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Boundary value problems for modified Dirac operators in Clifford analysis

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Abstract

In this paper, we discuss two kinds of Riemann type boundary value problems for the operator $\widetilde{D}_\lambda$, where λ is a complex number. Furthermore, we establish the Almansi type expansion for the operator $\widetilde{D}_\lambda^k$, where $k \in \mathbf{N}$. As applications of the expansion, we investigate the Riemann type boundary value problem and the generalized Riquier problem for the operator $\widetilde{D}_\lambda^k$.

MSC: 30G35; 45J05

Keywords: modified Dirac operator; Almansi type expansion; Riemann type boundary problem; Riquier problem

1 Introduction

The uniqueness and existence theorems for the solutions of boundary value problems for systems of partial differential equations are sufficiently well known. Such problems have remarkable applications in mathematical physics, the mechanics of deformable bodies, electromagnetism, relativistic quantum mechanics, and some of their natural generalizations. Almost all such problems can be set in the context of Clifford analysis (see [1, 2]). Clifford analysis is centered around the concept of monogenic functions, *i.e.* null solutions of a first order vector valued rotation invariant differential operator called the Dirac operator which factorizes the Laplace operator (see [3, 4]). As to the mathematical study of boundary value problems in Clifford analysis, there are several different approaches known in the literature. Without claiming completeness, we mention some of them. First of all, we have the approach originating with Bernstein, whose approach is to translate boundary value problems to the corresponding singular integral equations, then use the properties of the Fredholm operator to discuss the solvability of singular integral equations (see [5]). Another important approach is based on complex analysis. In this case, first we use analytic function theory to solve these kinds of boundary value problems, then we use the results of boundary value problems to solve singular integral equations (see [6, 7]). The advantage of this method is that the explicit representation of solutions can be obtained, but in the higher dimensional space there still exist many obstacles to generalize this method. In this paper, we continue to use the method in [6, 7] to solve boundary value problems for the modified Dirac operators.

The paper is organized as follows. In Section 2, we review some results on the theory of Clifford analysis. In Section 3, applying the Plemelj formula for the modified Dirac operator [6], we consider Riemann type boundary value problems for the operator $\widetilde{D}_\lambda$. In Section 4, using the Euler operator in Clifford analysis, we obtain the Almansi type expansion for the operator $\widetilde{D}_\lambda^k$. In Section 5, as applications of the expansion, we investigate the Riemann type boundary value problem and the generalized Riquier problem for the operator $\widetilde{D}_\lambda^k$.

2 Preliminaries

2.1 Clifford analysis

Let $\mathbf{R}_{0,m}$ be the real associative Clifford algebra generated by $\{e_1, e_2, \ldots, e_m\}$, where the basic vectors $e_1, e_2, \ldots, e_m$ satisfy the relations $e_ie_j + e_je_i = -2\delta_{i,j}$, $i, j = 1, \ldots, m$. Let $\varepsilon_i = -e_1e_i$, $i = 1, \ldots, m$, then the universal Clifford algebra $\mathbf{R}_{0,m-1}$ for $\mathbf{R}^{m-1}$ is generated by $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$, where the vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ satisfy the following relations:

$$\begin{cases} \varepsilon_1\varepsilon_i = \varepsilon_i\varepsilon_1, & i = 1, \ldots, m, \\ \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = -2\delta_{i,j}, & i, j = 2, \ldots, m. \end{cases}$$

Each of the elements in $\mathbf{R}_{0,m-1}$ may be written as $a = \sum_A a_A\varepsilon_A$, where a_A are real numbers and $\varepsilon_A = \varepsilon_{\alpha_1}\varepsilon_{\alpha_2}\cdots\varepsilon_{\alpha_h}$ with $A = \{\alpha_1, \ldots, \alpha_h\} \subset \{2, \ldots, m\}$. We define the norm of a as $|a| = (\sum_A |a_A|^2)^{\frac{1}{2}}$. If there exists $b \in \mathbf{R}_{0,m-1}$ such that $ab = ba = \varepsilon_1$, then b is called the inverse of a, which is denoted as a^{-1}.

