Zhao and Shi *Boundary Value Problems* (2015) 2015:162
DOI 10.1186/s13661-015-0422-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Structures on self-adjoint vertex conditions of local Sturm-Liouville operators on graphs

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Abstract

We study the vertex conditions of local Sturm-Liouville operators on metric graphs. Our aim is to give a new description of vertex conditions defining the self-adjoint Sturm-Liouville operators and to clarify the natural geometric structure on the space of complex vertex conditions. Based on this description, we give the self-adjointness results for local Sturm-Liouville operators on finite graphs and the Povzner-Wienholtz-type self-adjointness results for local Sturm-Liouville operators on infinite graphs.

Keywords: directed graph; local operators; vertex conditions

1 Introduction

A graph Γ we consider in this paper is an ordered pair of disjoint sets (V, E), where V is a countable vertex set and E is a countable edge set. Moreover, the graph is connected and each edge e_n has a positive length $|e_n|$.

A Sturm-Liouville operator on the graph Γ is actually a system of Sturm-Liouville operators on intervals complemented by appropriate matching conditions at inner vertices and some boundary conditions at the boundary vertices. The matching conditions and the boundary conditions are collectively called vertex conditions. The Sturm-Liouville operators on $L^2_w(\Gamma, \mathbb{C})$ are generated by the expression

$$Lf(x) := \frac{1}{w(x)}\big(-\big(p(x)f'(x)\big)' + q(x)f(x)\big), \quad x \in \Gamma, \tag{1.1}$$

where $1/p, q, w \in L^1_{\mathrm{loc}}(\Gamma, \mathbb{R})$ and $w > 0$ a.e. on Γ.

If V and E are finite sets, then the description of the self-adjoint vertex conditions can be treated as the description of the boundary conditions in self-adjoint multi-interval Sturm-Liouville problems. (The results about multi-interval Sturm-Liouville problems can be found in [1].) For example, in [2], Harmer described the self-adjoint boundary conditions for the Schrödinger operators on the finite graphs in terms of a unitary matrix. When the graph has infinitely many vertices, the general treatments for Sturm-Liouville operators on intervals and Sturm-Liouville operators on finite graphs are deficient.

For regular Sturm-Liouville operators defined on (a, b), in [3], Kong *et al.* clarified the natural geometric structure on the space of complex boundary conditions, which provides the basis for studying the dependence of Sturm-Liouville eigenvalues on boundary conditions. But for Sturm-Liouville operators on graphs, the geometric structure on the space of complex vertex conditions is not clear. Carlson provided a description of the domains of local essential self-adjoint differential operators on weighted directed graphs with the coefficients of the operators smooth enough [4]. And there are several other descriptions of the vertex conditions that one can add to the Schrödinger expression in order to create a self-adjoint operator on graph Γ (see, *e.g.*, [5–7] for details). Based on the methods in [4] and [3], we give a new description of all vertex conditions defining the domains of local (essentially) self-adjoint Sturm-Liouville operators on weighted directed graphs. The new description allows us to clarify a natural geometric structure on the space of vertex conditions.

The paper is organized as follows. Some results of the differential operators on graphs are introduced in Section 2. In Section 3 we give some properties of self-adjoint vertex conditions for local Sturm-Liouville operators on the graph Γ. Based on these properties we get the necessary conditions for local Sturm-Liouville operators to be self-adjoint. Moreover, we prove that each self-adjoint complex vertex condition at vertex v has a normalized form and all the normalized forms are contained in a finite set with cardinal number $2^{\delta(v)}$, where $\delta(v)$ denotes the degree of the vertex v. In the fourth section we give the sufficient conditions for local Sturm-Liouville operators to be self-adjoint, which are Povzner-Wienholtz-type self-adjointness results for Sturm-Liouville operators on graph. In the final section, we give an example to show how to obtain the proper self-adjoint restriction of a given Sturm-Liouville operator by using the results obtained in Section 3.

2 Notation and prerequisite results

Assume that the graph Γ is connected and each vertex in Γ appears in only finitely many edges. A metric graph Γ may be constructed from the graph data as follows. For each directed edge e_n, let (a_n, b_n) be a real interval of length $|e_n|$ with $a_n < b_n$, then we could see e_n as (a_n, b_n). Define the distance between two points in the graph as the length of the shortest path connecting them. For a function $f \in L^2_w(\Gamma, \mathbb{C})$, f_n denotes the restriction $f \upharpoonright e_n$. Let $L^2_w(\Gamma, \mathbb{C})$ denote the Hilbert space $\oplus_n L^2_{w_n}((a_n, b_n), \mathbb{C})$ with the inner product

$$(f,g)_w = \int_\Gamma f\bar{g}w = \sum_n \int_{a_n}^{b_n} f_n \bar{g}_n w_n \,\mathrm{d}x.$$

In this paper, we restrict our considerations to local Sturm-Liouville operators, so that we could describe the adjoint and self-adjoint extensions of Sturm-Liouville operators on the graph Γ in terms of appropriate conditions on each single vertex of the graph. The definition of local operator is given by Carlson in [4]. Let $\phi : \Gamma \to \mathcal{C}$ denote a C^∞ function which has compact support in Γ and is constant in an open neighborhood of each vertex. An operator $\mathcal{L}$ is a local operator if for every ϕ, ϕf is in the domain of $\mathcal{L}$ whenever f is.

Fix the vertex v, and let $\delta(v)$ be its degree. Identify interval endpoints as α_m if the corresponding edge endpoints are the same vertex v, in which case we will write $\alpha_m \sim v$, $m = 1, \ldots, \delta(v)$. Since $1/p, q, w \in L^1_{\mathrm{loc}}(\Gamma, \mathbb{R})$, for any solution f of $Lf = \lambda f$, f and pf' are locally absolutely continuous in each edge e_n in Γ. Hence, for every vertex v, $f(\alpha_m)$ and $(pf')(\alpha_m)$, $\alpha_m \sim v$, can be defined via appropriate limits. We define the maximal operator $\mathcal{L}_{\max}$ as follows:

$$\begin{aligned}
&\mathcal{L}_{\max} f = Lf = \frac{1}{w}\bigl(-(pf')' + qf\bigr),\\
&\operatorname{Dom}(\mathcal{L}_{\max}) = \bigl\{f \in L^2_w(\Gamma,\mathbb{C}) : f, pf' \in AC_{\mathrm{loc}}\bigl((a_n,b_n),\mathbb{C}\bigr)\\
&\qquad \text{for each } n, n \in \mathbb{N}, Lf \in L^2_w(\Gamma,\mathbb{C})\bigr\}.
\end{aligned}$$

For $f \in \operatorname{Dom}(\mathcal{L}_{\max})$, let $\hat{f}_v \in \mathbb{C}^{2\delta(v)}$ be the vector with the $(2k+j-1)$th component defined by

$$\hat{f}_v^{(2k+j-1)} = f^{[j]}(\alpha_k), \quad j = 0,1, k = 1,\ldots,\delta(v),$$

where $f^{[0]}(x) = f(x)$ and $f^{[1]}(x) = (pf')(x)$. The vector $\hat{f}_v \in \mathbb{C}^{2\delta(v)}$ is called the boundary value of f at v, $\alpha_m \sim v$.

