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The local well-posedness and stability to a nonlinear generalized Degasperis-Procesi equation

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Abstract

A nonlinear generalized Degasperis-Procesi equation is investigated. The local well-posedness of a strong solution for the equation in the Sobolev space $H^s(R)$ with $s > \frac{3}{2}$ is established. The $L^1(R)$ stability is obtained under certain assumptions on the initial data.

MSC: 35G25; 35L05

Keywords: nonlinear equation; Sobolev space $H^s(R)$; $L^1(R)$ stability

1 Introduction

Degasperis and Procesi [1] investigated a family of third order dispersive nonlinear equations

$$u_t + \alpha_0 u_x + \gamma u_{xxx} - \beta^2 u_{txx} = \left(\alpha_1 u^2 + \alpha_2 u_x^2 + \alpha_3 u u_{xx}\right)_x, \tag{1}$$

where constants α_i ($i = 0, 1, 2, 3$), β and $\gamma \in R$.

Applying the method of asymptotic integrability to Eq. (1), it is found in [1] that only three equations satisfy asymptotic integrability conditions: the KdV equation, the Camassa-Holm (CH) equation and one new equation of the form

$$u_t + u_x + 6uu_x + u_{xxx} - \beta^2 \left(u_{txx} + \frac{9}{2} u_x u_{xx} + \frac{3}{2} u u_{xxx}\right) = 0, \tag{2}$$

which can be transformed to the dispersionless form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + u u_{xxx}, \quad t > 0, x \in R. \tag{3}$$

Equation (3) is called Degasperis-Procesi (DP) equation which represents a model for shallow water dynamics, and its asymptotic accuracy is similar to the CH equation.

Here, we set the coefficients of Eq. (1) $\gamma = -\alpha_0 \beta^2$, $\alpha_2 = \alpha_3$ and $\beta > 0$. The objective of this work is to study the following nonlinear equation:

$$u_t + \alpha_0 u_x - \beta^2(\alpha_0 u_{xxx} + u_{txx}) = 2\alpha_1 uu_x + 3\alpha_2 u_x u_{xx} + \alpha_2 u u_{xxx}. \tag{4}$$

Obviously, DP equation (3) is the special case of Eq. (4).

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Degasperis *et al.* [2] proved the formal integrability of Eq. (3), derived the infinite sequence of conserved quantities with a bi-Hamiltonian structure and obtained the existence of the non-smooth solutions by constructing a Lax pair. Lenells [3] classified all weak traveling wave solutions of Eq. (3), while Coclite and Karlsen [4] established the well-posedness of periodic weak solutions for Eq. (3) and studied their long-time asymptotic behavior. It was shown in [5] that several numerical schemes were constructed and proved that they converged to weak solutions of Eq. (3). Escher *et al.* [6] discussed several qualitative properties of the DP equation. The existence and uniqueness of global weak solutions for Eq. (3) have been established provided that the initial data satisfy appropriate conditions in [6].

In fact, the KdV equation modeling weakly nonlinear unidirectional long waves and the CH equation modeling the unidirectional shallow water waves have been extensively investigated (see [7–11]). For other approaches to study related partial differential equations, the reader is referred to [12–20] and the references therein.

In this paper, assuming that the initial value $u(0, x)$ of Eq. (4) belongs to $H^s(R)$ ($s > \frac{3}{2}$), we will prove the existence and uniqueness of the local solution for Eq. (4) in the space $C([0, \infty); H^s(R)) \cap C^1([0, \infty); H^{s-1}(R))$ by using the Kato theorem (see [20]) and use the approaches presented in Kruzkov [19] to establish the $L^1(R)$ local stability of the solution for this nonlinear equation (4). From our knowledge, Eq. (4) has not been discussed in the literature.

In the following, we will give the conservation law in some case for Eq. (4) in Section 2. The proof of existence for the local strong solution and its stability to Eq. (4) are given in Section 3 and Section 4, respectively. For simplicity, we let c denote any positive constants.

2 Preliminaries

Applying the operator $\Lambda^{-2} = (1 - \beta^2 \partial_x^2)^{-1}$ to Eq. (4), we have

$$u_t + (\alpha_0 + b_1 u)u_x + b_2 \Lambda^{-2} \partial_x u^2 = 0, \tag{5}$$

where $b_1 = \frac{\alpha_2}{\beta^2}$, $b_2 = -(\alpha_1 + \frac{\alpha_2}{2\beta^2})$.