A typical element of $\mathbf{R}^m$ is denoted by $x = x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_m\varepsilon_m$ with $x_i \in \mathbf{R}$. We define $\overline{x} = x_1\varepsilon_1 - x_2\varepsilon_2 - \cdots - x_m\varepsilon_m$, then $x\overline{x} = \overline{x}x = |x|^2$. Obviously, for $x \neq 0$, we have $x^{-1} = \frac{\overline{x}}{|x|^2}$.

One of the main aims of Clifford analysis is to construct a first order operator, the so-called Dirac operator, factorizing the Laplace operator and to study the function-theoretical properties of the null solutions of this operator. When working over $\mathbf{R}^m$, this Dirac operator is defined by

$$D = \sum_{i=1}^m e_i\partial_{x_i}. \tag{1}$$

Then the modified Dirac operator is defined as

$$\widetilde{D} = \sum_{i=1}^m \varepsilon_i\partial_{x_i}. \tag{2}$$

When studying the modified Dirac operator in this setting, we consider functions f which are *e.g.* elements of spaces such as $C^k(\Omega) \otimes \mathbf{R}_{0,m-1}$ with Ω some open domain in $\mathbf{R}^m$. This means that f can be written as

$$f = \sum_A f_A(x)\varepsilon_A \tag{3}$$

with $f_A(x) \in C^k(\Omega)$. Denote by $|f| = (\sum_A |f_A(x)|)^{\frac{1}{2}}$ the norm of $f \in C^k(\Omega) \otimes \mathbf{R}_{0,m-1}$.

3 Boundary value problems for the operator $\widetilde{D}_\lambda$

3.1 Riemann type problem for the operator $\widetilde{D}_\lambda$

Let

$$E(x) = \frac{1}{\omega_m}\frac{\overline{x}}{|x|}, \tag{4}$$

where $\omega_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ is the surface area of the unit sphere in $\mathbf{R}^m$. Then $E(x)$ satisfies the equation $\widetilde{D}f = 0$.

Let f be a Hölder continuous function on $\partial\Omega$ and take its Cauchy transform

$$f(x) = \frac{1}{A_m}\int_{\partial\Omega} E(y-x)\,d\sigma_y f(y), \quad x \in \mathbf{R}^m \setminus \partial\Omega. \tag{5}$$

Then $f(x)$ satisfies the equation $\widetilde{D}f = 0$ in $\mathbf{R}^m \setminus \partial\Omega$ as was proved in [6].

In [6], the following Plemelj formulas hold for $s \in \partial\Omega$:

$$F^+(s) = \lim_{x\in\Omega, x\to s} f(x) = \frac{1}{2}f(s) + \frac{1}{A_m}\int_{\partial\Omega} E(y-s)\,d\sigma_y f(y) \tag{6}$$

and

$$F^-(s) = \lim_{x\in\mathbf{R}^m\setminus\overline{\Omega}, x\to s} f(x) = -\frac{1}{2}f(s) + \frac{1}{A_m}\int_{\partial\Omega} E(y-s)\,d\sigma_y f(y), \tag{7}$$

where $\overline{\Omega} = \Omega \cup \partial\Omega$.

In order to obtain the main result in this section, we need the following lemma.

Lemma 3.1 *Let x_1 be a nonzero finite real number and $\lambda \in C$. Then*

$$\ker\widetilde{D}_\lambda = e^{\lambda x_1}\ker\widetilde{D}, \tag{8}$$

where $\ker\widetilde{D}_\lambda = \{f | f \in C^1(\Omega)\otimes R_{0,m-1}, (\widetilde{D}-\lambda)f = 0\}$, *and* $\ker\widetilde{D} = \ker\widetilde{D}_\lambda$ *for* $\lambda = 0$.

Proof Letting $f \in \ker\widetilde{D}$, we have

$$\begin{aligned}
\widetilde{D}_\lambda\big(e^{\lambda x_1}f\big) &= (\widetilde{D}-\lambda)e^{\lambda x_1}f \\
&= \lambda e^{\lambda x_1}f + e^{\lambda x_1}\widetilde{D}f - \lambda e^{\lambda x_1}f \\
&= 0,
\end{aligned}$$

which implies that $e^{\lambda x_1}\ker\widetilde{D} \subset \ker\widetilde{D}_\lambda$.