It is easy to verify that the expression L is symmetric. Suppose $f, g \in \operatorname{Dom}(\mathcal{L}_{\max})$, with the support of g in an open ball containing at most one vertex v. Then integration by parts leads to

$$(Lf,g)_w - (f,Lg)_w = [f,g]_v,$$

where $[f,g]_v$ is a nondegenerate form in the boundary values of f and g at v, *i.e.*, there is an invertible $2\delta(v) \times 2\delta(v)$ matrix S_v such that

$$[f,g]_v = \hat{g}_v^* S_v \hat{f}_v.$$

A maximal independent set of vertex conditions at v may be written as $B_v\hat{f}_v = 0$, where B_v is a $K(v) \times 2\delta(v)$ matrix with linearly independent rows.

Denote the local Sturm-Liouville operator $\mathcal{L}$ as follows:

$$\mathcal{L}f = Lf \quad \text{for } f \in \operatorname{Dom}(\mathcal{L}), \tag{2.1}$$

with domain $\operatorname{Dom}(\mathcal{L}) \subset \operatorname{Dom}(\mathcal{L}_{\max})$, and functions in $\operatorname{Dom}(\mathcal{L})$ satisfy the vertex conditions $B_v\hat{f}_v = 0$ at each $v \in V$. By working on one edge e_i, using the classical theory in [8], pp.169-171, and the results given in [4], we can get the following lemmas.

Lemma 1 *Suppose that the operator $\mathcal{L}$ defined by* (2.1) *is self-adjoint and local. The vertex conditions at v annihilating the domain of $\mathcal{L}$ are written as*

$$B_v\hat{f}_v = 0,$$

where B_v is a $K(v) \times 2\delta(v)$ matrix with linearly independent rows. Then each B_v is a $\delta(v) \times 2\delta(v)$ matrix, and

$$B_v\bigl[S_v^*\bigr]^{-1}\bigl(B_v^*\bigr) = 0.$$

Lemma 2 *Suppose that* $\inf_{e_n\in E}|e_n|>0$, $w\equiv 1$, $p\in AC_{\mathrm{loc}}(\Gamma,\mathbb{R})$, $|p|\geq C>0$, p' *and* q *are uniformly bounded. The* $\delta(v)\times 2\delta(v)$ *matrix* B_v *is given with linearly independent rows and satisfies*

$$B_v\big[S_v^*\big]^{-1}\big(B_v^*\big)=0$$

at each $v\in V$. *If the operator* $\mathcal{L}$ *induced by* L *has the domain*

$$\mathrm{Dom}(\mathcal{L})=\big\{f\in\mathrm{Dom}(\mathcal{L}_{\max}):f\textit{ has compact support on }\Gamma,B_v\hat{f}_v=0,v\in V\big\},$$

then $\mathcal{L}$ *is essentially self-adjoint. Conversely, every local self-adjoint operator* $\mathcal{L}_1$ *formally given by* L *is the closure of one operator* $\mathcal{L}$.

3 Spaces of vertex conditions

A single vertex condition at v can be written as $\sum b_{j,k}f^{[j]}(\alpha_k)=0$. A maximal independent set of single vertex conditions at v can be written as $B_v\hat{f}_v=0$, where B_v is a $K(v)\times 2\delta(v)$ matrix with linearly independent rows. In the following, we will call such an independent set of single vertex conditions at v a vertex condition at v. We introduce the notation

$$F_v(x_i)=\begin{pmatrix}f(x_i)\\ f^{[1]}(x_i)\end{pmatrix},\quad x_i\in e_i, v\text{ is an endpoint of }e_i,$$

and write the matrix B_v into the block matrix $(B_1\mid\cdots\mid B_{\delta(v)})$, in which B_i are $K(v)\times 2$ matrices, $i=1,\dots,\delta(v)$. Then the vertex conditions at v, $B_v\hat{f}_v=0$ may be rewritten as

$$\big(B_1\mid\cdots\mid B_{\delta(V)}\big)\begin{pmatrix}F_v(\alpha_1)\\ \dots\\ F_v(\alpha_{\delta(v)})\end{pmatrix}=0,\tag{3.1}$$

and will be denoted by $(B_1\mid\cdots\mid B_{\delta(v)})$. The systems

$$\big(B_1\mid\cdots\mid B_{\delta(v)}\big)\begin{pmatrix}F_v(\alpha_1)\\ \dots\\ F_v(\alpha_{\delta(v)})\end{pmatrix}=0,\qquad\big(A_1\mid\cdots\mid A_{\delta(v)}\big)\begin{pmatrix}F_v(\alpha_1)\\ \dots\\ F_v(\alpha_{\delta(v)})\end{pmatrix}=0$$

represent the same complex vertex conditions at vertex v if and only if there is a matrix $T\in GL(K(v),\mathbb{C})$ such that

$$\big(B_1\mid\cdots\mid B_{\delta(v)}\big)=\big(TA_1\mid\cdots\mid TA_{\delta(v)}\big),$$

where $GL(\delta(v),\mathbb{C})$ is the set of $K(v)\times K(v)$ invertible matrices over $\mathbb{C}$. Denote the space $\mathfrak{B}_v^{\mathbb{C}}$ of complex vertex conditions at vertex v as the quotient space

$$M^*_{\delta(v)\times 2\delta(v)}(\mathbb{C})\big/GL\big(\delta(v),\mathbb{C}\big),$$

where $M_{\delta(v)\times 2\delta(v)}(\mathbb{C})$ stands for the set of $\delta(v)\times 2\delta(v)$ matrices over $\mathbb{C}$, and $M^*_{\delta(v)\times 2\delta(v)}(\mathbb{C})$ stands for the set of matrices in $M_{\delta(v)\times 2\delta(v)}(\mathbb{C})$ with rank $\delta(v)$. We give the space

$M_{\delta(\nu)\times 2\delta(\nu)}(\mathbb{C})$ the usual topology on $\mathbb{C}^{\delta(\nu)\times 2\delta(\nu)}$, then $M^*_{\delta(\nu)\times 2\delta(\nu)}(\mathbb{C})$ is an open subset of $M_{\delta(\nu)\times 2\delta(\nu)}(\mathbb{C})$. In this way, the quotient space $\mathfrak{B}^{\mathbb{C}}_\nu$ inherits the quotient topology.

