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Multiplicity of solutions for fractional Schrödinger equations with perturbation

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Abstract
In this paper, we investigate a class of fractional Schrödinger equations with perturbation. By using the mountain pass theorem and Ekeland's variational principle, we see that such equations possess two solutions. Recent results in the literature are generalized and significantly improved.

MSC: 34B15

Keywords: fractional Schrödinger equations; mountain pass theorem; critical point

1 Introduction

In this paper, we consider the following class of fractional Schrödinger equations:

$$(-\Delta)^\alpha u + V(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < \alpha < 1$, $2\alpha < N$, $1 \leq p < 2$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $h \in L^{\frac{2}{2-p}}(\mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$, and $(-\Delta)^\alpha u$ is defined pointwise for x in $\mathbb{R}^N$ by

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2\alpha}}\,dy$$

along any rapidly decaying function u of class $C^\infty(\mathbb{R}^N)$; see Lemma 3.5 of [1].

Recently, a lot of attention has been focused on the study of fractional and non-local problems; see [2–6]. This may be due to its concrete applications in different fields, such as the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, deblurring and denoising of images, and so on; see [1, 7–10]. For standing wave solutions of fractional Schrödinger equations in the whole space $\mathbb{R}^N$, there were also many works; see [11–20]. The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. The fractional quantum mechanics has been discovered as a result of expanding the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths. In [11], Laskin formulated the fractional Schrödinger equations as follows:

$$i\partial_t\psi = (-\Delta)^\alpha\psi + V(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{1.2}$$

where $0 < \alpha < 1$, ψ is the wavefunction and $V(x)$ denotes the potential energy. We let $\psi(x,t) = e^{i\omega t}u(x)$ be standing waves solutions for (1.2). Then u is a solution of an equation of type of (1.1).

For the fractional Schrödinger equations, variational methods are available. In [14], Felmer *et al.* studied the existence and regularity of solutions for a class of fractional Schrödinger equations under the Ambrosetti-Rabinowitz condition, *i.e.*, there exists $\theta > 2$ such that

$$0 < \theta F(x,t) \leq tf(x,t).$$

In [15], Secchi obtained the existence of ground state solutions of a class of fractional Schrödinger equations under the Ambrosetti-Rabinowitz condition and the following condition:

(V_0) $V \in C(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$ and $\lim_{|x|\to\infty} V(x) = \infty$.

In [19], Torres studied the existence of solutions for the following equations:

$$(-\Delta)^\alpha u + V(x)u = f(u) + h(x), \quad x \in \mathbb{R}^N, \tag{1.3}$$

under the conditions of (V_0) and the Ambrosetti-Rabinowitz condition for f.

As far as we know, there are few works on problem (1.1), of which nonlinearity involves a combination of superlinear or asymptotically linear terms and a sublinear perturbation. Motivated by the above facts, we investigate this case in this paper.

Before stating our results we introduce some notations. Throughout this paper, we denote by $\|\|_r$ the L^r-norm, $2 \leq r \leq \infty$, and $h^\pm = \max\{\pm h, 0\}$. If we take a subsequence of a sequence $\{u_n\}$ we shall denote it again by $\{u_n\}$.

Now we state our main result.

Theorem 1.1 *Assume that* $h \in L^{\frac{2}{2-p}} \setminus \{0\}$ *with* $h^+ \not\equiv 0$, (V_0), *and the following conditions are satisfied*:

(F_1) $f(x,s)$ *is a continuous function on* $\mathbb{R}^N \times \mathbb{R}$ *such that* $f(x,s) \equiv 0$ *for all* $s < 0$ *and* $x \in \mathbb{R}^N$. *Moreover, there exists* $b \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ *with* $|b|_\infty < \frac{S_2^2}{2}$ *such that*

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^k} = b(x) \quad \textit{uniformly in } x \in \mathbb{R}^N,$$

and

$$\frac{f(x,s)}{s^k} \geq b(x) \quad \forall s > 0 \textit{ and } x \in \mathbb{R}^N,$$

where S_r *is the best constant for the embedding of* X *in* $L^r(\mathbb{R}^N)$; *see Lemma* 2.2 *and Remark* 2.1 *in Section* 2;