On the contrary, for $f \in \widetilde{D}_\lambda$, we can see that

$$\widetilde{D}\big(e^{-\lambda x_1}f\big) = e^{-\lambda x_1}(\widetilde{D}-\lambda)f = 0,$$

which means that $\ker\widetilde{D}_\lambda \subset e^{\lambda x_1}\ker\widetilde{D}$. □

Therefore, we obtain the conclusion.

Theorem 3.2 *Let f be a Hölder continuous function on $\partial\Omega$ and let $G \in Z(\mathbf{R}_{0,m-1})$ be invertible with inverse G^{-1}. Then the Riemann type problem*

$$\begin{cases}
\widetilde{D}_\lambda\Phi(x) = 0, & x \in \mathbf{R}^m \setminus \partial\Omega, \\
\Phi^+(t) = G\Phi^-(t) + f(t), & t \in \partial\Omega, \\
\Phi^-(\infty) = 0,
\end{cases} \tag{I}$$

has a solution Φ *given by*

$$\Phi(x) = \frac{X(x)}{A_m} \int_{\partial\Omega} e^{\lambda x_1} E(y-x)\, d\sigma_y G^{-1} f(y), \tag{9}$$

where

$$X(x) = \begin{cases} G, & x \in \Omega, \\ 1, & x \in \mathbf{R}^m \setminus \overline{\Omega}. \end{cases} \tag{10}$$

Note that the center $Z(\mathbf{R}_{0,m-1})$ of $\mathbf{R}_{0,m-1}$ is the set of elements in $\mathbf{R}_{0,m-1}$ which commute with all elements of $\mathbf{R}_{0,m-1}$ (see *e.g.* [6])

Proof First, it follows by Lemma 3.1 that the function $\Phi(x)$ determined by (9) satisfies the equation

$$\widetilde{D}_\lambda \Phi(x) = 0.$$

Secondly, let $G \in Z(\mathbf{R}_{0,m-1})$ be invertible with inverse G^{-1}. It follows by (6) and (7) that

$$\begin{aligned} &\Phi^+(s) - G\Phi^-(s) \\ &\quad = \left(\frac{X^+(s)}{2} G^{-1} f(s) + \frac{1}{A_m} \int_{\partial\Omega} e^{\lambda x_1} E(y-s)\, d\sigma_y G^{-1} f(y) \right) \\ &\qquad - G\left(-\frac{X^-(s)}{2} G^{-1} f(s) + \frac{1}{A_m} \int_{\partial\Omega} e^{\lambda x_1} E(y-s)\, d\sigma_y G^{-1} f(y) \right) \\ &\quad = f(s), \end{aligned}$$

where $s \in \partial\Omega$.

Finally, it is obvious that the function $\Phi(x)$ vanishes at infinity.

Thus, we obtain the conclusion. □

3.2 Riemann type boundary value problem (II)

In this section, using the Plemelj formulas, we consider the following Riemann type boundary value problem (II).

Suppose that f is a Hölder continuous function on $\partial\Omega$. Find a function $\Psi \in C^1(\Omega) \otimes \mathbf{R}_{0,m-1}$ that satisfies

$$\begin{cases} \widetilde{D}_\lambda \Psi(x) = 0, & x \in \mathbf{R}^m \setminus \partial\Omega, \\ \Psi^+(t)a - \Psi^-(t)b = f(t), & t \in \partial\Omega, \\ \lim_{|x|\to+\infty} \frac{|\Psi(x)|}{|x|^l} < +\infty, \end{cases} \tag{II}$$

where

$$\Psi^+(t) = \lim_{x\to t} \Psi(x), \quad x \in \Omega; \qquad \Psi^-(t) = \lim_{x\to t} \Psi(x), \quad x \in \mathbf{R}^m \setminus \overline{\Omega}.$$

a, b are given $\mathbf{R}_{0,m-1}$ valued constants whose inverses are a^{-1}, b^{-1}.