The complex vertex condition $B_\nu \hat{f}_\nu = 0$ in $\mathfrak{B}^{\mathbb{C}}_\nu$ could be represented by the $\delta(\nu) \times 2\delta(\nu)$ matrix B_ν with rank $\delta(\nu)$. Up to the elementary row transformations, the $\delta(\nu)$-dimensional column vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are $\delta(\nu)$ columns in B_ν. Then the space $\mathfrak{B}^{\mathbb{C}}_\nu$ can be divided into $\binom{2\delta(\nu)}{\delta(\nu)}$ canonical atlas of local coordinate systems on the Grassmann manifold of $\delta^2(\nu)$-dimensional complex subspaces in $\mathbb{C}^{\delta^2(\nu)}$ through the origin [3], where $\binom{2\delta(\nu)}{\delta(\nu)}$ is the binomial coefficient

$$\binom{2\delta(\nu)}{\delta(\nu)} = \frac{(2\delta(\nu))!}{\delta(\nu)!\delta(\nu)!}.$$

Theorem 1 *The space $\mathfrak{B}^{\mathbb{C}}_\nu$ of complex vertex conditions at vertex ν is a connected and compact complex manifold of complex dimension $\delta^2(\nu)$. The space $\mathfrak{B}^{\mathbb{C}}_\nu$ is a connected and real manifold of dimension $2\delta^2(\nu)$ over the number field $\mathbb{R}$.*

Proof The proof is similar to Theorem 3.1 in [3]. □

For each edge e_n, let (a_n, b_n) be the corresponding real interval. Consider the operator $\mathcal{L}_{\max}$ on (a_n, b_n) with $f, g \in \operatorname{Dom}(\mathcal{L}_{\max})$, we have

$$\int_{a_n}^{b_n} (\bar{g}Lf - f\overline{Lg}) = -\bigl[f^{[1]}(b_n)\overline{g(b_n)} - f(b_n)\overline{g^{[1]}(b_n)}\bigr] + \bigl[f^{[1]}(a_n)\overline{g(a_n)} - f(a_n)\overline{g^{[1]}(a_n)}\bigr].$$

Then if $f, g \in \operatorname{Dom}(\mathcal{L}_{\max})$ and g vanishes outside of a small neighborhood of ν, we have

$$\int_\Gamma (\bar{g}Lf - f\overline{Lg}) = \sum_m (-1)^{\sigma_m}\bigl[f^{[1]}(\alpha_m)\overline{g(\alpha_m)} - f(\alpha_m)\overline{g^{[1]}(\alpha_m)}\bigr], \quad \alpha_m \sim \nu, \tag{3.2}$$

with

$$\sigma_m = \begin{cases} 1, & \alpha_m = b_m, \\ 0, & \alpha_m = a_m. \end{cases}$$

For $f, g \in \operatorname{Dom}(\mathcal{L}_{\max})$, define $[f, g]_\nu$ as the right-hand side of equality (3.2).

Assume that $e_k = (a_k, b_k)$, $k = 1, \ldots, \delta(\nu)$, are the edges in which ν is one endpoint, *i.e.*, $\nu = a_k$ or $\nu = b_k$. For $\alpha_m \sim \nu$, assume that $\alpha_1, \ldots, \alpha_s$ are b_k in the corresponding edges $e_k = (a_k, b_k)$ respectively, $k = 1, \ldots, s$, and $\alpha_{s+1}, \ldots, \alpha_{\delta(\nu)}$ are a_k in the corresponding edges $e_k =$

(a_k, b_k) respectively, $k = s+1, \ldots, \delta(\nu)$. Through a direct calculation we can obtain that

$$S_\nu = \begin{pmatrix} \underbrace{\begin{matrix} E_2 & & \\ & \ddots & \\ & & E_2 \end{matrix}}_{2s} & \\ & \underbrace{\begin{matrix} -E_2 & & \\ & \ddots & \\ & & -E_2 \end{matrix}}_{2\delta(\nu)-2s} \end{pmatrix}_{2\delta(\nu)\times 2\delta(\nu)}, \qquad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

Based on Lemma 1, we know that for the vertex condition $(B_1 \mid \cdots \mid B_{\delta(\nu)})\hat{f}_\nu = 0$ of a self-adjoint operator $\mathcal{L}$, the coefficient matrix is a $\delta(\nu) \times 2\delta(\nu)$ matrix and satisfies the condition $B_\nu[S_\nu^*]^{-1}B_\nu^* = 0$. Since S_ν is the matrix given in (3.3), the condition $B_\nu[S_\nu^*]^{-1}B_\nu^* = 0$ is equivalent to

$$\sum_{i=1}^{s} B_i E_2 B_i^* = \sum_{j=s}^{\delta(\nu)} B_j E_2 B_j^*. \tag{3.4}$$

If $B_\nu = (B_1 \mid \cdots \mid B_{\delta(\nu)})$ is a $\delta(\nu) \times 2\delta(\nu)$ matrix with linearly independent rows and satisfies equality (3.4), the vertex condition $B_\nu\hat{f}_\nu = 0$ is called a self-adjoint vertex condition at ν.

Obviously, the elementary row transformation on B_ν does not change the vertex conditions at ν, while the column transformations on B_ν change the vertex conditions. But there is a class of column transformations that would not change the self-adjointness of the vertex conditions.

Lemma 3 *For the local operator $\mathcal{L}$ defined by* (2.1) *and*

$$B_\nu = \bigl(B_1 \mid \cdots \mid B_s \mid B_{s+1} \mid \cdots \mid B_{\delta(\nu)}\bigr),$$

$B_\nu\hat{f}_\nu = 0$ *is a self-adjoint vertex condition at ν if and only if for any matrix A in the set*

$$\begin{aligned} &\bigl\{A : A = \bigl(B_1(E_2)^{i_1} \mid \cdots \mid B_s(E_2)^{i_s} \mid B_{s+1}(E_2)^{i_{s+1}} \mid \cdots \mid B_{\delta(\nu)}(E_2)^{i_{\delta(\nu)}}\bigr), \\ &\quad i_k = 1 \text{ or } i_k = 0, k = 1, \ldots, \delta(\nu)\bigr\}, \end{aligned} \tag{3.5}$$

$A\hat{f}_\nu = 0$ *is a self-adjoint vertex condition at ν.*

Proof Since $E_2^* = -E_2$, $(E_2)^2 = -I_2$, where $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then we have

$$(B_iE_2)E_2(B_iE_2)^* = (B_i)E_2(B_i)^*.$$

Therefore, the matrix operation-multiplication by E_2 does not change the rank of B_i, and for A in the set (3.5), $A\hat{f}_\nu = 0$ is a self-adjoint vertex condition at ν whenever $B_\nu\hat{f}_\nu = 0$ is a self-adjoint vertex condition. □

Definition 1 The column transformations used in Lemma 3 are called the self-adjoint column transformations.