Here we consider the Cauchy problem of Eq. (4)

$$\begin{cases} u_t + \alpha_0 u_x - \beta^2(\alpha_0 u_{xxx} + u_{txx}) = 2\alpha_1 u u_x + 3\alpha_2 u_x u_{xx} + \alpha_2 u u_{xxx}, \\ u(0, x) = u_0(x), \end{cases} \tag{6}$$

which is equivalent to the problem

$$\begin{cases} u_t + (\alpha_0 + b_1 u)u_x + b_2 \Lambda^{-2} \partial_x u^2 = 0, \\ u(0, x) = u_0(x). \end{cases} \tag{7}$$

Lemma 2.1 *Let* $y_1 = u - \beta^2 \partial_x^2 u$ *and* $\bar{y} = (1 - \beta^2 \partial_x^2)^{-1} u$. *The solutions of Eq.* (4) *in case* $\alpha_1 = -\frac{\alpha_2}{2\beta^2}$ *conform with the following conservation law*:

$$\int_R y_1 \bar{y} \, dx = \int_R |\hat{u}(t, \xi)|^2 \, d\xi = \int_R |\hat{u}_0(\xi)|^2 \, d\xi, \tag{8}$$

where $u_0(x) = u(0, x)$, $\hat{u}(t, \xi)$ *is the Fourier transform of* $u(t, x)$ *with respect to* x.

Proof For $y_1 = u - \beta^2 \partial_x^2 u$, $\bar{y} = (1 - \beta^2 \partial_x^2)^{-1} u$, we have $\frac{\partial y_1}{\partial t} = u_t - \beta^2 \partial_x^2 u_{txx}$ and $u = \bar{y} - \beta^2 \bar{y}_{xx}$. If $\alpha_1 = -\frac{\alpha_2}{2\beta^2}$, it has

$$
\begin{aligned}
\frac{d}{dt} \int_R y_1 \bar{y} \, dx &= 2 \int_R \frac{\partial y_1}{\partial t} \bar{y} \, dx \\
&= 2 \int_R \left(-\alpha_0 u_t + \alpha_0 \beta^2 u_{xxx} + 2\alpha_1 u u_x + 3\alpha_2 u_x u_{xx} + \alpha_2 u u_{xxx} \right) \bar{y} \, dx \\
&= 2 \int_R \alpha_0 \left(u \bar{y}_x - \beta^2 u \bar{y}_{xxx} \right) - \left(\alpha_1 u^2 \bar{y}_x + \frac{\alpha_2}{2} u^2 \bar{y}_{xxx} \right) dx \\
&= 2 \int_R \left(\alpha_0 u - \alpha_1 u^2 \right) u_x \, dx \\
&= 0.
\end{aligned}
$$

This completes the proof of Lemma 2.1. $\qquad\square$

3 Existence of the local solution for problem (6)

Firstly, we need to state some notions. Let $H^s(R)$ (where s is a real number) denote the Sobolev space with the norm defined by

$$
\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2\right)^s \left| \hat{h}(t, \xi) \right|^2 \right)^{\frac{1}{2}} < \infty,
$$

where $\hat{h}(t, \xi) = \int_R e^{-ix\xi} h(t, x) \, dx$. For $T > 0$ and $s \geq 0$, we let $C([0, \infty); H^s(R))$ denote the Fréchet space of all continuous H^s-valued functions on $[0, T)$.

We introduce the abstract quasi-linear evolution equation

$$
\begin{cases}
\frac{d\phi}{dt} + Q(\phi)\phi = g(\phi), & t \geq 0, \\
\phi(0) = \phi_0.
\end{cases}
\tag{9}
$$

Let X and Y be Hilbert spaces, where Y is continuously and densely embedded in X, and $W : Y \to X$ be a topological isomorphism. We define $L(Y, X)$ to be the space of all bounded linear operators from Y to X. We denote $L(X, X)$ by $L(X)$. Note that ρ_1, ρ_2, ρ_3 and ρ_4 in the following depend on $\max\{\|y\|_Y, \|z\|_Y\}$.

(I) $Q(y) \in L(Y, X)$ for $y \in X$ with

$$
\left\| \left(Q(y) - Q(z) \right) w \right\|_X \leq \rho_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y,
$$

$Q(y) \in G(X, 1, \alpha)$ $(\alpha > 0)$ and uniformly on bounded sets in Y.

(II) $WQ(y)W^{-1} = Q(y) + \bar{Q}(y)$, in which $\bar{Q} \in L(X)$ is bounded and uniformly on bounded sets in Y and

$$
\left\| \left(\bar{Q}(y) - \bar{Q}(z) \right) w \right\|_X \leq \rho_2 \|y - z\|_X \|w\|_Y, \quad y, z \in Y, w \in X.
$$

(III) $g : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y and satisfies

$$
\left\| g(y) - g(z) \right\|_Y \leq \rho_3 \|y - z\|_Y, \quad y, z \in Y,
$$

$$
\left\| g(y) - g(z) \right\|_X \leq \rho_4 \|y - z\|_X, \quad y, z \in Y.
$$

Kato theorem ([20]) *Assume that* (I), (II) *and* (III) *hold. If* $\phi_0 \in Y$, *there is a maximal* $T > 0$ *depending only on* $\|\phi_0\|_Y$ *and a unique solution* u *to problem* (9) *such that*

$$\phi = \phi(\cdot, \phi_0) \in C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

Moreover, the map $\phi_0 \rightarrow \phi(\cdot, \phi_0)$ is a continuous map from Y to the space $C([0, T); Y) \cap C^1([0, T); X)$.

