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Boundary value problems for impulsive multi-order Hadamard fractional differential equations

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Abstract
In this paper, we study the existence and uniqueness of solutions for impulsive multi-orders Caputo-Hadamard fractional differential equations equipped with boundary and integral conditions. The Banach, Schaefer, and Rothe fixed point theorems and degree theory are used to establish our main results. Examples illustrating the main results are presented.

MSC: 34A08; 34B10; 34A37

Keywords: impulsive differential equations; Hadamard fractional calculus; fixed point theorems; existence

1 Introduction

During the last years, fractional calculus has gained considerable importance due to the applications in almost all applied sciences. It was pointed out that fractional derivatives and integrals are more convenient for describing real materials, and some physical problems were treated by using derivatives of non-integer orders. For details, and some recent results on the subject we refer to [1–9] and the references cited therein.

It has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [10], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [1, 11–15]. However, this calculus with Hadamard derivatives is still studied less than that of Riemann-Liouville.

On the other hand, integer order impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and the social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance [16–20].

Recently in [21], Wang *et al.* studied existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multi-order fractional derivatives and a deviating argument:

$$\begin{cases} {}^cD^{\alpha_k}_{t_k^+} u(t) = f(t, u(t), u(\theta(t))), \quad 1 < \alpha_k \leq 2, \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}^{\beta_k}_{t_k^+} u(\eta_k), \qquad u'(0) = 0, \quad t_k < \eta_k < t_{k+1}, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha_k}_{t_k^+}$ is the *Caputo fractional derivative* of order α_k, $\mathcal{J}^{\beta_k}_{t_k^+}$ is Riemann-Liouville fractional integral of order $\beta_k > 0$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $J = [0, T]$ $(T > 0)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u(t_k^+)$, $u'(t_k^+)$ and $u(t_k^-)$, $u'(t_k^-)$ denote the right and left hand limits of $u(t)$ and $u'(t)$ at $t = t_k$ $(k = 1, 2, \ldots, p)$.

In 2015, Wang *et al.* [22] established the existence of solutions for a class of nonlinear impulsive Hadamard fractional differential equations with initial condition of the form

$$\begin{cases} {}_HD^{\alpha}_{1^+} u(t) = f(t, u(t)), \quad \alpha \in (0,1), t \in (1, e] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t_i) = {}_HJ^{1-\alpha}_{1^+} u(t_i^+) - {}_HJ^{1-\alpha}_{1^+} u(t_i^-) = p_i, \quad p_i \in \mathbb{R}, i = 1, 2, \ldots, m, \\ {}_HJ^{1-\alpha}_{1^+} u(1^+) = u_0, \quad u_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

where ${}_HD^{\alpha}_{1^+}$ is the left-side *Hadamard fractional derivative* of order α with the lower limit 1 and ${}_HJ^{1-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $1-\alpha$. The existence results were obtained by using the Banach contraction principle and Schauder's fixed point theorem on the weight spaces of piecewise continuous functions.

The Hadamard and Riemann-Liouville fractional derivatives have one similar property, which is the fact that the derivative of a constant is not equal to zero. It is caused by the definitions of them containing the usual derivative outside the integrals. In 2012, Jarad *et al.* [23] presented the modifications of the Hadamard fractional derivative into a more suitable one having physically interpretable initial conditions similar to the Caputo sense. In 2014, Gambo *et al.* [24] proved the fundamental theorem of fractional calculus, some interesting results and also semigroup properties of Caputo-Hadamard operators.

In this paper we are concerned with the existence of solutions for boundary value problems of impulsive Hadamard fractional differential equations of the form

$$\begin{cases} {}^C\mathcal{D}^{p_k}_{t_k} x(t) = f(t, x(t)), \quad t \in J_k \subset [t_0, T], t \neq t_k, \\ \Delta x(t_k) = \varphi_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \alpha x(t_0) + \beta x(T) = \sum_{i=0}^{m} \gamma_i \mathcal{J}^{q_i}_{t_i} x(t_{i+1}), \end{cases} \tag{1.3}$$

where ${}^C\mathcal{D}^{p_k}_{t_k}$ is the Hadamard fractional derivative of Caputo type of order $0 < p_k \leq 1$ on intervals $J_k := (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, with $J_0 = [t_0, t_1]$, $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ are the impulse points, $J := [t_0, T]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{J}^{q_i}_{t_i}$ is the Hadamard fractional integral of order $q_i > 0$, $i = 0, 1, \ldots, m$. The jump conditions are defined by $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$, $k = 1, 2, 3, \ldots, m$.

The paper is organized as follows: Section 2 contains some preliminary notations, definitions and lemmas that we need in the sequel. In Section 3 we present the main results for the problem (1.3), where existence and uniqueness results are proved by using Banach and Rothe fixed point theorems, Leray-Schauder alternative and degree theory. Examples illustrating the obtained results are also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of Hadamard fractional calculus (see [1]) and present preliminary results needed in our proofs later.