Theorem 2 *For the local operator $\mathcal{L}$ defined by* (2.1), *up to the elementary row transformations and the self-adjoint column transformations, the coefficient matrices of self-adjoint vertex conditions at v are*

$$\begin{pmatrix} 1 & c_{11} & 0 & c_{12} & \cdots & 0 & c_{1\delta(v)} \\ 0 & c_{21} & 1 & c_{22} & \cdots & 0 & c_{2\delta(v)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{(\delta(v)-1)1} & 0 & c_{(\delta(v)-1)2} & \cdots & 0 & c_{(\delta(v)-1)\delta(v)} \\ 0 & c_{\delta(v)1} & 0 & c_{\delta(v)2} & \cdots & 1 & c_{\delta(v)\delta(v)} \end{pmatrix}, \tag{3.6}$$

where the complex matrix $(c_{ij})_{\delta(v)\times\delta(v)}$ consisting of the even numbered columns of matrix (3.6) *has the following properties*:

(1) *for $i = j$, $c_{ij} \in \mathbb{R}$*,

(2) *for $i < j \leq s$ or $s < i < j$, $c_{ij} = \overline{c_{ji}}$*,

(3) *for $i \leq s < j$, $(c_{ij})_{0\leq i\leq s, s<j\leq\delta(v)} = -(c_{ji})^*_{0\leq i\leq s, s<j\leq\delta(v)}$*.

Proof For $(a_{ij}) = (A_1 \mid \cdots \mid A_s \mid A_{s+1} \mid \cdots \mid A_{\delta(v)})$ is a coefficient matrix of self-adjoint vertex conditions at v, there must be one of the elements a_{i1} and a_{i2}, $i = 1, \ldots, \delta(v)$, that is not zero. Through the elementary row transformations and the self-adjoint column transformations, the matrix (a_{ij}) becomes

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1(2\delta(v)-1)} & a_{1(2\delta(v))} \\ 0 & a_{22} & \cdots & a_{2(2\delta(v)-1)} & a_{2(2\delta(v))} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{(\delta(v)-1)2} & \cdots & a_{(\delta(v)-1)(2\delta(v)-1)} & a_{(\delta(v)-1)(2\delta(v))} \\ 0 & a_{\delta(v)2} & \cdots & a_{\delta(v)(2\delta(v)-1)} & a_{\delta(v)(2\delta(v))} \end{pmatrix}. \tag{3.7}$$

For the convenience of writing, the elements changed are still written as a_{ij}, $1 \leq i, j \leq \delta(v)$. Since (a_{ij}) is a matrix with rank $\delta(v)$, up to the elementary row transformations, the $\delta(v) - 1$ column vectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{3.8}$$

are $(\delta(v) - 1)$ columns in (3.7). Then (3.7) could be written as

$$\begin{pmatrix} 1 & a_{12} & 0 & a_{14} & a_{15} & \cdots & a_{1(2\delta(v))} \\ 0 & a_{22} & 1 & a_{24} & a_{25} & \cdots & a_{2(2\delta(v))} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{(\delta(v)-1)2} & 0 & a_{(\delta(v)-1)4} & a_{(\delta(v)-1)5} & \cdots & a_{(\delta(v)-1)(2\delta(v))} \\ 0 & a_{\delta(v)2} & 0 & a_{\delta(v)4} & a_{\delta(v)5} & \cdots & a_{\delta(v)(2\delta(v))} \end{pmatrix};$$

otherwise, matrix (3.7) is in the following two cases. (1) The elements a_{i3} and a_{i4} are all zero for all $1 \le i \le \delta(v)$. (2) One of the elements a_{13} and a_{14} is not zero, the rest of the elements in the third and fourth columns are zero. Then up to the elementary row transformations, the first two vectors in (3.8) appear in one block A_i, $i \ge 3$, of (3.7). In these two cases, (a_{ij}) could not be the coefficient matrix of a self-adjoint vertex condition at v. Then (a_{ij}) could be changed into

$$\begin{pmatrix} 1 & c_{11} & 0 & c_{12} & \cdots & 0 & c_{1\delta(v)} \\ 0 & c_{21} & 1 & c_{22} & \cdots & 0 & c_{2\delta(v)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{(\delta(v)-1)1} & 0 & c_{(\delta(v)-1)2} & \cdots & 0 & c_{(\delta(v)-1)\delta(v)} \\ 0 & c_{\delta(v)1} & 0 & c_{\delta(v)2} & \cdots & 1 & c_{\delta(v)\delta(v)} \end{pmatrix}. \tag{3.9}$$

The matrix (a_{ij}) is the coefficient matrix of a self-adjoint vertex condition at v, (3.9) is the coefficient matrix of the same self-adjoint vertex condition at v, then (3.9) must satisfy (3.4), *i.e.*,

$$\sum_{i=1}^{s} B_i E_2 B_i^* = \sum_{j=s}^{\delta(v)} B_j E_2 B_j^*.$$

After a direct calculation, we get that the matrix $(c_{ij})_{\delta(v)\times\delta(v)}$ has the following properties:

(1) for $i = j$, $c_{ij} = \overline{c_{ji}}$, *i.e.*, $c_{ii} \in \mathbb{R}$;

(2) for $i < j \le s$ or $s < i < j$, $c_{ij} = \overline{c_{ji}}$, *i.e.*,

$$\operatorname{Re} c_{ij} = \operatorname{Re} c_{ji}, \qquad \operatorname{Im} c_{ij} = -\operatorname{Im} c_{ji};$$

(3) for $i \le s < j$, the block matrix $(c_{ij})_{0\le i\le s, s<j\le\delta(v)}$ is equal to the $s \times (\delta(v) - s)$ matrix $-(c_{ji})^*_{0\le i\le s,s<j\le\delta(v)}$. In other words, the matrix $(c_{ij})_{\delta(v)\times\delta(v)}$ has the following form:

$$\begin{pmatrix} \underbrace{\begin{matrix} c_{11} & \overline{c_{21}} & \cdots & \overline{c_{s1}} \\ c_{21} & c_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \overline{c_{s2}} \\ c_{s1} & c_{s2} & \cdots & c_{ss} \end{matrix}}_{s} & -(c_{ji})^*, i \le s < j \\ (c_{ji}), i \le s < j & \underbrace{\begin{matrix} c_{(s+1)(s+1)} & \overline{c_{(s+2)(s+1)}} & \cdots & \overline{c_{\delta(v)(s+1)}} \\ c_{(s+2)(s+1)} & c_{(s+2)(s+2)} & \cdots & \overline{c_{\delta(v)(s+2)}} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\delta(v)(s+1)} & c_{\delta(v)(s+2)} & \cdots & c_{\delta(v)\delta(v)} \end{matrix}}_{\delta(v)-s} \end{pmatrix}.$$

