Boundary Value Problems
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Multiple solutions for biharmonic equations with improved subcritical polynomial growth and subcritical exponential growth

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Abstract

The main purpose of this paper is to establish the existence of three nontrivial solutions for a class of fourth-order elliptic equations with subcritical polynomial growth and subcritical exponential growth by using the minimax method and Morse theory.

Keywords: Morse theory; Adams-type inequality; subcritical polynomial growth; subcritical exponential growth

1 Introduction

Consider the following Navier boundary value problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u), & \text{in } \Omega; \\ u = \Delta u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Δ^2 is the biharmonic operator and Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 4$).

In problem (1), let $f(x, u) = b[(u + 1)^+ - 1]$, then we get the following Dirichlet problem:

$$\begin{cases} \Delta^2 u + c\Delta u = b[(u + 1)^+ - 1], & \text{in } \Omega; \\ u = \Delta u = 0, & \text{in } \partial\Omega, \end{cases} \tag{2}$$

where $u^+ = \max\{u, 0\}$ and $b \in \mathbb{R}$. We let λ_k ($k = 1, 2, \ldots$) denote the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$.

Fourth-order problems of this class with $N > 4$ have been studied by many authors. In [1], Lazer and Mckenna pointed out that this type of nonlinearity provides a model to study traveling waves in suspension bridges. Since then, more general nonlinear fourth-order elliptic boundary value problems have been studied. For problem (2), Lazer and Mckenna [2] proved the existence of $2k - 1$ solutions when $N = 1$, and $b > \lambda_k(\lambda_k - c)$ by the global bifurcation method. In [3], Tarantello found a negative solution when $b \geq \lambda_1(\lambda_1 - c)$ by a degree argument. For problem (1) when $f(x, u) = bg(x, u)$, Micheletti and Pistoia [4] proved

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that there exist two or three solutions for a more general nonlinearity g by the variational method. Xu and Zhang [5] discussed the problem when f satisfies the local superlinearity and sublinearity. Zhang [6] proved the existence of solutions for a more general nonlinearity $f(x, u)$ under some weaker assumptions. Zhang and Li [7] proved the existence of multiple nontrivial solutions by means of Morse theory and local linking. An and Liu [8] and Liu and Wang [9] also obtained the existence result for nontrivial solutions when f is asymptotically linear at positive infinity. In [10], Zhang and Wei obtained the existence of infinitely many solutions when the nonlinearity involves a combination of superlinear and asymptotically linear terms. As far as the problem (1) is concerned, existence results of sign-changing solutions were also obtained (see, *e.g.*, [11, 12]).

We notice that almost all the works (see [4–12]) mentioned above involve the nonlinear term $f(x, u)$ of a subcritical (polynomial) growth, say,

(SCP): there exist positive constants c_1 and c_2 and $q_0 \in (1, p^* - 1)$ such that

$$\left| f(x, t) \right| \leq c_1 + c_2 |t|^{q_0} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega,$$

where $p^* = 2N/(N - 4)$ denotes the critical Sobolev exponent. One of the main reasons to assume this condition (SCP) is to use the Sobolev compact embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ $(1 \leq q < p^*)$. In that case, it is easy to see that seeking weak solutions of problem (1) is equivalent to finding nonzero critical points of the following functional on $H^2(\Omega) \cap H_0^1(\Omega)$:

$$I(u) = \frac{1}{2} \int_\Omega \left(|\Delta u|^2 - c |\nabla u|^2 \right) dx - \int_\Omega F(x, u) \, dx, \tag{3}$$

where $F(x, u) = \int_0^u f(x, t) \, dt$. In this paper, inspired by Lam and Lu [13], our first main result will be to study problem (1) in the improved subcritical polynomial growth

(SCPI): $\lim_{t \to \infty} \dfrac{f(x, t)}{|t|^{p^* - 1}} = 0 \quad \text{uniformly on } x \in \Omega,$

which is weaker than (SCP). Note that, in this case, we do not have the Sobolev compact embedding anymore. Our work is to study problem (1) when nonlinearity f does not satisfy the Ambrosetti-Rabinowitz condition, *i.e.*, for some $\theta > 2$ and $\gamma > 0$,

$$0 < \theta F(x, t) \leq f(x, t)t \quad \text{for all } |t| \geq \gamma \text{ and } x \in \Omega. \tag{AR}$$

In fact, this condition was studied by Liu and Wang in [14] in the case of Laplacian by the Nehari manifold approach. However, we will use a suitable version of the mountain pass theorem and Morse theory to get three nontrivial solutions for problem (1) in the general case $N > 4$. Indeed, in this case, we have obtained two nontrivial solutions for problem (1) in [15] via the mountain pass theorem and a truncated technique.