Definition 2.1 For an at least n-times differentiable function $g : [a,b] \to \mathbb{R}$, $a,b > 0$, the Caputo type Hadamard derivative of fractional order α is defined as

$$ {}^{C}\mathcal{D}_{a}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \delta^n g(s)\frac{ds}{s}, \quad n-1 < \alpha < n, n = [\alpha]+1, $$

where $\delta = t\frac{d}{dt}$, $t \in [a,b]$, and $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 The Hadamard fractional integral of order α is defined as

$$ \mathcal{J}_a^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}, \quad \alpha > 0, $$

provided the integral exists on $[a,b]$.

Lemma 2.1 [23] *Let $x \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\alpha \in \mathbb{C}$, where $X_\delta^n[a,b] = \{g : [a,b] \to \mathbb{C} : \delta^{n-1}g(t) \in X[a,b]\}$. Then we have*

$$ \mathcal{J}_a^{\alpha}\left({}^{C}\mathcal{D}_a^{\alpha}\right)x(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \log\left(\frac{t}{a}\right)^k. $$

The key tools for proving of our results are based on the following fixed point theorems.

Theorem 2.1 [25] *Suppose that $A : \bar{\Omega} \to E$ is a completely continuous operator. If one of the following conditions is satisfied*:

(i) (*Altman*) $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$, *for all* $x \in \partial\Omega$,

(ii) (*Rothe*) $\|Ax\| \leq \|x\|$, *for all* $x \in \partial\Omega$,

(iii) (*Petryshyn*) $\|Ax\| \leq \|Ax - x\|$, *for all* $x \in \partial\Omega$,

then $\deg(I - A, \Omega, \theta) = 1$, *and hence A has at least one fixed point in* Ω.

Theorem 2.2 [25] *Suppose that $A : \bar{\Omega} \to E$ is completely continuous operator. If*

$$ Ax \neq \lambda x, \quad \forall x \in \partial\Omega, \lambda \geq 1, $$

then $\deg(I - A, \Omega, \theta) = 1$ *and A has at least one fixed point in* $\bar{\Omega}$.

Theorem 2.3 [26] *Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set*

$$V = \{u \in E : u = \lambda Tu, 0 < \lambda < 1\}$$

is bounded. Then T has a fixed point in E.

Lemma 2.2 *Assume that $\Phi = \alpha + \beta - \sum_{i=1}^{m} \frac{\gamma_i(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \neq 0$. Then the solution of the problem* (1.3) *is equivalent to the following integral equation*:

$$\begin{aligned} x(t) &= \mathcal{J}_{t_k}^{p_k} f\big(t, x(t)\big) + \sum_{i=0}^{k-1} \big(\mathcal{J}_{t_i}^{p_i} f\big(t_{i+1}, x(t_{i+1})\big) + \varphi_{i+1}\big(x(t_{i+1})\big)\big) \\ &\quad + \frac{1}{\Phi}\Bigg[\sum_{i=0}^{m} \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f\big(t_{i+1}, x(t_{i+1})\big) - \beta \mathcal{J}_{t_m}^{p_m} f\big(T, x(T)\big) \\ &\quad - \beta \sum_{i=0}^{m-1} \big(\mathcal{J}_{t_i}^{p_i} f\big(t_{i+1}, x(t_{i+1})\big) + \varphi_{i+1}\big(x(t_{i+1})\big)\big) \\ &\quad + \sum_{i=1}^{m} \bigg(\frac{\gamma_i(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\bigg)\bigg(\sum_{j=0}^{i-1} \big(\mathcal{J}_{t_j}^{p_j} f\big(t_{j+1}, x(t_{j+1})\big) + \varphi_{j+1}\big(x(t_{j+1})\big)\big)\bigg)\Bigg]. \end{aligned} \tag{2.1}$$

Proof By Lemma 2.1, the solution of (1.3) on interval J_0 can be written as

$$x(t) = \mathcal{J}_{t_0}^{p_0} f\big(t, x(t)\big) + x_0,$$

where $x_0 \in \mathbb{R}$. For $t \in J_1$, by using Lemma 2.1 and the impulse condition $\Delta x(t_1) = \varphi_1(x(t_1))$, we obtain

$$\begin{aligned} x(t) &= \mathcal{J}_{t_1}^{p_1} f\big(t, x(t)\big) + x\big(t_1^+\big) \\ &= \mathcal{J}_{t_1}^{p_1} f\big(t, x(t)\big) + \mathcal{J}_{t_0}^{p_0} f\big(t_1, x(t_1)\big) + \varphi_1\big(x(t_1)\big) + x_0. \end{aligned}$$

Again, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= \mathcal{J}_{t_2}^{p_2} f\big(t, x(t)\big) + x\big(t_2^+\big) \\ &= \mathcal{J}_{t_2}^{p_2} f\big(t, x(t)\big) + \mathcal{J}_{t_1}^{p_1} f\big(t_2, x(t_2)\big) + \varphi_2\big(x(t_2)\big) + \mathcal{J}_{t_0}^{p_0} f\big(t_1, x(t_1)\big) + \varphi_1\big(x(t_1)\big) + x_0. \end{aligned}$$

Repeating the above process, for $t \in J$, we obtain

$$x(t) = \mathcal{J}_{t_k}^{p_k} f\big(t, x(t)\big) + \sum_{i=0}^{k-1} \big(\mathcal{J}_{t_i}^{p_i} f\big(t_{i+1}, x(t_{i+1})\big) + \varphi_{i+1}\big(x(t_{i+1})\big)\big) + x_0. \tag{2.2}$$