□

Write matrix (3.6) into the block matrix and denote it as $(C_1 \mid \cdots \mid C_{\delta(v)})$, then the set

$$\begin{aligned} S = \bigl\{ A : A = \bigl(C_1(E_2)^{i_1} \mid \cdots \mid C_s(E_2)^{i_s} \mid C_{s+1}(E_2)^{i_{s+1}} \mid \cdots \mid C_{\delta(v)}(E_2)^{i_{\delta(v)}} \bigr), \\ i_k = 1 \text{ or } i_k = 0, k = 1, \ldots, \delta(v) \bigr\} \end{aligned} \tag{3.10}$$

is a set of matrices transformed from $(C_1 \mid \cdots \mid C_{\delta(v)})$ through the self-adjoint column transformations. For $A \in S$, we call the columns which are even columns in $(C_1 \mid \cdots \mid C_{\delta(v)})$ the unnormalized columns in A. And we call the elements in the set (3.10) normalized forms. If $(C_1 \mid \cdots \mid C_{\delta(v)})$ is the coefficient matrix of a self-adjoint vertex condition at v, then the $2^{\delta(v)}$ elements in the set S are also the coefficient matrices of self-adjoint vertex conditions.

Theorem 3 *If we only allow the elementary row transformations on the matrix* $B_v = (B_1 \mid \cdots \mid B_{\delta(v)})$, *where* $B_v\hat{f}_v = 0$ *is a self-adjoint vertex condition, then* B_v *can be normalized to one of the* $2^{\delta(v)}$ *forms in* (3.10), *and the complex matrix consisting of the unnormalized columns has the properties* (1)-(3) *in Theorem* 2.

Proof From Lemma 3 we can obtain that the vertex conditions in (3.10) satisfy (3.4) if and only if $(C_1 \mid \cdots \mid C_{\delta(v)})$ satisfies (3.4). According to the proof of Theorem 2, we could reach the conclusion. □

Definition 2 The $2^{\delta(v)}$ elements in the set (3.10) with $(C_1 \mid \cdots \mid C_{\delta(v)})$ that have the properties (1)-(3) in Theorem 2 are called the normalized forms of the coefficient matrices of self-adjoint vertex conditions at v.

The elementary row transformations on the coefficient matrices of self-adjoint vertex conditions allow us to bring them into their corresponding normalized forms, and the self-adjoint column transformations on these matrices give us simple one-to-one correspondences between the coefficient matrices in different normalized forms.

Theorem 4 *For the local operator* $\mathcal{L}$ *defined by* (2.1), *the space* $\mathfrak{S}_v^{\mathbb{C}}$ *of self-adjoint complex vertex conditions at vertex* v *is a connected, closed and analytic real submanifold of* $\mathfrak{B}_v^{\mathbb{C}}$ *and has dimension* $\delta^2(v)$ *over the number field* $\mathbb{R}$. *Therefore,* $\mathfrak{S}_v^{\mathbb{C}}$ *is also compact.*

Proof For an arbitrary complex vertex condition $(B_1 \mid \cdots \mid B_{\delta(v)})$ at vertex v, it is in one equivalence class of quotient space $M^*_{\delta(v)\times 2\delta(v)}(\mathbb{C})/GL(\delta(v),\mathbb{C})$. The set $\mathfrak{S}_v^{\mathbb{C}}$ of self-adjoint complex vertex conditions at vertex v is a subset of $\mathfrak{B}_v^{\mathbb{C}}$ which can be divided into $2^{\delta(v)}$ canonical atlas of local coordinate systems. The proof of the analyticity and the connectivity of $\mathfrak{S}_v^{\mathbb{C}}$ is similar to Theorem 3.11 in [3]. Thus $\mathfrak{S}_v^{\mathbb{C}}$ is an analytic real submanifold of $\mathfrak{B}_v^{\mathbb{C}}$ and has dimension $\delta^2(v)$ over the number field $\mathbb{R}$. The canonical atlas of local coordinate systems of $\mathfrak{S}_v^{\mathbb{C}}$ is internal connected and $\mathfrak{S}_v^{\mathbb{C}}$ is connected.

Next we prove that the space $\mathfrak{S}_v^{\mathbb{C}}$ is closed. Let $\{(B_1^{(n)} \mid \cdots \mid B_{\delta(v)}^{(n)})\}_{n=1}^{+\infty}$ be a sequence in $\mathfrak{S}_v^{\mathbb{C}}$ that converges to $\tilde{B} = (B_1' \mid \cdots \mid B_{\delta(v)}') \in \mathfrak{B}_v^{\mathbb{C}}$. Without loss of generality, we can assume that $\tilde{B}$ has the normalized form (3.9), *i.e.*,

$$\tilde{B} = \begin{pmatrix} 1 & c'_{11} & \cdots & 0 & c'_{1\delta(v)} \\ 0 & c'_{21} & \cdots & 0 & c'_{2\delta(v)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c'_{(\delta(v)-1)1} & \cdots & 0 & c'_{(\delta(v)-1)\delta(v)} \\ 0 & c'_{\delta(v)1} & \cdots & 1 & c'_{\delta(v)\delta(v)} \end{pmatrix}.$$

Denote the matrix constructed by the even numbered columns of $\tilde{B}$ as D. Then we will show that D has the properties (1)-(3) in Theorem 2.

For sufficiently large n, $B_\nu^{(n)} = (B_1^{(n)} \mid \cdots \mid B_{\delta(\nu)}^{(n)}) \in \mathcal{O}_1^s$, and hence

$$B_\nu^{(n)} = \begin{pmatrix} 1 & c_{11}^{(n)} & \cdots & 0 & c_{1\delta(\nu)}^{(n)} \\ 0 & c_{21}^{(n)} & \cdots & 0 & c_{2\delta(\nu)}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{(\delta(\nu)-1)1}^{(n)} & \cdots & 0 & c_{(\delta(\nu)-1)\delta(\nu)}^{(n)} \\ 0 & c_{\delta(\nu)1}^{(n)} & \cdots & 1 & c_{\delta(\nu)\delta(\nu)}^{(n)} \end{pmatrix}.$$