Let us now state our results. In this paper, we always assume that $f(x, t) \in C^1(\bar{\Omega} \times \mathbb{R})$. The conditions imposed on $f(x, t)$ are as follows:

(H$_1$) $f(x, t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$;
(H$_2$) $\lim_{|t| \to 0} \frac{f(x,t)}{t} = f_0$ uniformly for $x \in \Omega$, where f_0 is a constant;
(H$_3$) $\lim_{|t| \to \infty} \frac{f(x,t)}{t} = +\infty$ uniformly for $x \in \Omega$;
(H$_4$) $\frac{f(x,t)}{|t|}$ is nondecreasing in $t \in \mathbb{R}$ for any $x \in \Omega$.

Let $0 < \mu_1$ be the first eigenvalue of $(\Delta^2 - c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ $(c < \lambda_1)$ and $\varphi_1(x) > 0$ be the eigenfunction corresponding to μ_1. Throughout this paper, we denote by $| \cdot |_p$ the $L^p(\Omega)$ norm and the norm of u in $H^2(\Omega) \cap H_0^1(\Omega)$ will be defined by

$$\|u\| := \left(\int_\Omega \left(|\Delta u|^2 - c|\nabla u|^2 \right) dx \right)^{\frac{1}{2}}.$$

We also define $E = H^2(\Omega) \cap H_0^1(\Omega)$. In fact, the norm $\| \cdot \|$ is equivalent to another norm $\| \cdot \|_E$ defined by

$$\|u\|_E := \left(\int_\Omega |\Delta u|^2 \, dx \right)^{\frac{1}{2}}$$

on E, *i.e.*, there exist two positive constants C^*, C^{**} such that

$$C^{**} \|u\| \leq \|u\|_E \leq C^* \|u\|.$$

Theorem 1.1 *Let $N > 4$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (H$_1$)-(H$_4$). If $f_0 < \mu_1$, then problem (1) has at least three nontrivial solutions.*

In the case of $N = 4$, we have $p^* = +\infty$. So it is necessary to introduce the definition of the subcritical (exponential) growth in this case. By the improved Adams inequality (see [16]) for the fourth-order derivative, we have

$$\sup_{u \in E, \|u\|_E \leq 1} \int_\Omega e^{32\pi^2 u^2} \, dx \leq C|\Omega|,$$

where C is a positive constant (see Lemma 2.2). So, we now define the subcritical (exponential) growth in this case as follows:

(SCE): f has subcritical (exponential) growth on Ω, *i.e.*, $\lim_{t \to \infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0$ uniformly on $x \in \Omega$ for all $\alpha > 0$.

When $N = 4$ and f has the subcritical (exponential) growth (SCE), our work is still to study problem (1) without the (AR) condition. Our result is as follows.

Theorem 1.2 *Let $N = 4$ and assume that f has the subcritical exponential growth on Ω (condition (SCE)) and satisfies (H$_1$)-(H$_4$). If $f_0 < \mu_1$, then problem (1) has at least three nontrivial solutions.*

Remark 1.1 Indeed, in this case we have already obtained two nontrivial solutions for problem (1) in [15] via the mountain pass theorem and a truncated technique. So, this paper is a completion for our previous work (see [15]).

2 Preliminaries and auxiliary lemmas

Definition 2.1 Let $(E, \| \cdot \|_E)$ be a real Banach space with its dual space $(E^*, \| \cdot \|_{E^*})$ and $I \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(C)_c$ condition (Cerami condition) if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c, \qquad \|DI(x_n)\|_{E^*} \left(1 + \|x_n\|_E \right) \to 0,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E.

We have the following version of the mountain pass theorem (see [17]).

Proposition 2.1 *Let E be a real Banach space and suppose that $I \in C^1(E, R)$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let $c \geq \beta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \beta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Consider the following problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f_+(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

where

$$f_+(x, t) = \begin{cases} f(x, t), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Define a functional $I_+ : E \to \mathbb{R}$ by

$$I_+(u) = \frac{1}{2} \int_\Omega \left(|\Delta u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F_+(x, u) \, dx,$$

where $F_+(x, t) = \int_0^t f_+(x, s) \, ds$; then $I_+ \in C^1(E, \mathbb{R})$.