Lemma 3.3 *Let $Z_k = x_k\varepsilon_1 - x_1\varepsilon_k$, where $2 \leq k \leq m$. Then we have the polynomials of order p*

$$V_{k_1\cdots k_p} = \frac{1}{p!}\sum_{\pi(k_1\cdots k_p)} Z_{k_1}\cdots Z_{k_p} \in \ker\widetilde{D},$$

where the sum runs over all distinguishable permutations of all of $(k_1,\ldots,k_p)$.

The proof of Lemma 3.3 is similar to Proposition 11.2.3 in [3].

Theorem 3.4 *The boundary value problem* (II) *has a solution.*

Proof We will prove the function

$$\Psi(x) = \begin{cases} (\int_{\partial\Omega} e^{\lambda x_1}E(y-x)\,d\sigma_y f(y))a^{-1} \\ \quad + \sum_{p=1}^{l}\sum_{\pi(k_1\cdots k_p)} V_{k_1\cdots k_p}e^{\lambda x_1}a^{-1}, & x \in \Omega, l \in \mathbf{N}, \\ (\int_{\partial\Omega} e^{\lambda x_1}E(y-x)\,d\sigma_y f(y))b^{-1} \\ \quad + \sum_{p=1}^{l}\sum_{\pi(k_1\cdots k_p)} V_{k_1\cdots k_p}e^{\lambda x_1}b^{-1}, & x \in R^m\backslash\overline{\Omega}, l \in \mathbf{N}, \end{cases}$$

is a solution of the boundary value problem (II).

Denote

$$\Phi(x) = \begin{cases} \Psi(x)a, & x \in \Omega, \\ \Psi(x)b, & x \in R^m\backslash\overline{\Omega}. \end{cases}$$

Then

$$\begin{cases} \Phi^+(t) = \Psi^+(t)a, & t \in \partial\Omega, \\ \Phi^-(t) = \Psi^-(t)b, & t \in \partial\Omega, \end{cases}$$

where a, b have the inverses a^{-1}, b^{-1}, respectively. The boundary value problem (II) is equivalent to

$$\Phi^+(t) - \Phi^-(t) = f(t), \quad t \in \partial\Omega.$$

Note that

$$\bigl(T[f]\bigr)(x) = \int_{\partial\Omega} e^{\lambda x_1}E(y-x)\,d\sigma_y f(y)$$

for $x \in \mathbf{R}^m\backslash\partial\Omega$ is meaningful and satisfies the boundary properties

$$\bigl(T[f]\bigr)^+(t) - \bigl(T[f]\bigr)^-(t) = f(t), \quad t \in \partial\Omega.$$

Thus

$$\Phi^+(t) - \bigl(T[f]\bigr)^+(t) = \Phi^-(t) - \bigl(T[f]\bigr)^-(t), \quad t \in \partial\Omega,$$

which means that $\Phi(x) - (T[f])(x) = g(x) \in \ker\widetilde{D}_\lambda$ in $\mathbf{R}^m$ by the Painlevé theorem. By Lemma 3.3, we put $g(x) = \sum_{p=1}^{l}\sum_{\pi(k_1\cdots k_p)} V_{k_1\cdots k_p}e^{\lambda x_1}$. Thus we have the conclusion. □

4 Almansi type expansion for the operator $\widetilde{D}_\lambda^k$

In 1899, the Almansi expansion for polyharmonic functions was established, which was equivalent to the Fischer decomposition for polynomials (see [8]). One can find important applications and generalizations of this result in the case of several complex variables in the monograph of Aronszajn *et al.* [9], *e.g.* concerning functions holomorphic in the neighborhood of the origin in C^n. Also for the case of a Clifford analysis, one can refer to [10, 11]. But all these cases are limited to star-like domains. In this section, we consider the difficult case that Ω is some open domain in $\mathbf{R}^m$ not limited to star-like domains.

Definition 4.1 We define the generalized Euler operator by

$$\mathbf{E}_s = s\mathbf{I} + \mathbf{E} = s\mathbf{I} + \sum_{i=1}^{m} x_i \partial_{x_i},$$

where s is a complex constant, $\mathbf{I}$ is the identity operator, and $\mathbf{E}$ is the Euler operator.