(F_2) *there exists* $q \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ *with* $|q|_\infty > c_0$ *such that*

$$\lim_{s\to\infty} \frac{f(x,s)}{s^k} = q(x) \quad \textit{uniformly in } x \in \mathbb{R}^N,$$

where c_0 *is defined by* (2.1) *in Section* 2;

(F_3) *there exist two constants* θ, d_0 *satisfying* $\theta > 2$ *and* $0 \leq d_0 < \frac{S_2^2(\theta-2)}{2\theta}$ *such that*

$$F(x,s) - \frac{1}{\theta} f(x,s)s \leq d_0 s^2 \quad \forall s > 0 \text{ and } x \in \mathbb{R}^N,$$

where $F(x,s) = \int_0^s f(x,\tau)\,d\tau$.

Then we have the following results:

(i) *if* $k = 1$ *and* $\mu < 1$ *with*

$$\mu = \inf\biggl\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)^2\,dx \Bigm| u \in H^\alpha(\mathbb{R}^N), \int_{\mathbb{R}^N} q(x)u(x)^2\,dx = 1 \biggr\},$$

then there exists $\Lambda > 0$ *such that for every* $\lambda \in (0,\Lambda)$, *problem* (1.1) *has at least two nontrivial solutions*;

(ii) *if* $1 < k < 2_\alpha^* - 1$, *then there exists* $\Lambda > 0$ *such that for every* $\lambda \in (0,\Lambda)$, *problem* (1.1) *has at least two nontrivial solutions, where* $2_\alpha^* = \frac{2N}{N-2\alpha}$.

Remark 1.1 Theorem 1.1 extends the perturbation h in [19] to the case $\lambda h(x)|u|^{p-2}u$ and (F_3) is weaker than the Ambrosetti-Rabinowitz condition. Moreover, our f is allowed to be asymptotically linear at infinity when $k = 1$, which is not the same as that in [20], where they need $\limsup_{s\to 0^+} \frac{f(x,s)}{s} < \liminf_{s\to +\infty} \frac{f(x,s)}{s}$.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of our main results.

2 Preliminaries

In order to prove our main results, we first give some properties of space X on which the variational setting for problem (1.1) is defined. Let

$$H = H^\alpha(\mathbb{R}^N) := \biggl\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz < \infty \biggr\}$$

with the inner product and the norm

$$\langle u,v\rangle_H = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} u(x)v(x)\,dx, \qquad \|u\|_H = \langle u,u\rangle_H^{\frac{1}{2}}.$$

Letting

$$X = \biggl\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u^2(x)\,dx < +\infty \biggr\},$$

then X is a Hilbert space with the inner product

$$\langle u,v\rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx$$

and the corresponding norm $\|u\|^2 = \langle u,u\rangle$. Note that

$$X \subset H^\alpha(\mathbb{R}^N)$$

and

$$X \subset L^r(\mathbb{R}^N)$$

for all $r \in [2,2_\alpha^*]$ with the embedding being continuous. It is easy to get the following lemma.

Lemma 2.1 *Assume that the condition* (V_0) *holds. Then there exists a constant* $c_0 > 0$ *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u^2\,dx \geq c_0\|u\|_H^2, \quad \forall u \in H^\alpha(\mathbb{R}^N). \tag{2.1}$$

Lemma 2.2 (see [15, 19]) *Assume that the condition* (V_0) *holds. Then* X *is compactly embedded in* $L^r(\mathbb{R}^N)$ *for all* $r \in [2,2_\alpha^*)$.

Remark 2.1 By Lemma 2.2, we have

$$S_r\|u\|_r \leq \|u\|,$$

where S_r is the best constants for the embedding of X in $L^r(\mathbb{R}^N)$.

Now we begin describing the variational formulation of problem (1.1). Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} h(x)|u|^p\,dx. \tag{2.2}$$

By the continuity of f, g and Lemma 2.2, $J \in C^1(X,\mathbb{R})$ and its derivative is given by

$$\begin{aligned} J'(u)v ={}& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx + \\ & - \int_{\mathbb{R}^N} f\bigl(x,u(x)\bigr)v(x)\,dx - \lambda\int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\,dx \end{aligned} \tag{2.3}$$

for all $u,v \in X$. In addition, any critical point of J on X is a solution of problem (1.1).