Lemma 2.1 *Let $N > 4$ and $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that (H_2), (H_3) and (SCPI) hold. If $f_0 < \mu_1$, then*

  (i) *there exist $\rho, \alpha > 0$ such that $I_+(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*
  (ii) *$I_+(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCPI), (H_2) and (H_3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$ and $l > 2\mu_1$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F_+(x, s) \leq \frac{1}{2}(f_0 + \varepsilon)s^2 + A_1 s^{p^*}, \tag{4}$$

$$F_+(x, s) \geq \frac{1}{2}ls^2 - B_1. \tag{5}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (4), the Poincaré inequality and the Sobolev inequality $|u|_{p^*}^{p^*} \leq K \|u\|^{p^*}$, we get

$$
\begin{aligned}
I_+(u) &\geq \frac{1}{2}\|u\|^2 - \frac{f_0 + \varepsilon}{2}|u|_2^2 - A_1|u|_{p^*}^{p^*} \\
&\geq \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right)\|u\|^2 - A_1 K\|u\|^{p^*}.
\end{aligned}
$$

So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, from (5) we have

$$
I_+(t\varphi_1) \leq \frac{1}{2}\left(1 - \frac{l}{\mu_1}\right)t^2 + B_1|\Omega| \to -\infty \quad \text{as } t \to +\infty.
$$

Thus part (ii) is proved. □

Lemma 2.2 (see [16]) *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$
\sup_{u \in E, \|u\|_E \leq 1} \int_\Omega e^{32\pi^2 u^2} \, dx \leq C|\Omega|,
$$

where $\|u\|_E = (\int_\Omega |\Delta u|^2 \, dx)^{\frac{1}{2}}$. This inequality is sharp.

Lemma 2.3 *Let $N = 4$ and $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that (H_2), (H_3) and (SCE) hold. If $f_0 < \mu_1$, then*

  (i) *there exist $\rho, \alpha > 0$ such that $I_+(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*

  (ii) *$I_+(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof By (SCE), (H_2) and (H_3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$, $\kappa > 0$, $q > 2$ and $l > 2\mu_1$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$
F_+(x,s) \leq \frac{1}{2}(f_0 + \varepsilon)s^2 + A_1 \exp(\kappa|s|^2)s^q, \tag{6}
$$

$$
F_+(x,s) \geq \frac{1}{2}ls^2 - B_1. \tag{7}
$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (6), the Hölder inequality and Lemma 2.2, we get

$$
\begin{aligned}
I_+(u) &\geq \frac{1}{2}\|u\|^2 - \frac{f_0 + \varepsilon}{2}|u|_2^2 - A_1 \int_\Omega \exp(\kappa|u|^2)|u|^q \, dx \\
&\geq \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right)\|u\|^2 - A_1\left(\int_\Omega \exp\left(\kappa r\|u\|_E^2\left(\frac{|u|}{\|u\|_E}\right)^2\right) dx\right)^{\frac{1}{r}}\left(\int_\Omega |u|^{r'q} \, dx\right)^{\frac{1}{r'}} \\
&\geq \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right)\|u\|^2 - C\|u\|^q,
\end{aligned}
$$

where $r > 1$ is sufficiently close to 1, $\|u\|_E \leq \sigma$ and $\kappa r \sigma^2 < 32\pi^2$. So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough since $\|u\|_E \leq C^*\|u\|$.

On the other hand, from (7) we have

$$I_+(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{l}{\mu_1}\right)|t|^2 + B_1|\Omega| \to -\infty \quad \text{as } t \to +\infty.$$

Thus part (ii) is proved. □

Lemma 2.4 *For the functional I_+, if condition* (H$_4$) *holds, and for any* $\{u_n\} \in E$ *with*

$$\langle I'_+(u_n), u_n \rangle \to 0 \quad \text{as } n \to \infty,$$

then there is a subsequence, still denoted by $\{u_n\}$, *such that*

$$I_+(tu_n) \le \frac{1 + t^2}{2n} + I_+(u_n) \quad \text{for all } t \in \mathbb{R} \text{ and } n \in N.$$