For problem (7), we set $Q(u) = (\alpha_0 + b_1 u)\partial_x$, $Y = H^s(R)$, $X = H^{s-1}(R)$, $g(u) = -b_2\partial_x\Lambda^{-2}u^2$ and $W = \Lambda$. Then we will verify that $Q(u)$ and $g(u)$ satisfy conditions (I)-(III). We cite several conclusions presented in [11].

Lemma 3.1 *The operator* $Q(u) = (\alpha_0 + b_1 u)\partial_x$ *with* $\phi \in H^s(R)$ $(s > \frac{3}{2})$ *belongs to* $G(H^{s-1}, 1, \alpha)$.

Lemma 3.2 *For* $u, z, w \in H^s(R)$ *with* $s > \frac{3}{2}$, $Q(\phi) \in L(H^s(R), H^{s-1}(R))$, *it holds that*

$$\big\|\big(Q(u) - Q(z)\big)w\big\|_{H^{s-1}} \leq \rho_1 \|u - z\|_{H^{s-1}} \|w\|_{H^s}.$$

Lemma 3.3 *For* $u, z \in H^s(R)$ *and* $w \in H^{s-1}(R)$ $(s > \frac{3}{2})$, *it holds that* $\bar{Q}(u) = [\Lambda, (1 - b_1 u)\partial_x]\Lambda^{-1} \in L(H^{s-1})$ *and*

$$\big\|\big(\bar{Q}(u) - \bar{Q}(z)\big)w\big\|_{H^{s-1}} \leq \rho_2 \|u - z\|_{H^s} \|w\|_{H^{s-1}}.$$

Lemma 3.4 ([20]) *Let* r *and* q *be real numbers such that* $-r < q \leq r$. *Then*

$$\|u_1 u_2\|_{H^q} \leq c\|u_1\|_{H^r}\|u_2\|_{H^q} \quad \text{if } r > \frac{1}{2},$$

$$\|u_1 u_2\|_{H^{r+q-\frac{1}{2}}} \leq c\|u_1\|_{H^r}\|u_2\|_{H^q} \quad \text{if } r < \frac{1}{2}.$$

Lemma 3.5 *Let* $u, z \in H^s(R)$ *with* $s > \frac{3}{2}$ *and* $g(u) = -b_2\partial_x\Lambda^{-2}u^2$. *Then* g *is bounded on bounded sets in* H^s *and satisfies*

$$\big\|g(u) - g(z)\big\|_{H^s} \leq \rho_3 \|u - z\|_{H^s}, \tag{10}$$

$$\big\|g(u) - g(z)\big\|_{H^{s-1}} \leq \rho_4 \|u - z\|_{H^{s-1}}. \tag{11}$$

Proof For $s > \frac{3}{2}$, we have $\|u\|_{L^\infty} \leq c\|u\|_{H^s}$ and $\|u\|_{H^{s-1}} \leq c\|u\|_{H^s}$. Applying the algebra property of $H^s(R)$ and Lemma 3.4, we get

$$\begin{aligned}
\big\|g(u) - g(z)\big\|_{H^s} &\leq |b_2| \big\|\partial_x\Lambda^{-2}\big(u^2 - z^2\big)\big\|_{H^s} \\
&\leq c\big\|u^2 - z^2\big\|_{H^{s-1}} \\
&\leq c\|u - z\|_{H^{s-1}} \\
&\leq \rho_3 \|u - z\|_{H^s},
\end{aligned}$$

which completes the proof of (10). Similarly, we get (11). $\square$

Now we give the following theorem.

Theorem 3.6 *Let $u_0(x) \in H^s(R)$ with $s > \frac{3}{2}$. There exists $T > 0$ depending on $\|u_0\|_{H^s(R)}$ such that problem (6) or (7) has a unique solution $u(t,x) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{s-1}(R))$.*

Proof Using Lemmas 3.1-3.3 and 3.5, we know that conditions (I), (II) and (III) hold. Applying the Kato theorem, we find that problem (6) or (7) has a unique local solution

$$u = u(t,x) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{s-1}(R)),$$

where $T > 0$ depends on $\|u_0\|_{H^s}$. □

Remark 3.7 Let $T > 0$ be described in Theorem 3.6. Using the Sobolev embedding theorem, we ensure the boundedness of solution $u(t,x)$ to problem (7) in the domain $[0,T) \times R$. Namely, provided that $u_0 \in H^s(R)$ with $s > \frac{3}{2}$, we have $\|u\|_{L^\infty(R)} \leq M_T$, where M_T is a positive constant.