Lemma 4.2 *Let Ω be as stated before. For $f(x) \in C^2(\Omega) \otimes \mathbf{R}_{0,m-1}$,*

$$\widetilde{D}\mathbf{E}_s f(x) = \mathbf{E}_{s+1}\widetilde{D}f(x), \tag{11}$$

where $s \in C$.

Proof For $s = 0$, from Definition 4.1 it follows that, for $f(x) \in C^2(\Omega) \otimes \mathbf{R}_{0,m-1}$,

$$\widetilde{D}\mathbf{E}f(x) = \sum_{i=1}^{m} \varepsilon_i \partial_{x_i} \left(\sum_{j=1}^{m} x_j \partial_{x_j} f(x) \right) = \sum_{i=1}^{m} \varepsilon_i \partial_{x_i} f(x) + \sum_{i,j=1}^{m} \varepsilon_i \partial_{x_j} \frac{\partial^2 f}{\partial x_j \, \partial x_i} = \widetilde{D}f(x) + \mathbf{E}\widetilde{D}f(x).$$

This implies that $\widetilde{D}\mathbf{E} = \mathbf{E_1}\widetilde{D}$. For $s \neq 0$,

$$\widetilde{D}\mathbf{E}_s = \widetilde{D}(s + \mathbf{E}) = s\widetilde{D} + \mathbf{E_1}\widetilde{D} = \mathbf{E_{s+1}}\widetilde{D}.$$

This completes the lemma. □

Note that the proof of Lemma 4.2 is inspired by Ren in [11].

Lemma 4.3 *If $f \in \ker(\widetilde{D}_\lambda)$, then*

$$C_k \widetilde{D}_\lambda^k \mathbf{E}_\lambda^k f = f, \tag{12}$$

where $C_k = \frac{1}{k!\lambda^k}$ and $k \in \mathbf{N}$.

Proof Note that $f \in \ker \widetilde{D}_\lambda$. For $k = 1$, Lemma 4.2 implies that

$$\begin{aligned}
\widetilde{D}_\lambda \mathbf{E}_\lambda f &= \widetilde{D}_\lambda \mathbf{E}_\lambda f \\
&= \widetilde{D}\mathbf{E}_\lambda f - \lambda \mathbf{E}_\lambda f \\
&= \mathbf{E}_{\lambda+1}\widetilde{D}f - \lambda \mathbf{E}_\lambda f
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}_{\lambda+1}\widetilde{D}f - \lambda\mathbf{E}_{\lambda+1}f + \lambda f \\
&= \mathbf{E}_{\lambda+1}(\widetilde{D} - \lambda)f + \lambda f \\
&= \lambda f.
\end{aligned}
$$

Suppose that, for $k = l$,

$$C_l\widetilde{D}_\lambda^l\mathbf{E}_\lambda^l f = f,$$

where $C_l = \frac{1}{l!\lambda^l}$. For $k = l + 1$,

$$\widetilde{D}_\lambda^{l+1}\mathbf{E}_\lambda^l f = \widetilde{D}_\lambda\widetilde{D}_\lambda^l E_\lambda^l f = \frac{1}{C_l}\widetilde{D}_\lambda f = 0.$$