Applying the boundary condition of (1.3), it follows that

$$\begin{aligned} \alpha x(t_0) + \beta x(T) &= (\alpha + \beta)x_0 + \beta \mathcal{J}_{t_m}^{p_m} f\big(T, x(T)\big) \\ &\quad + \beta \sum_{i=0}^{m-1} \big(\mathcal{J}_{t_i}^{p_i} f\big(t_{i+1}, x(t_{i+1})\big) + \varphi_{i+1}\big(x(t_{i+1})\big)\big) \end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=0}^{m} \gamma_i \mathcal{J}_{t_i}^{q_i} x(t_{i+1}) \\
&\quad = \sum_{i=0}^{m} \gamma_i \mathcal{J}_{t_i}^{p_i+q_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) + x_0 \sum_{i=0}^{m} \frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \\
&\qquad + \sum_{i=1}^{m} \biggl(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \biggr) \Biggl(\sum_{j=0}^{i-1} \bigl(\mathcal{J}_{t_j}^{j-1} f\bigl(t_{j+1}, x(t_{j+1})\bigr) + \varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr) \Biggr),
\end{aligned}$$

which leads to

$$\begin{aligned}
x_0 = \frac{1}{\Phi} \Biggl[&\sum_{i=0}^{m} \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) - \beta \mathcal{J}_{t_m}^{p_m} f\bigl(T, x(T)\bigr) \\
&- \beta \sum_{i=0}^{m-1} \bigl(\mathcal{J}_{t_i}^{p_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) + \varphi_{i+1}\bigl(x(t_{i+1})\bigr)\bigr) \\
&+ \sum_{i=1}^{m} \biggl(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \biggr) \Biggl(\sum_{j=0}^{i-1} \bigl(\mathcal{J}_{t_j}^{j-1} f\bigl(t_{j+1}, x(t_{j+1})\bigr) + \varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr) \Biggr) \Biggr].
\end{aligned}$$

Replacing the constant x_0 into (2.2), we obtain (2.1), as desired. □

3 Main results

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k), k = 1, 2, \ldots, m\}$. Obviously, $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\| = \sup\{|x(t)|; t \in J\}$. A function $x \in PC$ is called a solution of the problem (1.3) if it satisfies (1.3).

In this section, we investigate the existence and uniqueness of solutions for the problem (1.3) via a variety of fixed point theorems by defining an operator $\mathcal{K} : PC \to PC$ as

$$\begin{aligned}
\mathcal{K}x(t) = {} & \mathcal{J}_{t_k}^{p_k} f\bigl(t, x(t)\bigr) + \sum_{i=0}^{k-1} \bigl(\mathcal{J}_{t_i}^{p_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) + \varphi_{i+1}\bigl(x(t_{i+1})\bigr)\bigr) \\
& + \frac{1}{\Phi} \Biggl[\sum_{i=0}^{m} \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) - \beta \mathcal{J}_{t_m}^{p_m} f\bigl(T, x(T)\bigr) \\
& - \beta \sum_{i=0}^{m-1} \bigl(\mathcal{J}_{t_i}^{p_i} f\bigl(t_{i+1}, x(t_{i+1})\bigr) + \varphi_{i+1}\bigl(x(t_{i+1})\bigr)\bigr) \\
& + \sum_{i=1}^{m} \biggl(\frac{\gamma_i (\log(t_{i+1}/t_i)^{q_i})}{\Gamma(q_i+1)} \biggr) \Biggl(\sum_{j=0}^{i-1} \bigl(\mathcal{J}_{t_j}^{p_j} f\bigl(t_{j+1}, x(t_{j+1})\bigr) + \varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr) \Biggr) \Biggr].
\end{aligned}$$

Clearly, the boundary value problem (1.3) becomes a fixed point problem $x = \mathcal{K}x$.

For convenience, we set the notations of constants, thus

$$\Lambda_1 = \frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)} + \sum_{i=0}^{m-1} \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} + \frac{1}{|\Phi|}\Biggl\{\sum_{i=0}^{m} \frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i+p_i}}{\Gamma(q_i+p_i+1)} + |\beta|\sum_{i=0}^{m} \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} + |\beta|\frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)} + \sum_{i=1}^{m}\sum_{j=0}^{i-1}\biggl(\frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j+1)}\biggr)\biggl(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\biggr)\Biggr\},$$

$$\Lambda_2 = k + \frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)} + \frac{1}{\Phi}\Biggl[|\beta|m + \sum_{i=0}^{m} \frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} + |\beta|\frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)}\Biggr].$$

Theorem 3.1 *Assume that $f : J\times\mathbb{R}\to\mathbb{R}$ and $\varphi_k : \mathbb{R}\to\mathbb{R}, k = 1,2,\dots,m$, are continuous functions which satisfy the following conditions*:

(H_1) $|f(t,x) - f(t,y)| \le L_1|x-y|, \forall t\in J, L_1 > 0, x,y\in\mathbb{R}$;
(H_2) $|\varphi_k(u) - \varphi_k(v)| \le L_2|u-v|$, $L_2 > 0$, *for all* $u,v\in\mathbb{R}, \forall k = 1,2,\dots,m$.