4 L^1 Stability of the local solution for problem (6)

Letting $f'(u) = \alpha_0 + b_1 u$, $\Psi(t,x,u) = b_2 \Lambda^{-2} \partial_x u^2$ in the first equation of (7), we get

$$\begin{cases} u_t + f'(u)u_x + \Psi(t,x,u) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{12}$$

Assume that $u_1(t,x)$ and $u_2(t,x)$ are solutions of problem (12) in the domain $[0,T) \times R$ with initial functions $u_{10}(x)$ and $u_{20}(x) \in H^s(R)$ ($s > \frac{3}{2}$). Here we give several lemmas.

Lemma 4.1 *Let $u(t,x)$ be the solution of problem (12) and $u_0(x) \in H^s(R)$ with $s > \frac{3}{2}$. Then $\|\Psi(t,x,u)\|_{L^\infty(R)} \leq M_T$ and*

$$\|\Psi(t,x,u)\|_{L^\infty(R)} \leq cM_T^2,$$

where positive constant c depends on b_2, β, $\|u_0\|_{L^\infty}$.

Proof We have

$$\begin{aligned}
|\Psi(t,x,u)| &= |b_2 \Lambda^{-2} \partial_x u^2| \\
&\leq |b_2| \left| \frac{1}{2\beta} \int_R e^{-\frac{1}{\beta}|x-y|} \operatorname{sign}(y-x) u^2 \, dy \right| \\
&\leq cM_T^2.
\end{aligned}$$

Applying Remark 3.7 and the integral $\int_R e^{-\frac{1}{\beta}|x-y|} \, dy = 2\beta$, we complete the proof. □

Lemma 4.2 *Assume that $u_1(t,x)$ and $u_2(t,x)$ are solutions of problem (12) in the domain $[0,T) \times R$ with initial functions $u_{10}(x)$ and $u_{20}(x) \in H^s(R)$ ($s > \frac{3}{2}$), respectively. Then*

$$\int_{-\infty}^{+\infty} |\Psi(t,x,u_1) - \Psi(t,x,u_2)| \, dx \leq c \int_{-\infty}^{+\infty} |u_1 - u_2| \, dx, \tag{13}$$

where $c > 0$ depends on b_2, β, $\|u_{10}\|_{L^\infty(R)}$, $\|u_{20}\|_{L^\infty(R)}$ and T.

Proof Using the property of the operator Λ^{-2} and Remark 3.7, we get

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} \left| \Psi(t,x,u_1) - \Psi(t,x,u_2) \right| dx \\
&= \int_{-\infty}^{+\infty} \left| b_2 \partial_x \Lambda^{-2} \left(u_1^2 - u_2^2\right) \right| dx \\
&\leq c \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \left| e^{-\frac{1}{\beta}|x-y|} \operatorname{sign}(y-x) \left(u_1^2 - u_2^2\right) \right| dy \\
&\leq c \int_{-\infty}^{+\infty} \left| u_1^2 - u_2^2 \right| dx \int_{-\infty}^{+\infty} e^{-\frac{1}{\beta}|x-y|} dy \\
&\leq c \int_{-\infty}^{+\infty} |u_1 - u_2| \, dx,
\end{aligned}
$$

in which we apply the Tonelli theorem to complete the proof. □

We introduce a function $\delta(\sigma)$ which is infinitely differential on $(-\infty, +\infty)$ and $\delta(\sigma) \geq 0$, $\delta(\sigma) \equiv 0$ for $|\sigma| \geq 1$, $\int_{-\infty}^{+\infty} \delta(\sigma) \, d\sigma = 1$. Let $\delta_\varepsilon(\sigma) = \delta(\varepsilon^{-1}\sigma) / \varepsilon$, where ε is an arbitrary positive constant. It is found that $\delta_\varepsilon(\sigma) \in C_0^\infty(-\infty, +\infty)$ and

$$
\begin{cases}
\delta_\varepsilon(\sigma) \geq 0, & \delta_\varepsilon(\sigma) = 0 \quad \text{for } |\sigma| \geq \varepsilon, \\
|\delta_\varepsilon(\sigma)| \leq \frac{c}{\varepsilon}, & \int_{-\infty}^{+\infty} \delta_\varepsilon(\sigma) \, d\sigma = 1.
\end{cases}
\tag{14}
$$

Let the function $\phi(x)$ be defined and locally integrable on $(-\infty, +\infty)$. Set $\phi^\varepsilon(x)$ to denote the approximation function of $\phi(x)$ as

$$
\phi^\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \delta\left(\frac{x-y}{\varepsilon}\right) \phi(y) \, dy.
\tag{15}
$$

We call x_0 a Lebesgue point of the function $\phi(x)$ if

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left| \phi(x) - \phi(x_0) \right| dx = 0.
$$

At any Lebesgue point x_0, we get

$$
\lim_{\varepsilon \to 0} \phi^\varepsilon(x_0) = \phi(x_0).
$$

Since the set of points which are not Lebesgue points of $\phi(x)$ has measure zero, we have $\phi^\varepsilon(x) \to \phi(x)$ as $\varepsilon \to 0$ almost everywhere.