We calculate

$$
\begin{aligned}
\widetilde{D}_\lambda^{l+1}\mathbf{E}_\lambda^{l+1} f &= \widetilde{D}_\lambda^l\widetilde{D}_\lambda\mathbf{E}_\lambda\mathbf{E}_\lambda^l f \\
&= \widetilde{D}_\lambda^l(\mathbf{E}_{\lambda+1}\widetilde{D}_\lambda + \lambda)\mathbf{E}_\lambda^l f \\
&= \widetilde{D}_\lambda^l\mathbf{E}_{\lambda+1}\widetilde{D}_\lambda\mathbf{E}_\lambda^l f + \lambda D_\lambda^l\mathbf{E}_\lambda^l f \\
&= \widetilde{D}_\lambda^{l-1}\widetilde{D}_\lambda\mathbf{E}_{\lambda+1}\widetilde{D}_\lambda\mathbf{E}_\lambda^l f + \frac{\lambda}{C_l}f \\
&= \widetilde{D}_\lambda^{l-1}\mathbf{E}_{\lambda+2}\widetilde{D}_\lambda^2\mathbf{E}_\lambda^l f + \frac{2\lambda}{C_l}f \\
&= \cdots \\
&= \mathbf{E}_{\lambda+l+1}\widetilde{D}_\lambda^{l+1}\mathbf{E}_\lambda^l f + \frac{(l+1)\lambda}{C_l}f \\
&= \frac{1}{C_{l+1}}f,
\end{aligned}
$$

which implies the conclusion. □

Denote $\ker\widetilde{D}_\lambda^k = \{f \mid f \in C^k(\Omega) \otimes R_{0,m-1}, (\widetilde{D} - \lambda)^k f = 0, k \in \mathbf{N}\}$.

Theorem 4.4 *If $f(x) \in \ker\widetilde{D}_\lambda^k$, then there exist unique functions $f_0, \dots, f_{k-1} \in \ker\widetilde{D}_\lambda$ such that*

$$f(x) = f_0(x) + \mathbf{E}_\lambda f_1(x) + \mathbf{E}_\lambda^2 f_2(x) + \cdots + \mathbf{E}_\lambda^{k-1} f_{k-1}(x), \tag{13}$$

where $f_0, \dots, f_{k-1}$ are given as follows:

$$
\begin{cases}
f_0(x) = (\mathbf{I} - C_1\mathbf{E}_\lambda\widetilde{D}_\lambda)(\mathbf{I} - C_2\mathbf{E}_\lambda^2\widetilde{D}_\lambda^2)\cdots(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1})f(x), \\
f_1(x) = C_1\widetilde{D}_\lambda(\mathbf{I} - C_2\mathbf{E}_\lambda^2\widetilde{D}_\lambda^2)\cdots(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1})f(x), \\
\vdots \\
f_{k-2}(x) = C_{k-2}\widetilde{D}_\lambda^{k-2}(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1})f(x), \\
f_{k-1}(x) = C_{k-1}\widetilde{D}_\lambda^{k-1}f(x),
\end{cases} \tag{14}
$$

and $C_k = \frac{1}{k!\lambda^k}$.

Conversely, if functions $f_0,\dots,f_{k-1} \in \ker \widetilde{D}_\lambda$, *then the function* $f(x)$ *given by* (13) *satisfies the equation* $\widetilde{D}_\lambda^k f = 0$.

Proof If we let the operator $\widetilde{D}_\lambda^{k-1}$ act on (13), then by Lemma 4.3, we have

$$\begin{aligned}
&\widetilde{D}_\lambda^{k-1} f(x) \\
&\quad= \widetilde{D}_\lambda^{k-1}\left(f_0(x) + \sum_{i=1}^{k-1} \mathbf{E}_\lambda^i f_i(x)\right) \\
&\quad= \widetilde{D}_\lambda^{k-1}\mathbf{E}_\lambda^{k-1} f_{k-1}(x) \\
&\quad= \frac{1}{C_{k-1}} f_{k-1}(x).
\end{aligned}$$

Thus,

$$f_{k-1}(x) = C_{k-1}\widetilde{D}_\lambda^{k-1} f(x).$$

Similarly, if we let the operator $\widetilde{D}_\lambda^{k-2}$ act on $f(x) - \mathbf{E}_\lambda^{k-1} f_{k-1}(x)$, we have

$$\begin{aligned}
&\widetilde{D}_\lambda^{k-2}\left[f(x) - \mathbf{E}_\lambda^{k-1} f_{k-1}(x)\right] \\
&\quad= \widetilde{D}_\lambda^{k-2}\left(f_0(x) + \sum_{i=1}^{k-2} \mathbf{E}_\lambda^i f_i(x)\right) \\
&\quad= \widetilde{D}_\lambda^{k-2}\left(\mathbf{E}_\lambda^{k-2} f_{k-2}(x)\right) \\
&\quad= \frac{1}{C_{k-2}} f_{k-2}(x).
\end{aligned}$$