Proof This lemma is essentially due to [18]. We omit the proof. □

Lemma 2.5 *Under the assumptions of Theorem* 1.1, *then I_+ and I satisfy the* $(C)_c$ *condition.*

Proof We first do the proof for I_+. Let $\{u_n\} \subset E$ be a $(C)_c$ $(c \in \mathbb{R})$ sequence such that for every $n \in \mathbb{N}$,

$$I_+(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F_+(x, u_n)\,dx = c + o(1), \tag{8}$$

$$(1 + \|u_n\|)\|I'_+(u_n)\|_E \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Clearly, (9) implies that

$$\langle I'_+(u_n), u_n \rangle = \|u_n\|^2 - \int_\Omega f_+\big(x, u_n(x)\big)u_n\,dx = o(1). \tag{10}$$

To complete our proof, we first need to verify that $\{u_n\}$ is bounded in E. Assume $\|u_n\| \to +\infty$ as $n \to \infty$. Let

$$s_n = \frac{2\sqrt{|c| + 1}}{\|u_n\|}, \qquad w_n = s_n u_n = \frac{2\sqrt{|c| + 1}u_n}{\|u_n\|}. \tag{11}$$

Since $\{w_n\}$ is bounded in E, it is possible to extract a subsequence (denoted also by $\{w_n\}$) such that

$$w_n \rightharpoonup w_0 \quad \text{in } E,$$

$$w_n^+ \to w_0^+ \quad \text{in } L^2(\Omega),$$

$$w_n^+(x) \to w_0^+(x) \quad \text{a.e. } x \in \Omega,$$

$$|w_n^+(x)| \le h(x) \quad \text{a.e. } x \in \Omega,$$

where $w_n^+ = \max\{w_n, 0\}$, $w_0 \in E$ and $h \in L^2(\Omega)$.

We claim that if $\|u_n\| \to +\infty$ as $n \to +\infty$, then $w^+(x) \equiv 0$. In fact, we set $\Omega_1 = \{x \in \Omega : w^+ = 0\}$, $\Omega_2 = \{x \in \Omega : w^+ > 0\}$. Obviously, by (11), $u_n^+ \to +\infty$ a.e. in Ω_2; noticing condition

(H₃), then, for any given $K > 0$, we have

$$\lim_{n \to +\infty} \frac{f(x, u_n^+)}{u_n^+} \left(w_n^+(x) \right)^2 \geq K w^+(x)^2 \quad \text{for a.e. } x \in \Omega_2. \tag{12}$$

From (10), (11), and (12), we obtain

$$4\big(|c| + 1\big) = \lim_{n \to +\infty} \|w_n\|^2 = \lim_{n \to +\infty} \int_\Omega \frac{f(x, u_n^+)}{u_n^+} \left(w_n^+ \right)^2 dx$$

$$\geq \int_{\Omega_2} \lim_{n \to +\infty} \frac{f(x, u_n^+)}{u_n^+} \left(w_n^+ \right)^2 dx \geq K \int_{\Omega_2} \left(w^+ \right)^2 dx.$$

Noticing that $w^+ > 0$ in Ω_2 and $K > 0$ can be chosen large enough, so $|\Omega_2| = 0$ and $w^+ \equiv 0$ in Ω. However, if $w^+ \equiv 0$, then $\lim_{n \to +\infty} \int_\Omega F_+(x, w_n^+) \, dx = 0$ and consequently

$$I_+(w_n) = \frac{1}{2} \|w_n\|^2 + o(1) = 2\big(|c| + 1\big) + o(1). \tag{13}$$

By $\|u_n\| \to +\infty$ as $n \to +\infty$ and in view of (11), we observe that $s_n \to 0$, then it follows from Lemma 2.4 and (8) that

$$I_+(w_n) = I_+(s_n u_n) \leq \frac{1 + s_n^2}{2n} + I_+(u_n) \to c \quad \text{as } n \to +\infty. \tag{14}$$

Clearly, (13) and (14) are contradictory. So $\{u_n\}$ is bounded in E.