For any $T_0 \in [0, T)$, we denote the band $\{(t,x) : [0, T_0] \times R\}$ by π_{T_0}. Let $K_r = \{x : |x| \leq r\}$ and

$$
\Pi = \left\{ (t,x,\tau,y) : \left| \frac{t-\tau}{2} \right| \leq \varepsilon, \rho \leq \frac{t+\tau}{2} \leq T_0 - \rho, \left| \frac{x-y}{2} \right| \leq \varepsilon, \left| \frac{x+y}{2} \right| \leq r - \rho \right\},
$$

where $r > 0$, $\rho > 0$.

Lemma 4.3 ([19]) *Let the function $\phi(t,x)$ be bounded and measurable in cylinder $[0, T_0] \times K_r$. If for any $\rho \in (0, \min[r, T_0])$ and any $\varepsilon \in (0, \rho)$, the function*

$$
V_\varepsilon = \frac{1}{\varepsilon^2} \iiiint_\Pi \left| \phi(t,x) - \phi(\tau,y) \right| dt \, dx \, d\tau \, dy
$$

satisfies

$$\lim_{\varepsilon \to 0} V_\varepsilon = 0.$$

Lemma 4.4 ([19]) *If $\frac{\partial \Phi(\phi)}{\partial \phi}$ is bounded, then the function $H(\phi_1, \phi_2) = \text{sign}(\phi_1 - \phi_2)(\Phi(\phi_1) - \Phi(\phi_2))$ satisfies the Lipschitz condition in ϕ_1 and ϕ_1.*

We state the concept of a characteristic cone. Let T be described in Theorem 3.6 and $\|\phi\|_{L^\infty(R)} \leq M_T$. For any $T_0 \in [0, T)$ and $R_0 > 0$, we define

$$N > \max_{(t,x) \in [0,T_0] \times K_{R_0}} |f'(\phi)|.$$

Let Ω represent the cone $\{(t, x) : |x| \leq R_0 - Nt, 0 \leq t \leq T_1 = \min(T_0, R_0 N^{-1})\}$ and S_τ designate the cross-section of the cone Ω by the plane $t = \tau$, $\tau \in [0, T_1]$.

Lemma 4.5 *Let $u(t, x)$ be the solution of problem (12) on π_{T_0}, $q(t, x) \in C_0^\infty(\pi_{T_0})$. It holds that*

$$\iint_{\pi_{T_0}} \big\{ |u - k| q_t + \text{sign}(u - k)\big[f(u) - f(k)\big] q_x$$

$$- \text{sign}(u - k) q(t, x) \Psi(t, x, u) \big\} \, dt \, dx = 0, \tag{16}$$

where k is an arbitrary constant.

Proof Suppose that $F(u)$ is a twice differential function. Multiplying the first equation of problem (12) by $F'(u)q(t, x)$ and integrating over π_{T_0}, we get

$$\iint_{\pi_{T_0}} \big\{ F'(u)qu_t + F'(u)qf'(u)u_x + F'(u)q\Psi(t, x, u) \big\} \, dt \, dx = 0. \tag{17}$$

The application of the method of integration by parts yields

$$\iint_{\pi_{T_0}} F'(u)qu_t \, dt \, dx = - \iint_{\pi_{T_0}} F'(u)q_t \, dt \, dx. \tag{18}$$

Since

$$\left(\int_k^u F'(z)f'(z) \, dz \right)' = F'(u)f'(u)u_x,$$

we have

$$\iint_{\pi_{T_0}} F'(u)f'(u)u_x q \, dt \, dx = - \iint_{\pi_{T_0}} \left(\int_k^u F'(z)f'(z)dz \right) q_x \, dt \, dx$$

$$= - \iint_{\pi_{T_0}} \Big\{ F'(u)\big[f(u) - f(k)\big]$$

$$- \int_k^u F''(z)\big[f(z) - f(k)\big] dz \Big\} q_x \, dt \, dx. \tag{19}$$

Substitute Eqs. (18) and (19) into Eq. (17). Let $F^\varepsilon(u)$ be an approximation of the function $F(u) = |u - k|$. When $\varepsilon \to 0$, $F^\varepsilon(u) \to F(u)$, we obtain Eq. (16). $\qquad\square$

Set function $q(t,x) \in C_0^\infty(\pi_{T_0})$, $q(t,x) \equiv 0$ outside the cylinder $\varpi = \{(t,x)\} = [\rho, T_0 - 2\rho] \times K_{r-2\rho}$, where $K_{r-2\rho} = \{|x| : |x| \le r - 2\rho\}$, $r > 0$, $0 < 2\rho < \min(T_0, r)$. Now we give the proof of the local stability for problem (6) or (7).