Therefore, we have

$$f_{k-2}(x) = C_{k-2}\widetilde{D}_\lambda^{k-2}\left(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1}\right) f(x).$$

By induction, we have

$$\begin{cases}
f_{k-1}(x) = C_{k-1}\widetilde{D}_\lambda^{k-1} f(x), \\
f_{k-2}(x) = C_{k-2}\widetilde{D}_\lambda^{k-2}(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1}) f(x), \\
\vdots \\
f_1(x) = C_1\widetilde{D}_\lambda(\mathbf{I} - C_2\mathbf{E}_\lambda^2\widetilde{D}_\lambda^2)\cdots(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1}) f(x), \\
f_0(x) = (\mathbf{I} - C_1\mathbf{E}_\lambda\widetilde{D}_\lambda)(\mathbf{I} - C_2\mathbf{E}_\lambda^2\widetilde{D}_\lambda^2)\cdots(\mathbf{I} - C_{k-1}\mathbf{E}_\lambda^{k-1}\widetilde{D}_\lambda^{k-1}) f(x).
\end{cases} \tag{15}$$

Conversely, suppose that the functions $f_0,\dots,f_{k-1} \in \ker \widetilde{D}_\lambda$. Applying Lemma 4.3, we obtain

$$\widetilde{D}_\lambda^k f(x) = \widetilde{D}_\lambda^k\left[f_0(x) + \sum_{i=1}^{k-1} \mathbf{E}_\lambda^i f_i(x)\right] = 0,$$

which completes the proof. □

5 Boundary value problems for the operator $\widetilde{D}_\lambda^k$

5.1 Riemann type boundary value problem (III)

Now we consider the following Riemann type boundary value problem (III).

Suppose that $g_l(t)$, $l = 0,\dots,k-1$, are Hölder continuous functions on $\partial\Omega$. Find a function $\Psi \in C^k(\Omega) \otimes \mathbf{R}_{0,m-1}$ that satisfies

$$\begin{cases} \widetilde{D}_\lambda^k \Psi(x) = 0, \quad x \in \mathbf{R}^m \setminus \partial\Omega, \\ \Psi^+(t)a - \Psi^-(t)b = g_0(t), \\ [\widetilde{D}_\lambda \Psi]^+(t)a - [\widetilde{D}_\lambda \Psi]^-(t)b = g_1(t), \\ \vdots \\ [\widetilde{D}_\lambda^{k-1} \Psi]^+(t)a - [\widetilde{D}_\lambda^{k-1} \Psi]^-(t)b = g_{k-1}(t), \quad t \in \partial\Omega, \\ \lim_{|x| \to +\infty} \frac{|\Psi(x)|}{|x|^l} < +\infty, \end{cases} \tag{III}$$

where

$$\Psi^+(t) = \lim_{x \to t} x, \quad x \in \Omega, \qquad \Psi^-(t) = \lim_{x \to t} x, \quad x \in \mathbf{R}^m \backslash \overline{\Omega},$$

and a, b are given $\mathbf{R}_{0,m}$ valued constants whose inverses are a^{-1}, b^{-1}.

Theorem 5.1 *The boundary value problem* (III) *has a solution*.

Proof We will prove that the function

$$\Phi(x) = F_0(x) + \mathbf{E}_\lambda F_1(x) + \cdots + \mathbf{E}_\lambda^{k-1} F_{k-1}(x), \tag{16}$$

where

$$F_i(x) = \begin{cases} (\int_{\partial\Omega} e^{\lambda x_1} E(y-x)\, d\sigma_y g_i(y))[a]^{-1} \\ \qquad + \sum_{p=1}^{l} \sum_{\pi(k_1\cdots k_p)} V_{k_1\cdots k_p} e^{\lambda x_1} [a]^{-1}, \quad x \in \Omega, l \in \mathbf{N}, \\ (\int_{\partial\Omega} e^{\lambda x_1} E(y-x)\, d\sigma_y g_i(y))[b]^{-1} \\ \qquad + \sum_{p=1}^{l} \sum_{\pi(k_1\cdots k_p)} V_{k_1\cdots k_p} e^{\lambda x_1} [b]^{-1}, \quad x \in \mathbf{R}^m \setminus \overline{\Omega}, l \in \mathbf{N}, \end{cases} \tag{17}$$