Next, we give the variant version of the mountain pass theorem which is important for the proof of our main results.

Theorem 2.1 (see [21]) *Let* E *be a real Banach space with its dual space* E^*, *and suppose that* $I \in C^1(E,\mathbb{R})$ *satisfies*

$$\max\bigl\{I(0),I(e)\bigr\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\hat{c} \geq \eta$ be characterized by

$$\hat{c} = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I\big(\gamma(\tau)\big),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \hat{c} \geq \eta \quad \textit{and} \quad \big(1 + \|u_n\|\big)\big\|I'(u_n)\big\|_{E^*} \to 0, \quad \textit{as } n \to \infty.$$

3 Proof of the main results

To prove our main results, we first give the following lemma.

Lemma 3.1 *For any real number $2_\alpha^* - 1 > k \geq 1$, assume that the conditions* (V_0), (F_1)-(F_2) *hold. Then there exists $\Lambda > 0$ such that for every $\lambda \in (0, \Lambda)$ there are two positive constants ρ, η such that $J(u)|_{\|u\|=\rho} \geq \eta > 0$.*

Proof For any $\epsilon > 0$, it follows from the conditions (F_1)-(F_2) that there exist $C_\epsilon > 0$ and $2_\alpha^* > r > \max\{2, k\}$ such that

$$F(x,s) \leq \frac{|b|_\infty + \epsilon}{2} s^2 + \frac{C_\epsilon}{r}|s|^r, \quad \forall s \in \mathbb{R}. \tag{3.1}$$

By (2.2) and (3.1), Sobolev's inequality, and Hölder's inequality, one has

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} h(x)|u|^p\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \frac{|b|_\infty + \epsilon}{2} u(x)^2\,dx - \int_{\mathbb{R}^N} \frac{C_\epsilon}{r} u(x)^r\,dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} h(x)\big|u(x)\big|^p\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{|b|_\infty + \epsilon}{2S_2^2}\|u\|^2 - \frac{C_\epsilon}{rS_r^r}\|u\|^r - \frac{\lambda S_2^{-p}}{p}\|h\|_{\frac{2}{2-p}}\|u\|^p \\
&= \|u\|^p\left[\frac{1}{2}\left(1 - \frac{|b|_\infty + \epsilon}{2S_2^2}\right)\|u\|^{2-p} - \frac{C_\epsilon}{rS_r^r}\|u\|^{r-p} - \frac{\lambda S_2^{-p}}{p}\|h\|_{\frac{2}{2-p}}\right]
\end{aligned} \tag{3.2}$$

for all $u \in X$. Take $\epsilon = \frac{S_2^2}{2} - |b|_\infty$ and define

$$l(t) = \frac{1}{4}t^{2-p} - C_\epsilon r^{-1} S_r^{-r} t^{r-p}, \quad \forall t \geq 0.$$

It is easy to prove that there exists $\rho > 0$ such that

$$\max_{t \geq 0} l(t) = l(\rho) = \frac{r-2}{4(r-p)}\left[\frac{(2-p)rS_r^r}{4(r-p)C_\epsilon}\right]^{\frac{2-p}{r-2}}.$$

Then it follows from (3.2) that there exists $\Lambda > 0$ such that for every $\lambda \in (0, \Lambda)$ there exist two positive constants ρ, η such that $J(u)|_{\|u\|=\rho} \geq \eta > 0$. □

Consider the minimum problem

$$\mu = \inf\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)^2\,dx : u \in H^\alpha(\mathbb{R}^N),\right.$$
$$\left.\int_{\mathbb{R}^N} q(x)u(x)^2\,dx = 1\right\}. \tag{3.3}$$

Then we have the following results.