Theorem 4.6 *Assume that $u_1(t,x)$ and $u_2(t,x)$ are two local strong solutions of problem (6) or (7) with initial data $u_0(x), v_0(x) \in L^1(R) \cap H^s(R)$ ($s > \frac{3}{2}$), respectively. Let $T > 0$ be the maximum existence time of $u_1(t,x)$ and $u_2(t,x)$. For any $t \in [0, T)$, it holds that*

$$\big\| u_1(t,x) - u_2(t,x) \big\|_{L^1} \le e^{ct} \big\| u_{10}(x) - u_{20}(x) \big\|_{L^1},$$

where c is a positive constant depending on $\|u_{10}\|_{L^\infty(R)}$ and $\|u_{20}\|_{L^\infty(R)}$.

Proof We define

$$P(t,x,\tau,y) = q\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \delta_\varepsilon\left(\frac{t-\tau}{2}, \frac{x-y}{2}\right) = q(\cdots)\lambda_\varepsilon(\dot{:}), \tag{20}$$

in which $(\cdots) = (\frac{t+\tau}{2}, \frac{x+y}{2})$, $(\dot{:}) = (\frac{t-\tau}{2}, \frac{x-y}{2})$. Thus, we obtain

$$P_t + P_\tau = q_t(\cdots)\lambda_\varepsilon(\dot{:}), \qquad P_x + P_y = q_x(\cdots)\lambda_\varepsilon(\dot{:}).$$

Using Lemma 4.5 and the Kruzkov's device of doubling the variables [19], we get

$$\iiiint_{\pi_{T_0} \times \pi_{T_0}} \big\{ \big| u_1(t,x) - u_2(\tau,y) \big| P_t + \operatorname{sign}\big(u_1(t,x) - u_2(\tau,y)\big)$$
$$\times \big[f\big(u_1(t,x)\big) - f\big(u_2(\tau,y)\big) \big] P_x - \operatorname{sign}\big(u_1(t,x) - u_2(\tau,y)\big)$$
$$\times \Psi\big(t,x,u_1(t,x)\big) P \big\} \, dt \, dx \, d\tau \, dy = 0. \tag{21}$$

Similarly, we have

$$\iiiint_{\pi_{T_0} \times \pi_{T_0}} \big\{ \big| u_2(\tau,y) - u_1(t,x) \big| P_\tau + \operatorname{sign}\big(u_2(\tau,y) - u_1(t,x)\big)$$
$$\times \big[f\big(u_2(\tau,y)\big) - f\big(u_1(t,x)\big) \big] P_y - \operatorname{sign}\big(u_2(\tau,y) - u_1(t,x)\big)$$
$$\times \Psi\big(\tau,y,u_2(\tau,y)\big) P \big\} \, dt \, dx \, d\tau \, dy = 0. \tag{22}$$

Adding (21) and (22), we obtain

$$0 \le \iiiint_{\pi_{T_0} \times \pi_{T_0}} \big\{ \big| u_1(t,x) - u_2(\tau,y) \big| q_t \lambda_\varepsilon$$
$$+ \operatorname{sign}\big(u_1(t,x) - u_2(\tau,y)\big)$$
$$\times \big[f\big(u_1(t,x)\big) - f\big(u_2(\tau,y)\big) \big] q_x \lambda_\varepsilon \big\} \, dt \, dx \, d\tau \, dy$$

$$+ \left| \iiiint_{\pi_{T_0} \times \pi_{T_0}} \left\{ \text{sign}\big(u_1(t,x) - u_2(\tau,y)\big) \right. \right.$$

$$\left. \left. \times \big[\Psi\big(t,x,u_1(t,x)\big) - \Psi\big(\tau,y,u_2(\tau,y)\big) \big] q\lambda_\varepsilon \right\} dt \, dx \, d\tau \, dy \right|. \tag{23}$$

