palais-Smale condition and following assumptions*:

(i) *There exists a τ-dimensional subspace V of X and a positive constant $\rho > 0$ such that*

$$\sup_{x \in V \cap S_\rho} \mathcal{J}(x) < \mathcal{J}(0),$$

where $S_\rho = \{x \in X \mid \|x\| = \rho\}$.

(ii) *There exists a closed subspace X_2 of X such that the dimension of the complemented subspace of X_2 is κ, and*

$$\inf_{x \in X_2} \mathcal{J}(x) > -\infty,$$

then when $\tau > \kappa$, the functional $\mathcal{J}$ at least has '$\tau - \kappa$' pairs of different critical points.

Proof of Theorem 1.1 Due to Theorem 1.2 and Lemma 3.2, the even functional I (defined in (1.2)) is C^1 and satisfies the Palais-Smale condition on X. Thus we only need to check that the assumptions (i), (ii) of Lemma 3.4 are valid for our functional I.

For the case $1 < s < p < n$ *with* $\gamma \in \mathbb{R}^+$.

Choosing any integer $m > 0$, let V_m be an m-dimensional subspace of X. Denote $A_m = \{u \in V_m : \|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{L^r(\Omega)} = 1\}$, there exists $\delta_1 > 0$ (since A_m is compact in X) such that

$$\inf_{u \in A_m} \|u\|_{L^s(\Omega)}^s = \delta_1.$$

Thus, for $v = \epsilon u \in \epsilon A_m$ with $\epsilon A_m \triangleq \{\epsilon u : u \in A_m\}$, one can deduce that

$$\begin{aligned} I(v) &= \frac{1}{p}\int_\Omega \epsilon^p |\nabla u|^p \,dx + \frac{1}{r}\int_\Omega |u|^r \epsilon^r \,dx - \frac{\gamma}{s}\int_\Omega |u|^s \epsilon^s \,dx \\ &= \frac{1}{p}\epsilon^p \int_\Omega |\nabla u|^p \,dx + \frac{1}{r}\epsilon^r \int_\Omega |u|^r \,dx - \frac{\gamma}{s}\epsilon^s \int_\Omega |u|^s \,dx \\ &\leq \frac{1}{p}\epsilon^p + \frac{1}{r}\epsilon^r - \frac{\gamma}{s}\epsilon^s \delta_1. \end{aligned} \tag{3.35}$$

Due to $r > p^* > p > s > 1$, if we choose ϵ small enough, $I(v) < 0 = I(0)$ for any $v \in \epsilon A_m$. This implies that assumption (i) is valid for the functional I when $\tau = m$ and $\rho = \epsilon > 0$. And by virtue of the coercivity of functional I (see Lemma 3.2), we can choose $X_2 = X$ in assumption (ii). Therefore, due to Lemma 3.4 and the arbitrary of integer m, we finish the proof of the first part of our Theorem 1.1.

For the case $1 < s = p < n$ *with* $\lambda_m < \gamma \leq \lambda_{m+1}$.

Let v_j be the eigenfunctions of the operator $-\Delta_p$ with respect to the eigenvalue λ_j, *i.e.*,

$$\begin{cases} -\operatorname{div}(|\nabla v_j|^{p-2}\nabla v_j) = \lambda_j |v_j|^{p-2} v_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.36}$$

Note that $V = \operatorname{span}\{v_1, \dots, v_m\}$ is a subspace of $W_0^{1,p}(\Omega)$, and let $A = \{v : v \in V \text{ and } \|v\|_{L^p(\Omega)} = 1\}$. Obviously, the set $A \subseteq V$ is a closed ellipsoid surface centered at 0 such that $\Gamma(A) = m$ and it is also a compact set satisfying

$$\int_\Omega |\nabla u|^p d \leq \lambda_m \int_\Omega |u|^p \,dx = \lambda_m, \quad u \in A.$$

In order to obtain the similar inequality (3.35) to satisfy assumption (i), we need to give the estimate of the term $\int_\Omega |u|^r \,dx$. By applying the same argument as in [25] with Propositions 1.2, 1.3 in [13] about the regularity of v_j, one can conclude that the eigenfunctions $v_j \in L^\infty(\Omega)$. Therefore, setting $\epsilon A = \{\epsilon u : u \in A\}$ and $B \triangleq \bigcup_{0\leq\epsilon\leq 1} \epsilon A$, we have

$$\sup_{u\in B} \int_\Omega |u|^r \,dx = \eta_1 > 0.$$

So, for $v = \epsilon u \in \epsilon A$, we can get

$$\begin{aligned} I(v) =& \frac{1}{p}\int_\Omega \epsilon^p |\nabla u|^p \,dx + \frac{1}{r}\int_\Omega |u|^r \epsilon^r \,dx - \frac{\gamma}{p}\int_\Omega |u|^p \epsilon^p \,dx \\ &= \frac{1}{p}\epsilon^p \int_\Omega |\nabla u|^p \,dx + \frac{1}{r}\epsilon^r \int_\Omega |u|^r \,dx - \frac{\gamma}{p}\epsilon^p \int_\Omega |u|^p \,dx \\ &\leq \left(\frac{\lambda_m}{p} - \frac{\gamma}{p}\right)\epsilon^p + \frac{\eta_1}{r}\epsilon^r. \end{aligned} \tag{3.37}$$

Since $r > p^* > p$ and $\lambda_m < \gamma \leq \lambda_{m+1}$, there is some ϵ_0 small enough such that

$$I(v) < 0 = I(0) \quad \text{for any } v \in B_1 \triangleq \bigcup_{0<\epsilon\leq\epsilon_0} \epsilon A. \tag{3.38}$$

Moreover, by virtue of the regularity of $v_j \in L^\infty(\Omega)$ again, the following two norms are equivalent in the eigenspace $V = \operatorname{span}\{v_1, \ldots, v_m\}$

$$\|\cdot\|_{W_0^{1,p}(\Omega)} < \|\cdot\|_{L^r(\Omega)} + \|\cdot\|_{W_0^{1,p}(\Omega)} < C\|\cdot\|_{W_0^{1,p}(\Omega)},$$

where $C > 1$ is a constant. Hence, there exists some surface $S_\rho = \{x \in X \mid \|x\|_{W_0^{1,p}(\Omega)} + \|x\|_{L^r(\Omega)} = \rho\}$ $(\rho > 0)$ such that

$$0 \notin S_\rho \cap V \subset B_1.$$

This implies that formula (3.38) is true on the surface $S_\rho \cap V$. So, we obtain the condition of assumption (i). Similarly, the coercivity of the functional I implies that assumption (ii) is valid by choosing $X_2 = X$ in Lemma 3.4. Then we are done. □

Remark 3.2 If $p = 2$ with supercritical exponent $r > 2^* = \frac{2n}{n-2}$, our argument is valid as well.

Competing interests
The author declares that they have no competing interests.

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