for $0 \le i \le k-1$, and

$$F_i(t) = C_i \left[g_i(t) - \widetilde{D}_\lambda^i \sum_{l=i+1}^{k-i-1} E_\lambda^l F_l(t) \right], \quad t \in \partial\Omega, \tag{18}$$

is a solution of the boundary value problem (III).

From Theorem 3.4, we can see that $F_i(x) \in \ker \widetilde{D}_\lambda$. It follows by Theorem 4.4 that $\Phi(x) \in \ker \widetilde{D}_\lambda^k$.

Then, applying Lemma 4.3 and (18), we can see that

$$\left[\widetilde{D}_\lambda^i F\right]^+(t)a - \left[\widetilde{D}_\lambda^i F\right]^-(t)b = \frac{1}{C_i} F_i(t) + \widetilde{D}_\lambda^i \sum_{l=i+1}^{k-i-1} E_\lambda^l F_l(t) = g_i(t),$$

which completes the proof. □

5.2 Generalized Riquier problem for the operator $\widetilde{D}_\lambda^k$

In 1936, Nicolescu established Riquier problem for polyharmonic equations (see [12]). In 2003, applying the 0-normalized system of functions with respect to the Laplace operator, Karachik obtained a solution of the Riquier problem in harmonic analysis (see [13]). In this section, we will study the generalized Riquier problem for the operator $\widetilde{D}_\lambda^k$ by the expansion (13), as follows:

Find a function Φ such that $\widetilde{D}_\lambda^i \Phi \in C(\overline{\Omega}) \otimes \mathbf{R}_{0,m-1}$, for $i = 0, \dots, k-1$, and

$$\begin{cases} \widetilde{D}_\lambda^k \Phi = 0, \quad x \in \Omega, \\ \widetilde{D}_\lambda^i \Phi|_{\partial\Omega} = g_i(t), \quad t \in \partial\Omega. \end{cases} \tag{IV}$$

Theorem 5.2 *Suppose that the functions* $f_i(x) \in C(\overline{\Omega}) \otimes \mathbf{R}_{0,m-1}$, $i = 0, \dots, k-1$. *Then problem* (IV) *has a solution given by*

$$\Phi(x) = f_0(x) + \sum_{i=1}^{k-1} \mathbf{E}_\lambda^i f_i(x), \tag{19}$$

where the functions $f_i(x)$ *satisfy*

$$\begin{cases} \widetilde{D}_\lambda f_i(x) = 0, \quad x \in \Omega, \\ f_i(x)|_{\partial\Omega} = C_i[g_i(t) - \widetilde{D}_\lambda^i \sum_{j=i+1}^{k-i-1} \mathbf{E}_\lambda^j f_j(x)|_{\partial\Omega}], \quad t \in \partial\Omega. \end{cases} \tag{20}$$

Proof First, by Theorem 4.4, we can see that

$$\widetilde{D}_\lambda^k \Phi(x) = 0.$$

Then, for $0 \le i \le k-1$, Lemma 4.3 implies that

$$\widetilde{D}_\lambda^i \Phi(x) = \widetilde{D}_\lambda^i \left(f_0(x) + \sum_{j=1}^{k-1} \mathbf{E}_\lambda^j f_j(x) \right) = \frac{1}{C_i} f_i(x) + \widetilde{D}_\lambda^i \sum_{j=i+1}^{k-1-i} \mathbf{E}_\lambda^j f_j(x).$$

Letting $x \to t$, the formulas in (20) give $\widetilde{D}_\lambda^i \Phi|_{\partial\Omega} = g_i(t)$, $i = 0, \dots, k-1$, which implies the conclusion. □

Competing interests

The author declares that they have no competing interests.

Author's contributions

The author read and approved the final manuscript.

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