Denote the matrix constructed by the even numbered columns of $B_\nu^{(n)}$ as $D^{(n)}$. The matrix $D^{(n)}$ has the properties (1)-(3) in Theorem 2, and

$$D^{(n)} \to D, \quad \text{as } n \to +\infty,$$

under the norm $\|\cdot\|$ on $M_{\delta(\nu)\times 2\delta(\nu)}(\mathbb{C})$. Therefore,

$$\bigl(B_1' \mid \cdots \mid B_{\delta(\nu)}'\bigr) \in \mathfrak{S}_\nu^{\mathbb{C}}. \qquad \square$$

Example 1 When $\delta(\nu) = 2$, $s = 1$, the canonical atlas of local coordinate systems of $\mathfrak{B}_\nu^{\mathbb{C}}$ is as follows:

$$\begin{cases} \mathcal{O}_1 = \left\{ \begin{pmatrix} 1 & a_{12} & 0 & b_{12} \\ 0 & a_{22} & 1 & b_{22} \end{pmatrix}; a_{12}, a_{22}, b_{12}, b_{22} \in \mathbb{C} \right\}, \\ \mathcal{O}_2 = \left\{ \begin{pmatrix} a_{11} & -1 & 0 & b_{12} \\ a_{21} & 0 & 1 & b_{22} \end{pmatrix}; a_{11}, a_{21}, b_{12}, b_{22} \in \mathbb{C} \right\}, \\ \mathcal{O}_3 = \left\{ \begin{pmatrix} 1 & a_{12} & b_{11} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{pmatrix}; a_{12}, a_{22}, b_{11}, b_{21} \in \mathbb{C} \right\}, \\ \mathcal{O}_4 = \left\{ \begin{pmatrix} a_{11} & -1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{pmatrix}; a_{11}, a_{21}, b_{11}, b_{21} \in \mathbb{C} \right\}, \\ \mathcal{O}_5 = \left\{ \begin{pmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix}; b_{11}, b_{21}, b_{12}, b_{22} \in \mathbb{C} \right\}, \\ \mathcal{O}_6 = \left\{ \begin{pmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \end{pmatrix}; a_{11}, a_{21}, a_{12}, a_{22} \in \mathbb{C} \right\}. \end{cases}$$

The canonical atlas of local coordinate systems of $\mathfrak{S}_\nu^{\mathbb{C}}$ is as follows:

$$\begin{cases} \mathcal{O}_1^s = \left\{ \begin{pmatrix} 1 & a_1 & 0 & -\overline{a_2} \\ 0 & a_2 & 1 & a_3 \end{pmatrix}; a_1, a_3 \in \mathbb{R}, a_2 \in \mathbb{C} \right\}, \\ \mathcal{O}_2^s = \left\{ \begin{pmatrix} a_1 & -1 & 0 & -\overline{a_2} \\ a_2 & 0 & 1 & a_3 \end{pmatrix}; a_1, a_3 \in \mathbb{R}, a_2 \in \mathbb{C} \right\}, \\ \mathcal{O}_3^s = \left\{ \begin{pmatrix} 1 & a_1 & -\overline{a_2} & 0 \\ 0 & a_2 & a_3 & -1 \end{pmatrix}; a_1, a_3 \in \mathbb{R}, a_2 \in \mathbb{C} \right\}, \\ \mathcal{O}_4^s = \left\{ \begin{pmatrix} a_1 & -1 & -\overline{a_2} & 0 \\ a_2 & 0 & a_3 & -1 \end{pmatrix}; a_1, a_3 \in \mathbb{R}, a_2 \in \mathbb{C} \right\}. \end{cases}$$

4 The self-adjoint Sturm-Liouville operators

Assume that the graph Γ has a finite vertex set $V = \{v_1, \dots, v_n\}$ and a finite edge set E. The vertex conditions at v_i are written as $B_{v_i}\hat{f}_{v_i} = 0$. Then, for the local operator $\mathcal{L}$ defined by (2.1), the vertex conditions on Γ are

$$\begin{cases} B_{v_1}\hat{f}_{v_1} = 0, & \text{at } v_1, \\ B_{v_2}\hat{f}_{v_2} = 0, & \text{at } v_2, \\ \dots, \\ B_{v_n}\hat{f}_{v_n} = 0, & \text{at } v_n. \end{cases} \tag{4.1}$$

Denote B_Γ as $B_\Gamma = (B_{v_1}, \dots, B_{v_n})$, where B_{v_i} is a $\delta(v_i) \times 2\delta(v_i)$ matrix with linearly independent rows. In the following, B_Γ represents (4.1), *i.e.*, the vertex conditions on Γ. Define $\mathfrak{B}_\Gamma^{\mathbb{C}}$ as

$$\mathfrak{B}_\Gamma^{\mathbb{C}} = \bigotimes_{i=1}^{n} \mathfrak{B}_{v_i}^{\mathbb{C}}.$$

Then the complex self-adjoint vertex conditions space of the local operator $\mathcal{L}$ on the graph Γ is denoted as $\mathfrak{S}_\Gamma^{\mathbb{C}}$,

$$\mathfrak{S}_\Gamma^{\mathbb{C}} = \bigotimes_{i=1}^{n} \mathfrak{S}_{v_i}^{\mathbb{C}}.$$

Corollary 1 *The space $\mathfrak{B}_\Gamma^{\mathbb{C}}$ has dimension $\sum_{i=1}^{n} \delta^2(v_i)$ over the number field $\mathbb{C}$, and has dimension $\sum_{i=1}^{n} 2\delta^2(v_i)$ over the number field $\mathbb{R}$. The space $\mathfrak{S}_\Gamma^{\mathbb{C}}$ is a connected, closed and analytic real submanifold of $\mathfrak{B}_\Gamma^{\mathbb{C}}$ with dimension $\sum_{i=1}^{n} \delta^2(v_i)$ over the number field $\mathbb{R}$.*

For $x, y \in \Gamma$, denote the distance between x and y as $d(x,y)$.

Lemma 4 *Suppose that the graph Γ has finitely many edges and*

$$\sup_{x,y\in\Gamma} d(x,y) < \infty,$$

the operator $\mathcal{L}$ defined by (2.1) *with domain*

$$\bigl\{f \in \mathrm{Dom}(\mathcal{L}_{\max}) : B_v\hat{f}_v = 0, B_v \in \mathfrak{S}_v^{\mathbb{C}}, \textit{for all } v \in V\bigr\}$$

is self-adjoint.