Lemma 3.2 *There exist a constant* $c_1 > 0$ *and* $\phi_1 \in H^\alpha(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N} q(x)\phi_1(x)^2\,dx = 1$ *such that* $\mu \geq c_1$ *and*

$$\mu = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi_1(x)-\phi_1(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)\phi_1(x)^2\,dx, \tag{3.4}$$

i.e. the minimum (3.3) *is achieved.*

Proof For any $u \in H^\alpha(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} q(x)u(x)^2\,dx = 1$, by Lemma 2.1 and Sobolev's embedded theorem, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)^2\,dx \geq c_0\|u\|_H^2 \geq c_0\|u\|_2^2 \geq \frac{c_0}{|q|_\infty} > 0.$$

Therefore, there exists a constant $c_1 > 0$ such that $\mu \geq c_1$. Let $\{u_n\} \subset H^\alpha(\mathbb{R}^N)$ be a minimizing sequence of (3.3). Clearly, $\int_{\mathbb{R}^N} q(x)u_n(x)^2\,dx = 1$ and $\{u_n\}$ is bounded. Then there exist a subsequence $\{u_n\}$ and $\phi_1 \in H^\alpha(\mathbb{R}^N)$ such that $u_n \rightharpoonup \phi_1$ weakly in $H^\alpha(\mathbb{R}^N)$ and $u_n \to \phi_1$ strongly in $L^2(\mathbb{R}^N)$. So it is easy to verify that $\int_{\mathbb{R}^N} q(x)u_n(x)^2\,dx \to \int_{\mathbb{R}^N} q(x)\phi_1(x)^2\,dx$ as $n \to \infty$ and $\int_{\mathbb{R}^N} q(x)\phi_1(x)^2\,dx = 1$. Therefore,

$$\begin{aligned}\mu &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi_1(x)-\phi_1(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)\phi_1(x)^2\,dx \\ &\leq \liminf_{n\to\infty}\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)u(x)^2\,dx\right\} \\ &\leq \mu.\end{aligned} \tag{3.5}$$

This implies that

$$\mu = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi_1(x)-\phi_1(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)\phi_1(x)^2\,dx. \qquad \square$$

Lemma 3.3 *For any real number* $2_\alpha^* - 1 > k \geq 1$, *assume that the conditions* (V_0), (F_1)-(F_2) *hold. Let* $\rho, \Lambda > 0$ *be as in Lemma* 3.1. *Then we have the following results*:

(i) *If* $k = 1$ *and* $\mu < 1$, *then there exists* $e \in X$ *with* $\|e\| > \rho$ *such that* $J(e) < 0$ *for all* $\lambda \in (0, \Lambda)$.

(ii) *If* $k > 1$, *then there exists* $e \in X$ *with* $\|e\| > \rho$ *such that* $J(e) < 0$ *for all* $\lambda \in (0, \Lambda)$.

Proof (i) In case $k = 1$. Since $\mu < 1$, we can choose a nonnegative function $\varphi \in H^\alpha(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} q(x)\varphi(x)^2\,dx = 1$ such that

$$\|\varphi\|^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(z)|^2}{|x-z|^{N+2\alpha}}\,dx\,dz + \int_{\mathbb{R}^N} V(x)\varphi(x)^2\,dx < 1.$$

Therefore, by the condition (F_2) and Fatou's lemma, we have

$$\begin{aligned} \lim_{t\to+\infty}\frac{J(t\varphi)}{t^2} &\le \frac{1}{2}\|\varphi\|^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\varphi)}{t^2\varphi^2}\varphi^2\,dx - \lim_{t\to+\infty}\frac{\lambda}{pt^{2-p}}\int_{\mathbb{R}^N} h(x)|\varphi|^p\,dx \\ &\le \frac{1}{2}\|\varphi\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} q(x)\varphi^2\,dx \\ &= \frac{1}{2}\bigl(\|\varphi\|^2 - 1\bigr) < 0. \end{aligned} \tag{3.6}$$

So, $J(t\varphi) \to -\infty$ as $t \to +\infty$, then there exists $e \in X$ with $\|e\| > \rho$ such that $J(e) < 0$ for all $\lambda \in (0,\Lambda)$.