We note that the first two terms of the integrand of (23) have the form

$$G_\varepsilon = G\big(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\big)\lambda_\varepsilon(\dot{:}), \tag{24}$$

where G satisfies the Lipschitz condition in all its variables. Then

$$\iiiint_{\pi_{T_0} \times \pi_{T_0}} G_\varepsilon \, dt \, dx \, d\tau \, dy$$

$$= \iiiint_{\pi_{T_0} \times \pi_{T_0}} G\big(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\big)\lambda_\varepsilon \, dt \, dx \, d\tau \, dy$$

$$= \iiiint_{\pi_{T_0} \times \pi_{T_0}} \big\{ G\big(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\big)$$

$$\quad - G\big(t,x,t,x,u_1(t,x),u_2(t,x)\big) \big\} \lambda_\varepsilon \, dt \, dx \, d\tau \, dy$$

$$\quad + \iiiint_{\pi_{T_0} \times \pi_{T_0}} G\big(t,x,t,x,u_1(t,x),u_2(t,x)\big)\lambda_\varepsilon \, dt \, dx \, d\tau \, dy$$

$$= A_{11}(\varepsilon) + A_{12}.$$

As $G_\varepsilon = 0$ outside the region Π, applying the estimate $|\lambda_\varepsilon(\dot{:})| \le \frac{c}{\varepsilon^2}$ and Lemma 4.4, we get

$$\big|A_{11}(\varepsilon)\big| \le c\left[\varepsilon + \frac{1}{\varepsilon^2} \iiiint_\Pi \big|u_1(t,x) - u_2(\tau,y)\big| \, dt \, dx \, d\tau \, dy \right],$$

where c is a positive constant independent of ε. Using Lemma 4.3, we know $A_{11}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For the term A_{12}, we substitute $t = \eta_1$, $\frac{t-\tau}{2} = \mu_1$, $x = \eta_2$, $\frac{x-y}{2} = \mu_2$. Combining with the identity

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{+\infty} \lambda_\varepsilon(\mu_1,\mu_2) \, d\mu_1 \, d\mu_2 = 1,$$

we obtain

$$A_{12} = 2^2 \iint_{\pi_{T_0} \times \pi_{T_0}} G\big(\eta_1,\eta_2,\eta_1,\eta_2,u_1(\eta_1,\eta_2),u_2(\eta_1,\eta_2)\big)$$

$$\quad \times \left(\int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{+\infty} \lambda_\varepsilon(\mu_1,\mu_2) \, d\mu_1 \, d\mu_2 \right) d\eta_1 \, d\eta_2$$

$$= 4 \iint_{\pi_{T_0}} G\big(t,x,t,x,u_1(t,x),u_2(t,x)\big) \, dt \, dx.$$

Thus, we have

$$\lim_{\varepsilon \to 0} \iiiint_{\pi_{T_0} \times \pi_{T_0}} G_\varepsilon \, dt \, dx \, d\tau \, dy = 4 \iint_{\pi_{T_0}} G\big(t,x,t,x,u_1(t,x),u_2(t,x)\big) \, dt \, dx. \tag{25}$$

Similarly, the integrand of the third term in (23) can be represented as

$$B_\varepsilon = \text{sign}\big(u_1(t,x) - u_2(\tau,y)\big)\big[\Psi\big(t,x,u_1(t,x)\big) - \Psi\big(\tau,y,u_2(\tau,y)\big)\big]q\lambda_\varepsilon$$

$$= B\big(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\big)\lambda_\varepsilon(\dot{:}). \tag{26}$$

Then

$$\iiiint_{\pi_{T_0}\times\pi_{T_0}} B_\varepsilon \, dt\, dx\, d\tau\, dy$$

$$= \iiiint_{\pi_{T_0}\times\pi_{T_0}} \big\{B\big(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\big)$$

$$- B\big(t,x,t,x,u_1(t,x),u_2(t,x)\big)\lambda_\varepsilon\big\}\, dt\, dx\, d\tau\, dy$$

$$+ \iiiint_{\pi_{T_0}\times\pi_{T_0}} B\big(t,x,t,x,u_1(t,x),u_2(t,x)\big)\lambda_\varepsilon \, dt\, dx\, d\tau\, dy$$

$$= A_{21}(\varepsilon) + A_{22}.$$

Applying Lemma 4.4, we get

$$\big|A_{21}(\varepsilon)\big| \le c\left[\varepsilon + \frac{1}{\varepsilon^2}\iiiint_\Pi \big|u_1(t,x) - u_2(\tau,y)\big|\, dt\, dx\, d\tau\, dy\right].$$

By using Lemma 4.3, it yields $A_{21}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Repeating the steps as before, we have

$$A_{22} = 4\iint_{\pi_{T_0}} B\big(t,x,t,x,u_1(t,x),u_2(t,x)\big)\, dt\, dx. \tag{27}$$

From (23) to (27), we get

$$\iint_{\pi_{T_0}} \big\{\big|u_1(t,x) - u_2(t,x)\big|q_t + \text{sign}\big(u_1(t,x) - u_2(t,x)\big)\big[f(u_1) - f(u_2)\big]q_x\big\}\, dt\, dx$$

$$+ \left|\iint_{\pi_{T_0}} \text{sign}\big(u_1(t,x) - u_2(t,x)\big)\big[\Psi(t,x,\phi) - \Psi(t,x,\psi)\big]q\, dt\, dx\right| \ge 0. \tag{28}$$