Proof The proof is based on Theorem 3.4 in [4]. □

Theorem 5 *Suppose that the graph Γ has infinitely many edges and vertices, $\sup_{x,y\in\Gamma} d(x,y) = \infty$, and there is no finite accumulation point in V.*

If the operator $\mathcal{L}$ defined by (2.1) *satisfies the following two conditions*:

(1) $p > 0$, $p \in AC_{\mathrm{loc}}(\Gamma, \mathbb{R})$,

(2) $\frac{1}{w}, \frac{(p')^2}{w} \in L_{\mathrm{loc}}(\Gamma, \mathbb{R})$, $\frac{p}{w}$ *is essentially bounded on* Γ,

and $\mathcal{L}$ is lower bounded with domain

$$\mathrm{Dom}(\mathcal{L}) = \{f \in \mathrm{Dom}(\mathcal{L}_{\max}) : f \text{ has compact support on } \Gamma, B_v\hat{f}_v = 0, B_v \in \mathfrak{S}_v^{\mathbb{C}}, v \in V\},$$

then $\mathcal{L}$ is essentially self-adjoint. Conversely, every local self-adjoint operator $\mathcal{L}_1$ formally given by L is the closure of one of the operators $\mathcal{L}$.

Proof Using the theory in [9] on one edge e_i, we can get that the operator $\mathcal{L}$ is symmetric and the domain of $\mathcal{L}^*$ is contained in $\mathrm{Dom}(\mathcal{L}_{\max})$. By Theorem 3.1 and Corollary 3.2 in [4], we get that $f \in \mathrm{Dom}(\mathcal{L}^*)$ satisfies the same conditions $B_v\hat{f}_v = 0$ at the vertices. Then

$$\mathrm{Dom}(\mathcal{L}^*) = \{f \in \mathrm{Dom}(\mathcal{L}_{\max}) : B_v\hat{f}_v = 0, B_v \in \mathfrak{S}_v^{\mathbb{C}}, v \in V\}.$$

Without loss of generality, we assume that $\mathcal{L} \geq I$. Next we need to show that the equation

$$-(pf')'(t) + q(t)f(t) = 0, \quad t \in \Gamma \backslash V, f \in \mathrm{Dom}(\mathcal{L}^*) \tag{4.2}$$

has only a trivial solution (derivative is understood in a distribution sense).

Fix a point $o \in \Gamma$, we define a sequence of functions $\{\chi_n\}$, $\chi_n \in C_{\mathrm{comp}}^{\infty}(\Gamma)$ such that

$$\chi_n(t) := \begin{cases} 1, & 0 \leq d(o,t) \leq t_n, \\ 0, & d(o,t) \geq t_n + 1, \end{cases}$$

and

$$|\chi_n'| \leq 2, \qquad \chi_n'(v) = 0 \quad \text{for all } v \text{ satisfies } t_n \leq d(o,t) \leq t_n + 1,$$

where the function space $C_{\mathrm{comp}}^{\infty}(\Gamma)$ contains the functions belonging to $C^{\infty}(\Gamma)$ and having compact support on Γ. Assume that $\tilde{f} \neq 0$ is a solution of equation (4.2). Since $\tilde{f}$ satisfies the conditions

$$B_v\hat{\tilde{f}}_v = 0 \quad \text{for all } v \in V, \tag{4.3}$$

then for each v and $g := \tilde{f}\chi_n$,

$$B_v\hat{g}_v = 0 \quad \text{for all } v \in V.$$

Since $p \in AC_{\mathrm{loc}}(\Gamma)$, one verifies that

$$\frac{1}{w}\left[-\left(p(\tilde{f}\chi_n)'\right)' + q(\tilde{f}\chi_n)\right] = -\frac{1}{w}p'\chi_n'\tilde{f} - \frac{1}{w}p\chi_n''\tilde{f} - 2\frac{1}{w}p\chi_n'\tilde{f}'. \tag{4.4}$$

Let Γ_m be a subtree of Γ containing all $x \in \Gamma$, $|x| \leq m$.

$$\begin{aligned}\int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \frac{1}{w}(p')^2(\chi_n')^2\tilde{f}^2\,\mathrm{d}t &\leq \|(\chi_n')^2\tilde{f}^2\|_{L^{\infty}(\Gamma_{t_n+1}-\Gamma_{t_n})}\int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \frac{1}{w}(p')^2\,\mathrm{d}t \\ &< \infty,\end{aligned}$$

$$\begin{aligned}
\int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \frac{1}{w} p^2 (\chi_n'')^2 \tilde{f}^2 \,\mathrm{d}t &\le \left\| \frac{p}{w} \right\|^2_{L^\infty(\Gamma_{t_n+1}-\Gamma_{t_n})} \left\| (\chi_n'')^2 \right\|_{L^\infty(\Gamma_{t_n+1}-\Gamma_{t_n})} \\
&\quad \times \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} w\tilde{f}^2 \,\mathrm{d}t < \infty, \\
\int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \frac{1}{w} p^2 (\chi_n')^2 (\tilde{f}')^2 \,\mathrm{d}t &\le \left\| (\chi_n'')^2 \right\|_{L^\infty(\Gamma_{t_n+1}-\Gamma_{t_n})} \\
&\quad \times \left\| (p\tilde{f}')^2 \right\|_{L^\infty(\Gamma_{t_n+1}-\Gamma_{t_n})} \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \frac{1}{w} \,\mathrm{d}t < \infty,
\end{aligned}$$

and then the right-hand side of equality (4.4) belongs to $L^2_w(\Gamma, \mathbb{C})$. Then we have $\tilde{f}\chi_n \in \mathrm{Dom}(\mathcal{L})$ and

$$\begin{aligned}
\bigl(\mathcal{L}(\tilde{f}\chi_n), (\tilde{f}\chi_n)\bigr)_w &= \int_\Gamma \bigl[-p'\chi_n'\tilde{f} - p\chi_n''\tilde{f} - 2p\chi_n'\tilde{f}'\bigr]\overline{\tilde{f}\chi_n} \,\mathrm{d}t \\
&= \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} (\chi_n')^2 p\tilde{f}^2 \,\mathrm{d}t = \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} w(\chi_n')^2 \tilde{f}^2 \frac{p}{w} \,\mathrm{d}t \\
&\ge \bigl((\tilde{f}\chi_n), (\tilde{f}\chi_n)\bigr)_w = \int_\Gamma \tilde{f}^2(x)\chi_n^2(x) w \,\mathrm{d}t.
\end{aligned} \tag{4.5}$$

On the other hand,

$$\int_{\Gamma_{t_n+1}-\Gamma_{t_n}} w(\chi_n')^2 \tilde{f}^2 \frac{p}{w} \,\mathrm{d}t \le 4 \left\| \frac{p}{w} \right\|_{L^\infty(\Gamma)} \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \tilde{f}^2 w \,\mathrm{d}t. \tag{4.6}$$

Combining (4.5) with (4.6), we obtain

$$\int_{\Gamma_{t_n}} \tilde{f}^2(x) w \,\mathrm{d}t \le \int_\Gamma \tilde{f}^2(x)\chi_n^2(x) w \,\mathrm{d}t \le 4 \left\| \frac{p}{w} \right\|_{L^\infty(\Gamma)} \int_{\Gamma_{t_n+1}-\Gamma_{t_n}} \tilde{f}^2 w \,\mathrm{d}t.$$

Since $\tilde{f} \in L^2_w(\Gamma, \mathbb{C})$, $\tilde{f} = 0$. This completes the proof. □

Let $\mathcal{L}$ be the operator defined by (2.1). Assume that the functions in $\mathrm{Dom}(\mathcal{L})$ are continuous on Γ, *i.e.*,

$$f(\alpha_1) = f(\alpha_2) = \cdots = f(\alpha_{\delta(v)}), \quad \alpha_i \sim v, i = 1, 2, \ldots, \delta(V),$$

at each vertex $v \in V$. We get the following result.