Next, we prove that $\{u_n\}$ has a convergent subsequence. In fact, we can suppose that

$$u_n \rightharpoonup u \quad \text{in } E,$$

$$u_n \to u \quad \text{in } L^q(\Omega), \forall 1 \leq q < p^*,$$

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.$$

Now, since f has the improved subcritical growth on Ω, for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f_+(x, s) \leq C(\varepsilon) + \varepsilon |s|^{p^* - 1}, \quad \forall (x, s) \in \Omega \times \mathbb{R},$$

then

$$\left| \int_\Omega f_+(x, u_n)(u_n - u) \, dx \right|$$

$$\leq C(\varepsilon) \int_\Omega |u_n - u| \, dx + \varepsilon \int_\Omega |u_n - u| |u_n|^{p^* - 1} \, dx$$

$$\leq C(\varepsilon) \int_\Omega |u_n - u| \, dx + \varepsilon \left(\int_\Omega \left(|u_n|^{p^* - 1} \right)^{\frac{p^*}{p^* - 1}} dx \right)^{\frac{p^* - 1}{p^*}} \left(\int_\Omega |u_n - u|^{p^*} dx \right)^{\frac{1}{p^*}}$$

$$\leq C(\varepsilon) \int_\Omega |u_n - u| \, dx + \varepsilon C(\Omega).$$

Similarly, since $u_n \rightharpoonup u$ in E, $\int_\Omega |u_n - u| \, dx \to 0$. Since $\varepsilon > 0$ is arbitrary, we can conclude that

$$\int_\Omega \big(f_+(x, u_n) - f_+(x, u) \big)(u_n - u) \, dx \to 0 \quad \text{as } n \to \infty. \tag{15}$$

By (10), we have

$$\langle I'_+(u_n) - I'_+(u), (u_n - u) \rangle \to 0 \quad \text{as } n \to \infty. \tag{16}$$

From (15) and (16), we obtain

$$\int_\Omega \left[\left| \Delta(u_n - u) \right|^2 - c \left| \nabla(u_n - u) \right|^2 \right] dx \to 0 \quad \text{as } n \to \infty.$$

So we have $u_n \to u$ in E, which means that I_+ satisfies $(C)_c$.

Next we prove that I satisfies $(C)_c$. In fact, by (H_4), we have

$$f(x, t)t - 2F(x, t) \geq f(x, st)st - 2F(x, st)$$

for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$. By the proof of Lemma 2.1 in [19], we can similarly prove that $(C)_c$ sequence $\{u_n\}$ is bounded. The other part of the proof is similar to the case already proved and is omitted. □

Lemma 2.6 *Under the assumptions of Theorem* 1.2, I_+ *and* I *satisfy the* $(C)_c$ *condition.*

Proof We only do the proof for I_+. Similar to the first part in the proof of Lemma 2.5, we easily know that $(C)_c$ sequence $\{u_n\}$ is bounded in E. Next, we prove that $\{u_n\}$ has a convergent subsequence. Without loss of generality, suppose that

$$\|u_n\| \leq \beta_0,$$

$$u_n \rightharpoonup u \quad \text{in } E,$$

$$u_n \to u \quad \text{in } L^q(\Omega), \forall q \geq 1,$$

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.$$

By the equivalence of the norm on E, we have

$$\|u_n\|_E \leq C^* \|u_n\|.$$

Let $\beta = C^* \beta_0$. Now, since f_+ has the subcritical exponential growth (SCE) on Ω, we can find a constant $C_\beta > 0$ such that

$$\left| f_+(x, t) \right| \leq C_\beta \exp\left(\frac{32\pi^2}{2\beta^2} |t|^2 \right), \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

Thus, by the Adams-type inequality (see Lemma 2.2),

$$\left| \int_\Omega f_+(x, u_n)(u_n - u) \, dx \right|$$

$$\leq C \left(\int_\Omega \exp\left(\frac{32\pi^2}{\beta^2} |u_n|^2 \right) dx \right)^{\frac{1}{2}} |u_n - u|_2$$

$$\leq C \left(\int_\Omega \exp\left(\frac{32\pi^2}{\beta^2} \|u_n\|_E^2 \left| \frac{u_n}{\|u_n\|_E} \right|^2 \right) dx \right)^{\frac{1}{2}} |u_n - u|_2$$

$$\leq C |u_n - u|_2 \to 0.$$

Similar to the last part in the proof of Lemma 2.5, we have $u_n \to u$ in E, which means that I_+ satisfies $(C)_c$. □

3 Computation of the critical groups

It is well known that critical groups and Morse theory are the main tools in solving elliptic partial differential equations. Let us recall some results which will be used later. We refer the readers to the book [20] for more information on Morse theory.