If $L_1\Lambda_1 + L_2\Lambda_2 < 1$ *then the problem* (1.3) *has a unique solution on* J.

Proof We define a closed ball B_r by $B_r = \{x\in PC; \|x\|\le r\}$ where $r \ge (M_1\Lambda_1 + M_2\Lambda_2)(1 - L_1\Lambda_1 - L_2\Lambda_2)^{-1}$, where $M_1 = \sup_{t\in J}|f(t,0)|$ and $M_2 = \max\{|\varphi_i(0)|, i = 1,2,\dots,m\}$.

We will show that $\mathcal{K} : B_r \to B_r$. For any $x\in B_r$, we have

$$\begin{aligned}
\bigl|\mathcal{K}x(t)\bigr| &\le \mathcal{J}_{t_k}^{p_k}\bigl|f\bigl(t,x(t)\bigr)\bigr| + \sum_{i=0}^{k-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr)\bigr| + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr)\bigr|\bigr) \\
&\quad + \frac{1}{|\Phi|}\Biggl[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr)\bigr| + |\beta|\mathcal{J}_{t_m}^{p_m}\bigl|f\bigl(T,x(T)\bigr)\bigr| \\
&\quad + |\beta|\sum_{i=0}^{m-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr)\bigr| + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr)\bigr|\bigr) \\
&\quad + \sum_{i=1}^{m}\biggl(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\biggr)\Biggl(\sum_{j=0}^{i-1}\bigl(\mathcal{J}_{t_j}^{p_j}\bigl|f\bigl(t_{j+1},x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr|\bigr)\Biggr)\Biggr] \\
&\le \mathcal{J}_{t_k}^{p_k}\bigl(\bigl|f\bigl(t,x(t)\bigr) - f(t,0)\bigr| + \bigl|f(t,0)\bigr|\bigr) \\
&\quad + \sum_{i=0}^{k-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl(\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f(t_{i+1},0)\bigr| + \bigl|f(t_{i+1},0)\bigr|\bigr) \\
&\quad + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr) - \varphi_{i+1}(0)\bigr| + \bigl|\varphi_{i+1}(0)\bigr|\bigr) \\
&\quad + \frac{1}{|\Phi|}\Biggl[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\bigl(\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f(t_{i+1},0)\bigr| + \bigl|f(t_{i+1},0)\bigr|\bigr) \\
&\quad + |\beta|\mathcal{J}_{t_m}^{p_m}\bigl(\bigl|f\bigl(T,x(T)\bigr) - f(T,0)\bigr| + \bigl|f(T,0)\bigr|\bigr) \\
&\quad + |\beta|\sum_{i=0}^{m-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl(\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f(t_{i+1},0)\bigr| + \bigl|f(t_{i+1},0)\bigr|\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr) - \varphi_{i+1}(0)\bigr| + \bigl|\varphi_{i+1}(0)\bigr|\bigr) \\
&\quad + \sum_{i=1}^{m} \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\bigl(\mathcal{J}_{t_j}^{p_j}\bigl(\bigl|f\bigl(t_{j+1},x(t_{j+1})\bigr) - f(t_{j+1},0)\bigr| \right. \\
&\quad \left. + \bigl|f(t_{j+1},0)\bigr|\bigr) + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr) - \varphi_{j+1}(0)\bigr| + \bigl|\varphi_{j+1}(0)\bigr|\bigr)\right)\Bigg] \\
&\leq (L_1 r + M)\frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)} + \sum_{i=0}^{k-1}\left\{(L_1 r + M_1)\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} + (L_2 r + M_2)\right\} \\
&\quad + \frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|(L_1 r + M_1)\frac{(\log(t_{i+1}/t_i))^{q_i+p_i}}{\Gamma(q_i+p_i+1)} + |\beta|(L_1 r + M_1)\frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)}\right. \\
&\quad + |\beta|\sum_{i=0}^{m-1}\left\{(L_1 r + M_1)\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} + (L_2 r + M_2)\right\} \\
&\quad \left. + \sum_{i=1}^{m}\left\{\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}(L_1 r + M_1)\frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j+1)} + (L_2 r + M_2)\right)\right\}\right] \\
&\leq (L_1\Lambda_1 + L_2\Lambda_2)r + (M_1\Lambda_1 + M_2\Lambda_2) \leq r.
\end{aligned}
$$