(ii) In case $k > 1$. $q \in L^\infty(\mathbb{R}^N,\mathbb{R}^+)$ with $q^+ \not\equiv 0$, we can choose a function $\omega \in H^\alpha(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} q(x)|\omega|^{k+1}\,dx > 0.$$

Therefore, by the condition (F_2) and Fatou's lemma, we have

$$\begin{aligned} \lim_{t\to+\infty}\frac{J(t\omega)}{t^{k+1}} &\le \frac{\|\omega\|^2}{2t^{k-1}} - \lim_{t\to+\infty}\int_\Omega \frac{F(x,t\omega)}{t^{k+1}\omega^{k+1}}\varphi^{k+1}\,dx - \lim_{t\to+\infty}\frac{\lambda}{pt^{k+1-p}}\int_\Omega \xi(x)|\omega|^p\,dx \\ &\le -\frac{1}{k+1}\int_\Omega q(x)\omega^{k+1}\,dx \\ &< 0. \end{aligned} \tag{3.7}$$

So, $J(t\omega) \to -\infty$ as $t \to +\infty$; then there exists $e \in X$ with $\|e\| > \rho$ such that $J(e) < 0$ for all $\lambda \in (0,\Lambda)$. □

Next, we define

$$\beta = \inf_{\gamma\in\Gamma}\max_{0\le\tau\le1} J\bigl(\gamma(\tau)\bigr),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$. Then by Theorem 2.1, Lemma 3.1, and Lemma 3.3, there exists a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to \beta \quad\text{and}\quad \bigl(1 + \|u_n\|\bigr)\bigl\|J'(u_n)\bigr\|_{E^*} \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Then we have the following results.

Lemma 3.4 *For any real number $2_\alpha^* - 1 > k \ge 1$, assume that the conditions* (V_0), (F_1)-(F_3) *hold. Let $\Lambda > 0$ be as in Lemma* 3.1. *Then $\{u_n\}$ defined by* (3.8) *is bounded in X for all $\lambda \in (0,\Lambda)$.*

Proof For n large enough, by Hölder's inequality and Lemma 2.1, one has

$$\begin{aligned}
\beta + 1 &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \int_{\mathbb{R}^N}\left[F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\right]dx \\
&\quad - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} h(x)|u_n|^p\,dx \\
&\geq \frac{\theta-2}{2\theta}\|u_n\|^2 - d_0\int_{\mathbb{R}^N} u_n^2\,dx - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} h(x)|u_n|^p\,dx \\
&\geq \frac{\theta-2}{2\theta}\|u_n\|^2 - \frac{d_0}{S_2^2}\|u_n\|^2 - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)S_2^{-p}\|h\|_{\frac{2}{2-p}}\|u_n\|^p \\
&\geq \left(\frac{\theta-2}{2\theta} - \frac{d_0}{S_2^2}\right)\|u_n\|^2 - \Lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)S_2^{-p}\|h\|_{\frac{2}{2-p}}\|u_n\|^p, \qquad (3.9)
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in X, since $1 \leq p < 2$. □

Denote $B_\rho = \{u \in X : \|u\| < \rho\}$, where ρ is given by Lemma 3.1. Then by Ekeland's variational principle and Lemma 2.2, we have the following lemma, which shows that J has a local minimum if λ is small.

Lemma 3.5 *For any real number* $2_\alpha^* - 1 > k \geq 1$, *assume that the conditions* (V_0), (F_1)-(F_3). *Let* $\Lambda > 0$ *be as in Lemma* 3.1. *Then for every* $\lambda \in (0, \Lambda)$, *there exists* $u_0 \in X$ *such that*

$$J(u_0) = \inf\{J(u) : u \in \bar{B}_\rho\} < 0,$$

and u_0 *is a solution of problem* (1.1).

Proof Since $h \in L^{\frac{2}{2-p}} \setminus \{0\}$ with $h^+ \not\equiv 0$, we can choose a function $\psi \in H^\alpha(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} h(x)|\psi|^p\,dx > 0.$$

Hence, we have

$$\begin{aligned}
J(t\psi) &= \frac{t^2}{2}\|\psi\|^2 - \int_{\mathbb{R}^N} F(x,t\psi)\,dx - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} h(x)|\psi|^p\,dx \\
&\leq \frac{t^2}{2}\|\psi\|^2 - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} h(x)|\psi|^p\,dx < 0 \qquad (3.10)
\end{aligned}$$

for $t > 0$ small enough, which implies $\theta_0 := \inf\{J(u) : u \in \bar{B}_\rho\} < 0$. By Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \bar{B}_\rho$ such that $J(u_n) \to \theta_0$ and $J'(u_n) \to 0$ as $n \to \infty$. Hence Lemma 2.2 implies that there exists $u_0 \in X$ such that $J'(u_0) = 0$ and $J(u_0) = c_1 < 0$. □