Let H be a Hilbert space and $I \in C^1(H, \mathbb{R})$ be a functional satisfying the (PS) condition (or $(C)_c$ condition), and $H_q(X, Y)$ be the qth singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of I with $I(u_0) = c$, $c \in \mathbb{R}$, and let U be a neighborhood of u_0. The group

$$C_q(I, u_0) := H_q\big(I^c \cap U, I^c \cap U \setminus \{u_0\}\big), \quad q \in Z,$$

is said to be the qth critical group of I at u_0, where $I^c = \{u \in H : I(u) \leq c\}$.

Let $K := \{u \in H : I'(u) = 0\}$ be the set of critical points of I and $a < \inf I(K)$, the critical groups of I at infinity are formally defined by (see [21])

$$C_q(I, \infty) := H_q\big(H, I^a\big), \quad q \in Z.$$

For the convenience of our proof, we firstly prove two important propositions.

Proposition 3.1 *If the assumptions of Theorem* 1.1 (*or Theorem* 1.2) *hold, then*

$$C_q(I, \infty) = 0 \quad \text{for all integers } q \geq 0.$$

Proof Let $S^\infty = \{u \in E : \|u\| = 1\}$ be the unit sphere in E and $B^\infty = \{u \in E : \|u\| \leq 1\}$. By (H_3), for any $M > 0$ there exists $c > 0$, such that $F(x, t) \geq Mt^2 - c$, for $(x, t) \in \Omega \times \mathbb{R}$, which implies

$$I(tu) \to -\infty \quad \text{as } t \to +\infty$$

for any $u \in S^\infty$. Using (H_4), we have

$$f(x, t)t - 2F(x, t) \geq 0 \quad \text{for } (x, t) \in \Omega \times \mathbb{R}. \tag{17}$$

Choose

$$a < \min\Big\{ \inf_{u \in B^\infty} I(u), 0 \Big\}.$$

Then, for any $u \in S^\infty$, there exists $t > 1$ such that $I(tu) \leq a$, that is,

$$I(tu) = \frac{t^2}{2} - \int_\Omega F(x, tu)\, dx \leq a,$$

which implies

$$\frac{d}{dt}I(tu) = t - \int_\Omega f(x, tu)u \, dx$$

$$\leq \frac{1}{t}(2a) < 0 \quad \text{by (17)}.$$

The rest of the proof is similar to the proof of Lemma 2.6 in [19]. □

Proposition 3.2 *If the assumptions of Theorem* 1.1 (*or Theorem* 1.2) *hold, then*

$$C_q(I, 0) = \delta_{q,0}Z \quad \text{for all integers } q \geq 0.$$

Proof By Lemma 2.1 (Lemma 2.3), $u = 0$ is a local minimizer of I. So we have

$$C_q(I, 0) = \delta_{q,0}Z \quad \text{for all integers } q \geq 0.$$ □

4 Proof of the main results

Proofs of Theorem 1.1 *and Theorem* 1.2 By Lemma 2.1 (Lemma 2.3), Lemma 2.5 (Lemma 2.6), and Proposition 2.1, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \geq \beta$. Since $I_+(0) = 0$, $u_1 \neq 0$ and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1) and satisfies

$$C_1(I_+, u_1) \neq 0, \quad u_1 > 0. \tag{18}$$

Using Lemma 3.1 in [22], we obtain

$$C_q(I, u_1) = C_q(I_{C_0^1(\Omega)}, u_1) = C_q(I_+|_{C_0^1(\Omega)}, u_1) = C_q(I_+, u_1) = \delta_{q1}Z. \tag{19}$$

Similarly, we can obtain another negative critical point u_2 of I satisfying

$$C_q(I, u_2) = \delta_{q,1}Z. \tag{20}$$

Now, from Proposition 3.2, we have

$$C_q(I, 0) = \delta_{q,0}Z. \tag{21}$$

On the other hand, from Proposition 3.1, we have

$$C_q(I, \infty) = 0. \tag{22}$$

Then from (19), (20), (21), (22), and the Morse relation, we have

$$1 + 2t = (1 + t)Q(t)$$

with $Q(t) = \sum_{q \geq 0} d_q t^q$ where $d_q \in \mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ for all $q \geq 0$. Then $d_0 = d_1 = 1$ and so the right-hand side has a term t^2, a contradiction. This means that there is one more critical point of I, $u_3 \notin \{0, u_1, u_2\}$. Then $u_3 \in E$ is a nontrivial solution of problem (1). □

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