Then $\mathcal{K}B_r \subseteq B_r$. Next we will show that $\mathcal{K}$ is a contraction mapping. For $x, y \in B_r$, we get

$$
\begin{aligned}
&|\mathcal{K}x - \mathcal{K}y| \\
&\quad \leq \mathcal{J}_{t_k}^{p_k}\bigl|f\bigl(t,x(t)\bigr) - f\bigl(t,y(t)\bigr)\bigr| + \sum_{i=0}^{k-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f\bigl(t_{i+1},y(x_{i+1})\bigr)\bigr| \\
&\qquad + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr) - \varphi_{i+1}\bigl(y(t_{i+1})\bigr)\bigr|\bigr) \\
&\qquad + \frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f\bigl(t_{i+1},y(t_{i+1})\bigr)\bigr|\right. \\
&\qquad + |\beta|\mathcal{J}_{t_m}^{p_m}\bigl|f\bigl(T,x(T)\bigr) - f\bigl(T,y(T)\bigr)\bigr| \\
&\qquad + |\beta|\sum_{i=0}^{m-1}\bigl(\mathcal{J}_{t_i}^{p_i}\bigl|f\bigl(t_{i+1},x(t_{i+1})\bigr) - f\bigl(t_{i+1},y(t_{i+1})\bigr)\bigr| + \bigl|\varphi_{i+1}\bigl(x(t_{i+1})\bigr) - \varphi_{i+1}\bigl(y(t_{i+1})\bigr)\bigr|\bigr) \\
&\qquad + \sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\bigl(\mathcal{J}_{t_j}^{p_j}\bigl|f\bigl(t_{j+1},x(t_{j+1})\bigr) - f\bigl(t_{j+1},y(t_{j+1})\bigr)\bigr|\right. \\
&\qquad \left.\left. + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr) - \varphi_{j+1}\bigl(y(t_{j+1})\bigr)\bigr|\bigr)\right)\right] \\
&\quad \leq (L_1\Lambda_1 + L_2\Lambda_2)\|x - y\|.
\end{aligned}
$$

Since $(L_1\Lambda_1 + L_2\Lambda_2) < 1$, the operator $\mathcal{K}$ is contractive. Hence $\mathcal{K}$ has a unique fixed point on B_r. Therefore the problem (1.3) has a unique solution on J. □

Theorem 3.2 *Let f and φ_k, $k = 1, 2, \ldots, m$, be continuous functions. Assume that there are two positive real numbers N_1 and N_2 such that*:

(H_3) $|f(t,x)| \leq N_1$ *and* $|\varphi_k(x)| \leq N_2$, *for* $t \in J$, $x \in \mathbb{R}$ *and* $k = 1, 2, \ldots, m$.

Then the problem (1.3) *has at least one solution on* J.

Proof Define a ball $B_\omega = \{x \in PC; \|x\| < \omega\}$. The proof is divided into 3 steps.

Step 1. We will show that $\mathcal{K}$ is continuous. To prove this, we let $\{x_n\}$ be a sequence in PC such that $x_n \to x$ as $n \to \infty$. Then we have

$$\begin{aligned}
&\left|\mathcal{K}x_n(t) - \mathcal{K}x(t)\right| \\
&\quad \leq \mathcal{J}_{t_k}^{p_k}\left|f\left(t, x_n(t)\right) - f\left(t, x(t)\right)\right| \\
&\qquad + \sum_{i=0}^{k-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\left(t_{i+1}, x_n(t_{i+1})\right) - f\left(t_{i+1}, x(t_{i+1})\right)\right| + \left|\varphi_{i+1}\left(x_n(t_{i+1})\right) - \varphi_{i+1}\left(x(t_{i+1})\right)\right|\right) \\
&\qquad + \frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\left|f\left(t_{i+1}, x_n(t_{i+1})\right) - f\left(t_{i+1}, x(t_{i+1})\right)\right|\right. \\
&\qquad + |\beta|\mathcal{J}_{t_m}^{p_m}\left|f\left(T, x_n(T)\right) - f\left(T, x(T)\right)\right| \\
&\qquad + |\beta|\sum_{i=0}^{m-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\left(t_{i+1}, x_n(t_{i+1})\right) - f\left(t_{i+1}, x(t_{i+1})\right)\right|\right. \\
&\qquad + \left.\left|\varphi_{i+1}\left(x_n(t_{i+1})\right) - \varphi_{i+1}\left(x(t_{i+1})\right)\right|\right) \\
&\qquad + \sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathcal{J}_{t_j}^{p_j}\left|f\left(t_{j+1}, x_n(t_{j+1})\right) - f\left(t_{j+1}, x(t_{j+1})\right)\right|\right.\right. \\
&\qquad + \left.\left.\left.\left|\varphi_{j+1}\left(x_n(t_{j+1})\right) - \varphi_{j+1}\left(x(t_{j+1})\right)\right|\right)\right)\right].
\end{aligned}$$

Using the continuity of f and φ_k for $k = 1, 2, \ldots, m$, we have $|f(t,x_n) - f(t,x)|$ and $|\varphi_k(x_n) - \varphi_k(x)|$ vanish as $n \to \infty$. Therefore $\|\mathcal{K}x_n - \mathcal{K}x\| \to 0$, which yields the continuity of the operator $\mathcal{K}$.