Proof of Theorem 1.1 From Lemma 2.2 and Lemma 3.4, there is a $\bar{u} \in X$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ weakly in X, $u_n \to \bar{u}$ strongly in $L^s(\mathbb{R})$ for $s \in [2, 2_\alpha^*)$. By using a standard procedure, we can prove that $u_n \to \bar{u}$ strongly in X. Moreover, $J(\bar{u}) = \beta > 0$ and

$\bar{u}$ is another solution of problem (1.1). Thus, combining with Lemma 3.5, we prove that problem (1.1) has at least two solutions $u_0, \bar{u} \in X$ satisfying $J(u_0) < 0$ and $J(\bar{u}) > 0$. □

Competing interests
The author declares that they have no competing interests.

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References

1. Di Nezza, E, Palatucci, G, Valdinoci, E: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521-573 (2012)
2. Barrios, B, Colorado, E, de Pablo, A, Sánchez, U: On some critical problems for the fractional Laplacian operator. J. Differ. Equ. **252**, 6133-6162 (2012)
3. Brändle, C, Colorado, E, de Pablo, A: A concave-convex elliptic problem involving the fractional Laplacian. Proc. R. Soc. Edinb., Sect. A **143**, 39-71 (2013)
4. Di Blasio, G, Volzone, B: Comparison and regularity results for the fractional Laplacian via symmetrization methods. J. Differ. Equ. **253**, 2593-2615 (2012)
5. Servadei, R, Valdinoci, E: Mountain pass solutions for non-local elliptic operators. J. Math. Anal. Appl. **389**, 887-898 (2012)
6. Yu, X: The Nehari manifold for elliptic equation involving the square root of the Laplacian. J. Differ. Equ. **252**, 1283-1308 (2012)
7. Silvestre, L: Regularity of the obstacle problem for a fractional power of the Laplace operator. Commun. Pure Appl. Math. **60**, 67-112 (2007)
8. Tan, J, Wang, Y, Yang, J: Nonlinear fractional field equations. Nonlinear Anal. **75**, 2098-2110 (2012)
9. Caffarelli, L: Surfaces minimizing nonlocal energies. Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. **20**, 281-299 (2009)
10. Caffarelli, L, Vazquez, JL: Nonlinear porous medium flow with fractional potential pressure. Arch. Ration. Mech. Anal. **202**, 537-565 (2011)
11. Laskin, N: Fractional quantum mechanics and path integrals. Phys. Lett. A **269**, 298-305 (2000)
12. Chang, X, Wang, Z: Ground state solutions of scalar field equations involving fractional Laplacian with general nonlinearity. Nonlinearity **26**, 479-494 (2013)
13. Feng, B: Ground state solutions for the fractional equation. Electron. J. Differ. Equ. **2013**, 127 (2013)
14. Felmer, P, Quaas, A, Tan, J: Positive solutions of the nonlinear Schrödinger equation with fractional Laplacian. Proc. R. Soc. Edinb., Sect. A **142**, 1237-1262 (2012)
15. Secchi, S: Ground state solutions for the fractional equation in $\mathbb{R}^N$. J. Math. Phys. **54**, 031501 (2013)
16. Dipierro, S, Palatucci, G, Valdinoci, E: Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian. Matematiche **68**, 201-216 (2013)
17. Autuori, G, Pucci, P: Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$. J. Differ. Equ. **255**, 2340-2362 (2013)
18. Secchi, S: On fractional Schrödinger equations in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition. arXiv:1210.0755
19. Torres, C: Non-homogeneous fractional Schrödinger equation. arXiv:1311.0708
20. Chang, X: Ground state solutions of asymptotically linear fractional Schrödinger equation. J. Math. Phys. **54**, 061504 (2013)
21. Ekeland, I: Convexity Methods in Hamiltonian Mechanics. Springer, Berlin (1990)