Set

$$l(t) = \int_{-\infty}^{+\infty} \big|u_1(t,x) - u_2(t,x)\big|\, dx \tag{29}$$

and

$$v_\varepsilon(\sigma) = \int_{-\infty}^{\sigma} \delta_\varepsilon(z)\, dz. \tag{30}$$

Take two numbers $\rho, \tau \in (0, T_1)$ and $\rho < \tau$. In (28), we let

$$q(t,x) = \big[v_\varepsilon(t-\rho) - v_\varepsilon(t-\tau)\big]\chi(t,x), \quad \varepsilon < \min(\rho, T_1 - \tau), \tag{31}$$

in which

$$\chi(t,x) = \chi_\theta(t,x) = 1 - \nu_\theta\big(|x| + Nt - R_0 + \theta\big),$$ (32)

where θ is a small positive constant and $\chi(t,x) = 0$ outside the cone Ω. When $\theta \to 0$, $R_0 \to +\infty$, we observe that $\chi_\theta \to 1$. By the definition of the number N, we have

$$0 = \chi_t + N|\chi_x| \geq \chi_t + N\chi_x, \quad (t,x) \in \Omega.$$

According to (28)-(32), we get

$$\iint_{\pi_{T_0}} \big\{|u_1(t,x) - u_2(t,x)|\big[\delta_\varepsilon(t-\rho) - \delta_\varepsilon(t-\tau)\big]\chi_\theta(t,x)\big\}\,dt\,dx$$

$$+ \int_0^{T_1} dt \int_{-\infty}^{+\infty} \big\{|\Psi(t,x,u_1) - \Psi(t,x,u_2)|$$

$$\times \big[\nu_\varepsilon(t-\rho) - \nu_\varepsilon(t-\tau)\big]\chi_\theta(t,x)\big\}\,dx \geq 0.$$ (33)

For (33), sending $\theta \to 0$, $R_0 \to +\infty$ combining with Lemma 4.2, we obtain

$$\int_0^{T_1}\big[\delta_\varepsilon(t-\rho) - \delta_\varepsilon(t-\tau)\big]l(t)\,dt + c\int_0^{T_1}\big[\nu_\varepsilon(t-\rho) - \nu_\varepsilon(t-\tau)\big]l(t)\,dt \geq 0,$$ (34)

where c is independent of ε.

Applying the properties of the function δ_ε for $\varepsilon \leq \min(\rho, T_1 - \rho)$, we get

$$\left|\int_0^{T_1}\delta_\varepsilon(t-\rho)l(t) - l(\rho)\,dt\right| = \left|\int_0^{T_1}\delta_\varepsilon(t-\rho)\big[l(t) - l(\rho)\big]\,dt\right|$$

$$\leq \frac{c}{\varepsilon}\int_{\rho-\varepsilon}^{\rho+\varepsilon}|l(t) - l(\rho)|\,dt.$$

Consequently,

$$\int_0^{T_1}\delta_\varepsilon(t-\rho)l(t)dt \to l(\rho) \quad \text{as } \varepsilon \to 0.$$ (35)

Now set

$$K(\rho) = \int_0^{T_1}\nu_\varepsilon(t-\rho)l(t)\,dt = \int_0^{T_1} dt \int_{-\infty}^{t-\rho}\delta_\varepsilon(\sigma)l(t)\,d\sigma.$$

It is seen that

$$K'(\rho) = -\int_0^{T_1}\delta_\varepsilon(t-\rho)l(t)\,dt.$$

Letting $\varepsilon \to 0$, we derive

$$K'(\rho) \to -l(\rho)$$

and

$$K(\rho) \to K(0) - \int_0^\rho l(t)\,dt, \qquad K(\tau) \to K(0) - \int_0^\tau l(t)\,dt.$$

Hence, we get

$$K(\rho) - K(\tau) \to \int_\rho^\tau l(t)\,dt \quad \text{as } \varepsilon \to 0. \tag{36}$$

Applying (34)-(36), we obtain the inequality

$$l(\rho) + c \int_\rho^\tau l(t)\,dt \geq l(\tau).$$

As $\rho \to 0$, $\tau \to t$, we get the following result:

$$\int_{-\infty}^{+\infty} \left| u_1(0,x) - u_2(0,x) \right| dx + c \int_0^t dt \int_{-\infty}^{+\infty} \left| u_1(t,x) - u_2(t,x) \right| dx$$
$$\geq \int_{-\infty}^{+\infty} \left| u_1(t,x) - u_2(t,x) \right| dx.$$

Finally, we complete the proof of Theorem 4.6 by using the Gronwall inequality. □

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