Step 2. $\mathcal{K}$ maps a bounded set into a bounded set. For each $x \in \bar{B}_\omega$, we have

$$\begin{aligned}
|\mathcal{K}x| &\leq \mathcal{J}_{t_k}^{p_k}\left|f\left(t, x(t)\right)\right| + \sum_{i=0}^{k-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\left(t_{i+1}, x(t_{i+1})\right)\right| + \left|\varphi_{i+1}\left(x(t_{i+1})\right)\right|\right) \\
&\quad + \frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\left|f\left(t_{i+1}, x(t_{i+1})\right)\right| + |\beta|\mathcal{J}_{t_m}^{p_m}\left|f\left(T, x(T)\right)\right|\right. \\
&\quad + |\beta|\sum_{i=0}^{m-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\left(t_{i+1}, x(t_{i+1})\right)\right| + \left|\varphi_{i+1}\left(x(t_{i+1})\right)\right|\right) \\
&\quad + \left.\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathcal{J}_{t_j}^{p_j}\left|f\left(t_{j+1}, x(t_{j+1})\right)\right| + \left|\varphi_{j+1}\left(x(t_{j+1})\right)\right|\right)\right)\right] \\
&\leq \Lambda_1 N_1 + \Lambda_2 N_2,
\end{aligned}$$

which yields the boundedness of $\mathcal{K}\bar{B}_\omega$.

Step 3. $\mathcal{K}$ maps a bounded set into an equicontinuous set. Let $\tau_1, \tau_2 \in (t_k, t_{k+1})$, for each $k = 0, 1, 2, \dots, m$, we have

$$\left|\mathcal{K}x(\tau_1) - \mathcal{K}x(\tau_2)\right| \le \mathcal{J}_{t_k}^{p_k}\left|f\big(\tau_1, x(\tau_1)\big) - f\big(\tau_2, x(\tau_2)\big)\right|.$$

The continuity of x and f implies that $\mathcal{K}x(\tau_1) \to \mathcal{K}x(\tau_2)$ as $\tau_1 \to \tau_2$. Consequently $\mathcal{K}$ is completely continuous by applying the Azelá-Ascoli theorem.

Let $V = \{x \in B_\omega; \mu\mathcal{K}x = x \text{ for } \mu \in (0,1)\}$. For all $x \in V$, $x = \mu\mathcal{K}x$, we have

$$|x| \le \mu|\mathcal{K}x| \le \Lambda_1 N_1 + \Lambda_2 N_2.$$

Hence V is bounded. By Theorem 2.3, the problem (1.3) has at least one solution on J. □

Theorem 3.3 *Assume that*

(H$_4$) $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$ *and* $\lim_{x\to 0}\frac{\varphi_k(x)}{x} = 0$ *for* $k = 1, 2, \dots, m$.

Then the problem (1.3) *has at least one solution on* J.

Proof From (H$_4$), choosing $\epsilon = 1/(\Lambda_1 + \Lambda_2)$, there exist constants $\delta_1, \delta_2 \in \mathbb{R}^+$ such that

$$\left|f(t,x)\right| < \epsilon|x| \quad \text{where } |x| < \delta_1 \quad \text{and} \quad \left|\varphi(x)\right| < \epsilon|x| \quad \text{where } |x| < \delta_2.$$

Now, we define an open ball $\Omega = \{u \in PC; \|u\| < \min\{\delta_1, \delta_2\}\}$. By Theorem 3.2, the operator $\mathcal{K} : \bar{\Omega} \to PC$ is completely continuous. For any $x \in \partial\Omega$, we have

$$\begin{aligned}
|\mathcal{K}x| \le{}& \mathcal{J}_{t_k}^{p_k}\left|f\big(t, x(t)\big)\right| + \sum_{i=0}^{k-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\big(t_{i+1}, x(t_{i+1})\big)\right| + \left|\varphi_{i+1}\big(x(t_{i+1})\big)\right|\right) \\
&+ \frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|\mathcal{J}_{t_i}^{q_i+p_i}\left|f\big(t_{i+1}, x(t_{i+1})\big)\right| + |\beta|\mathcal{J}_{t_m}^{p_m}\left|f\big(T, x(T)\big)\right|\right. \\
&+ |\beta|\sum_{i=0}^{m-1}\left(\mathcal{J}_{t_i}^{p_i}\left|f\big(t_{i+1}, x(t_{i+1})\big)\right| + \left|\varphi_{i+1}\big(x(t_{i+1})\big)\right|\right) \\
&\left.+ \sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathcal{J}_{t_j}^{p_j}\left|f\big(t_{j+1}, x(t_{j+1})\big)\right| + \left|\varphi_{j+1}\big(x(t_{j+1})\big)\right|\right)\right)\right] \\
\le{}& (\Lambda_1\epsilon + \Lambda_2\epsilon)\|x\| = \|x\|.
\end{aligned}$$

It follows from Theorem 2.1, case (ii), that the problem (1.3) has at least one solution on J. □

Theorem 3.4 *Let* f *and* φ_k *for* $k = 1, 2, \dots, m$, *be continuous functions and satisfy the following inequalities*:

(H$_5$) $|f(t,x)| \le a|x| + b$, $\forall(t,x) \in J \times \mathbb{R}$ *and* $|\varphi_k(x)| \le c|x| + d$, $\forall x \in \mathbb{R}$, $k = 1, \dots, m$, *where constants* $a, c > 0$ *and* $b, d \ge 0$.

Then the problem (1.3) *has at least one solution on* J.