Corollary 2 *Suppose that the local operator $\mathcal{L}$ defined by* (2.1) *is self-adjoint, with functions in* $\mathrm{Dom}(\mathcal{L})$ *continuous on Γ. For a vertex v, let $\alpha_1, \ldots, \alpha_s \sim v$ be b_k in the corresponding edges $e_k = (a_k, b_k)$, $k = 1, \ldots, s$, and $\alpha_{s+1}, \ldots, \alpha_{\delta(v)}$ be a_k in the corresponding edges $e_k = (a_k, b_k)$, $k = s+1, \ldots, \delta(v)$. Then the functions f in* $\mathrm{Dom}(\mathcal{L})$ *satisfy the condition*

$$\sum_{i=1}^{s} (pf')(\alpha_i) + \sum_{j=s+1}^{\delta(v)} (-1)(pf')(\alpha_j) = rf(v), \quad r \in \mathbb{R},$$

at each vertex v.

Proof All the functions $f \in \mathrm{Dom}(\mathcal{L})$ satisfy the continuity conditions, that means

$$f(\alpha_m) + (-1)f(\alpha_{\delta(v)}) = 0, \quad m = 1, \dots, \delta(v) - 1,$$

at each vertex $v \in \Gamma$. For a fix vertex v, B_v is a $\delta(v) \times 2\delta(v)$ matrix with linearly independent rows. Then

$$B_v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c'_2 & 0 & c'_4 & 0 & c'_6 & \cdots & c'_{2\delta(v)-1} & c'_{2\delta(v)} \end{pmatrix}_{\delta(v)\times 2\delta(v)}$$

and $(c'_{2\delta(v)-1})^2 + (c'_{2\delta(v)})^2 \neq 0$. If $c'_{2\delta(v)} \neq 0$, we can rewrite the vertex condition as $A_v\hat{f}_v = 0$, where

$$A_v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c_2 & 0 & c_4 & 0 & c_6 & \cdots & c_{2\delta(v)-2} & c_{2\delta(v)-1} & -1 \end{pmatrix}_{\delta(v)\times 2\delta(v)}.$$

Therefore we can get that $A_v \in \{(C_1 \mid C_2 \mid \cdots \mid C_{\delta(v)}E_2)\}$, where the matrices $(C_1 \mid C_2 \mid \cdots \mid C_{\delta(v)}E_2)$ are introduced in (3.10). Since $B_v \in \mathfrak{S}_v^{\mathbb{C}}$, the complex matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_4 & \cdots & c_{2\delta(v)-2} & c_{2\delta(v)-1} \end{pmatrix}_{\delta(v)\times 2\delta(v)}$$

has the properties (1)-(3) in Theorem 2. Then we have

$$\begin{cases} c_2 = c_4 = \cdots = c_{2s} = 1, \\ c_{2s+2} = c_{2s+4} = \cdots = c_{2\delta(v)-2} = -1, \\ c_{2\delta(V)-1} \in \mathbb{R}, \end{cases}$$

i.e.,

$$\sum_{i=1}^{s} (pf')(\alpha_i) + \sum_{j=s+1}^{\delta(v)} (-1)(pf')(\alpha_j) = rf(v), \quad r \in \mathbb{R}.$$

If $c'_{2\delta(\nu)} = 0$ and $c'_{2\delta(\nu)-1} \neq 0$, through the elementary row transformations, the condition $B_\nu \hat{f}_\nu = 0$ is equal to the condition $A_\nu \hat{f}_\nu = 0$, where

$$A_\nu = \begin{pmatrix} 1 & c_2 & 0 & c_4 & \cdots & 0 & c_{2\delta(\nu)-2} & 0 & 0 \\ 0 & c_2 & 1 & c_4 & \cdots & 0 & c_{2\delta(\nu)-2} & 0 & 0 \\ 0 & c_2 & 0 & c_4 & \cdots & 0 & c_{2\delta(\nu)-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_2 & 0 & c_4 & \cdots & 1 & c_{2\delta(\nu)-2} & 0 & 0 \\ 0 & c_2 & 0 & c_4 & \cdots & 0 & c_{2\delta(\nu)-2} & 1 & 0 \end{pmatrix}.$$

Then we have $A_\nu \in \{(C_1 \mid C_2 \mid \cdots \mid C_{\delta(\nu)})\}$, the matrix $(C_1 \mid C_2 \mid \cdots \mid C_{\delta(\nu)})$ is introduced in (3.6). Since $B_\nu \in \mathfrak{S}_\nu^{\mathbb{C}}$, the complex matrix

$$\begin{pmatrix} c_2 & \cdots & c_{2\delta(\nu)-2} & 0 \\ c_2 & \cdots & c_{2\delta(\nu)-2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_2 & \cdots & c_{2\delta(\nu)-2} & 0 \end{pmatrix}_{\delta(\nu)\times\delta(\nu)}$$

has the properties (1)-(3) in Theorem 2. Then we have

$$c_2 = c_4 = \cdots = c_{2\delta(\nu)-2} = 0.$$

Therefore the vertex conditions $B_\nu \hat{f}_\nu = 0$ are the conditions

$$f(\alpha_1) = f(\alpha_2) = \cdots = f(\alpha_{\delta(\nu)}) = 0, \quad \alpha_i \sim \nu.$$

That is, corresponding to the conditions

$$\begin{cases} f(\alpha_1) = f(\alpha_2) = \cdots = f(\alpha_{\delta(\nu)}), \\ \sum_{i=1}^{s}(-1)(pf')(\alpha_i) + \sum_{j=s+1}^{\delta(\nu)}(pf')(\alpha_j) = rf(\nu), \end{cases}$$

for $r = \infty$. □

Remark 1 For the complex self-adjoint conditions B_ν including the continuity conditions, the space consisting of B_ν is a real submanifold of $\mathfrak{B}_\nu^{\mathbb{C}}$ with dimension 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Received: 20 April 2015 Accepted: 21 August 2015 Published online: 17 September 2015

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