Proof Define a unit ball as $\mathcal{O} = \{x \in PC; \|x\| < 1\}$. It is straightforward to show that the operator $\mathcal{K} : \bar{\mathcal{O}} \to PC$ is completely continuous. Suppose that there is $x^* \in \partial\mathcal{O}$. Then we choose $\lambda = (a + c)\Lambda_1 + (b + d)\Lambda_2 + 1$ such that $\mathcal{K}x^* = \lambda x^*$. By taking the norm in both sides of $\|\mathcal{K}x^*\| = \|\lambda x^*\|$, we obtain $\|\mathcal{K}\|\|x^*\| \geq \lambda\|x^*\|$. Then we have

$$\begin{aligned}
\|\mathcal{K}\| &= \sup_{\|x\|=1} |\mathcal{K}x| \\
&= \sup_{\|x\|=1} \Bigg\{ \mathcal{J}_{t_k}^{p_k} \big|f\big(t, x(t)\big)\big| + \sum_{i=0}^{k-1} \big(\mathcal{J}_{t_i}^{p_i} \big|f\big(t_{i+1}, x(t_{i+1})\big)\big| + \big|\varphi_{i+1}\big(x(t_{i+1})\big)\big| \big) \\
&\quad + \frac{1}{|\Phi|} \Bigg[\sum_{i=0}^{m} |\gamma_i| \mathcal{J}_{t_i}^{q_i+p_i} \big|f\big(t_{i+1}, x(t_{i+1})\big)\big| + |\beta| \mathcal{J}_{t_m}^{p_m} \big|f\big(T, x(T)\big)\big| \\
&\quad + |\beta| \sum_{i=0}^{m-1} \big(\mathcal{J}_{t_i}^{p_i} \big|f\big(t_{i+1}, x(t_{i+1})\big)\big| + \big|\varphi_{i+1}\big(x(t_{i+1})\big)\big| \big) \\
&\quad + \sum_{i=1}^{m} \bigg(\frac{|\gamma_i| (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \bigg) \bigg(\sum_{j=0}^{i-1} \big(\mathcal{J}_{t_j}^{p_j} \big|f\big(t_{j+1}, x(t_{j+1})\big)\big| + \big|\varphi_{j+1}\big(x(t_{j+1})\big)\big| \big) \bigg) \Bigg] \Bigg\} \\
&\leq (a + c)\Lambda_1 + (b + d)\Lambda_2 = \lambda - 1,
\end{aligned}$$

which contradicts $\|\mathcal{K}\| \geq \lambda$. Hence the assumptions of Theorem 2.2 hold. Therefore the problem (1.3) has at least one solution on J. □

4 Examples

In this section, we present four examples to illustrate our results.

Example 4.1 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases}
{}^C\mathcal{D}_{t_k}^{(\frac{k+1}{k+2})} x(t) = \frac{10-t^2}{8(t^2+24)} \big(\frac{(|x(t)|+2)^2}{|x(t)|+3} \big), & t \in [1, \frac{8e+1}{9}] \setminus \{t_k\}, \\
\Delta x(t_k) = \frac{\sin|x(t_k)|}{5(11-k)}, & t_k = \frac{ke+1}{k+1}, k = 1, 2, \ldots, 7, \\
\frac{3}{2} x(1) + \frac{4}{5} x\big(\frac{8e+1}{9}\big) = \sum_{i=0}^{7} (1 - e^{-i}) \mathcal{J}_{t_i}^{(\frac{i^2+5i+2}{i^2+4i+3})} x(t_{i+1}).
\end{cases} \tag{4.1}$$

Here $\alpha = 3/2$, $\beta = 4/5$, $m = 7$, $p_k = (k+1)/(k+2)$, $\gamma_k = 1 - e^{-k}$, $q_k = (k^2 + 5k + 2)/(k^2 + 4k + 3)$ for $k = 0, 1, \ldots, 7$. From the information, we find that $\Phi \approx 2.0961081$, $\Lambda_1 \approx 3.280445$, and $\Lambda_2 \approx 13.552466$. The functions f and φ_k are given by

$$f(t, x) = \frac{10 - t^2}{8(t^2 + 24)} \bigg(\frac{(|x| + 2)^2}{|x| + 3} \bigg), \qquad \varphi_k(x) = \frac{\sin|x|}{5(11 - k)},$$

which satisfy

$$\big|f(t, x) - f(t, y)\big| \leq \frac{2}{25}|x - y| \quad \text{and} \quad \big|\varphi_k(x) - \varphi_k(y)\big| \leq \frac{1}{20}|x - y|, \quad \forall k = 1, 2, \ldots, 7.$$

Then we get $L_1 = 2/25$ and $L_2 = 1/20$, which implies $L_1\Lambda_1 + L_2\Lambda_2 \approx 0.940059 < 1$. Therefore the problem (4.1) has a unique solution on $[1, (8e + 1)/9]$ due to Theorem 3.1.

Example 4.2 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{\log(\sum_{i=0}^{k+1}(1/(i+1)!))}x(t) = \frac{(2-e^{-t})\log(|x(t)|+1)}{|x(t)|+2} - 2, & t \in [\frac{\pi}{2}, 2\pi] \setminus \{t_k\}, \\ \Delta x(t_k) = e^{-\frac{k}{4}}\cos(kx(t_k)) + e^{\frac{k}{4}}\sin(kx(t_k)), & t_k = 2^{\frac{2k-9}{9}}\pi, k = 1,2,\ldots,8, \\ -e^{-\frac{\pi}{2}}x(\frac{\pi}{2}) + e^{-2\pi}x(2\pi) = \sum_{i=0}^{8}(-2)^i(i^2+1)\mathcal{J}_{t_i}^{(\frac{|5i-4|}{i+1})}x(t_{i+1}). \end{cases} \tag{4.2}$$

Here $\alpha = -e^{-\pi/2}$, $\beta = e^{-2\pi}$, $m = 8$, $p_k = \log(\sum_{i=0}^{k+1}(1/(i+1)!))$, $\gamma_k = (-2)^k(k^2+1)$, $q_k = |5k-4|/(k+1)$ for $k = 0,1,\ldots,8$. We find that $\Phi \approx -1.422922 \neq 0$. The functions $f(t,x) = (2-e^{-t})\log(|x|+1)/(|x|+2) - 2$ and $\varphi_k(x) = e^{-\frac{k}{4}}\cos(kx) + e^{\frac{k}{4}}\sin(kx)$ are bounded as

$$|f(t,x)| \le 4 \quad \text{and} \quad |\varphi_k(x)| \le \sqrt{e^{-4}+e^4}.$$

Hence the assumption (H_3) of Theorem 3.2 holds. Therefore the problem (4.2) has at least one solution on $[\pi/2, 2\pi]$.

Example 4.3 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{(\frac{2k+2}{k^2+2k+2})}x(t) = \frac{e^{tx(t)}(\sin x(t) - x(t))}{2t+1}, & t \in [\frac{4}{3}, 3] \setminus \{t_k\}, \\ \Delta x(t_k) = \frac{kx^3(t_k)}{\log(|x(t_k)|+2)}, & t_k = \frac{k+8}{6}, k = 1,2,\ldots,9, \\ \sqrt{3}x(\frac{4}{3}) + \frac{3}{5}x(3) = \sum_{i=0}^{9}(\frac{i^2+1}{i^2+2})\mathcal{J}_{t_i}^{\arctan i}x(t_{i+1}). \end{cases} \tag{4.3}$$

Here $\alpha = \sqrt{3}$, $\beta = 3/5$, $m = 9$, $p_k = 2(k+1)/(k^2+1)$, $\gamma_k = (k^2+1)/(k^2+2)$, $q_k = \arctan(k)$ for $k = 0,1,\ldots,9$. We find that $\Phi \approx 2.003684 \neq 0$. The functions $f(t,x) = e^{tx}(\sin x - x)/(2t+1)$ and $\varphi_k(x) = kx^3/\log(|x|+2)$ satisfy

$$\lim_{x\to 0}\frac{f(t,x)}{x} = \lim_{x\to 0}\frac{e^{xt}}{2t+1}\left(\frac{\sin x}{x} - 1\right) = 0$$

and

$$\lim_{x\to 0}\frac{\varphi_k(x)}{x} = \lim_{x\to 0}\frac{kx^2}{\log(|x|+2)} = 0, \quad \forall k = 1,2,\ldots,9.$$

Thus the condition (H_4) of Theorem 3.3 holds. Therefore, we conclude that the problem (4.3) has at least one solution on $[4/3, 3]$.

Example 4.4 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{\sqrt{1-\sin^2(k+1)}}x(t) = e^{\frac{2t}{3}}\sin x(t) + tx(t)\cos x(t) + 2, & t \in [\frac{3}{2}, 3] \setminus \{t_k\}, \\ \Delta x(t_k) = kx(t_k) - \log(|x(t_k)| + \frac{3}{5}), & t_k = 3 \cdot 2^{\frac{k-11}{11}}, k = 1,2,\ldots,10, \\ \frac{4}{3}x(\frac{3}{2}) - \frac{3}{4}x(3) = \sum_{i=0}^{10}\frac{(-1)^i}{i+1}\mathcal{J}_{t_i}^{(\frac{3k+2}{2k+3})}x(t_{i+1}). \end{cases} \tag{4.4}$$

Here $\alpha = 4/3$, $\beta = -3/4$, $m = 10$, $p_k = \sqrt{1 - \sin^2(k+1)}$, $\gamma_k = (-1)^k/(k+1)$, $q_k = (3k + 2)/(2k+3)$, for $k = 0, 1, \ldots, 10$. We find that $\Phi \approx 0.605503 \neq 0$. The functions $f(t,x) = e^{\frac{2t}{3}} \sin x + tx\cos x + 2$ and $\varphi(x) = kx - \log(|x| + (3/5))$ satisfy the inequalities

$$\left|f(t,x)\right| \leq t|x| + \left(2 + e^{\frac{2t}{3}}\right) \leq 3|x| + \left(2 + e^2\right)$$

and

$$\left|\varphi_k(x)\right| \leq |x|(k+1) + \frac{3}{5} \leq 11|x| + \frac{3}{5}.$$

Therefore (H_5) holds. According to Theorem 3.4, the problem (4.4) has at least one solution on $[3/2, 3]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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