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Renormalized weak solutions to the three-dimensional steady compressible magnetohydrodynamic equations

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Abstract

We are concerned with the Dirichlet problem of the three-dimensional steady viscous compressible magnetohydrodynamic (MHD) equations. It is proved that for any specific heat ratio $\gamma > 1$, the Dirichlet problem of the steady compressible MHD equations on a bounded domain $\Omega \subset \mathbb{R}^3$ admits a renormalized weak solution. Our method relies upon the weighted estimates of both pressure and kinetic energy for the approximate system, and the method of weak convergence developed by Lions and Feireisl.

MSC: 35Q60; 35J47

Keywords: steady compressible MHD equations; renormalized weak solutions; Dirichlet problem

1 Introduction

We consider the steady compressible magnetohydrodynamic equations in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\operatorname{div}\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P(\rho) - (\nabla\times\mathbf{H})\times\mathbf{H} = \rho\mathbf{f}, \tag{1.2}$$

$$-\nu\Delta\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = \mathbf{0}, \tag{1.3}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{1.4}$$

where $\rho \geq 0$, $\mathbf{u} = (u^1, u^2, u^3)$, and $P(\rho) = \rho^\gamma$ with $\gamma > 1$ being the specific heat ratio are the fluid density, velocity, and pressure, respectively. $\mathbf{H} = (H^1, H^2, H^3)$ is the magnetic field, $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The constant viscosity coefficients μ and λ satisfy

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{1.5}$$

$\mathbf{f}$ is a given vector field which models an outer force density.

Equations (1.1)-(1.4) are supplemented the following boundary conditions:

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{H} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\mathbf{g}$ is a given function in Ω. Moreover, the total mass is prescribed,

$$\int_{\Omega} \rho(x)\, dx = M > 0. \tag{1.7}$$

There is huge literature on the studies about global existence of renormalized weak solutions of steady compressible flows. The important progress in the spatial three-dimensional case is due to the seminal work of Lions [1], where he obtained the existence of renormalized weak solutions of the Navier-Stokes equations for $\gamma > \frac{5}{3}$, *i.e.*, (1.1)-(1.2) with $\mathbf{H} = \mathbf{0}$. However, as is well known, common fluids and gases in normal conditions (*i.e.*, no high densities or high temperatures) are often well described by the ideal gas model, *i.e.*, (1.1)-(1.2) with $\mathbf{H} = \mathbf{0}$ for $\gamma > 1$. Furthermore, it is possible to deduce from the kinetic theory of gases that $\gamma = \frac{5}{3}$ (for a monatomic gas). Hence, the most interesting region for physical applications is $\gamma \in (1, \frac{5}{3}]$. Later, by combining Lions' compactness framework of renormalized solutions and Feireisl's oscillation defect measure on nonsteady compressible Navier-Stokes equations [2], Novo and Novotný [3] improved Lions' result to $\gamma > \frac{3}{2}$ for the potential force (*i.e.*, $\mathbf{f} = \nabla\phi$). As emphasized in many papers (refer to [4–6] for instance), the condition on γ comes from the integrability of the density ρ in $L^p(\Omega)$. The higher integrability of ρ has, the smaller γ can be allowed. By deriving a new weighted estimate of the pressure, the improved estimates of density have been suggested independently by Plotnikov and Sokolowski [7–9] and by Frehse *et al.* [10]. Using L^∞ estimates for the inverse Laplacian of the pressure together with the nonlinear potential theory, Březina and Novotný [11] proved existence of weak solutions with $\gamma > \frac{1+\sqrt{13}}{3} \approx 1.53$ for space periodic boundary conditions to avoid the lack of estimates near the boundary. Recently, Frehse *et al.* [12] treated $\gamma > \frac{4}{3}$ for Dirichlet boundary conditions, where they relied on the momentum equation by a test function which provides the potential estimates for pressure, and by a bootstrap argument different from that used in [11]. Then, by obtaining weighted estimates for both the pressure P and the kinetic energy $\rho|\mathbf{u}|^2$, Jiang and Zhou [5] showed the existence of spatially periodic weak solutions to the three-dimensional steady compressible Navier-Stokes equations for any $\gamma > 1$. It is worth noticing that the Dirichlet problem in [12] has been revisited very recently by Plotnikov and Weigant [13] for $\gamma > 1$. There are also some studies on existence results for steady compressible full Navier-Stokes equations, refer to [14–16] and references therein.

The relevant background of magnetohydrodynamic fluids can be found in [17–22] and references cited therein. The steady compressible magnetohydrodynamic system is investigated in [6] with $\gamma > 1$ for the periodic case. Recently, Yan [23] considered the three-dimensional full magnetohydrodynamic equations under slip boundary conditions for $\gamma > \frac{4}{3}$. In fact, one of the important restrictions to the value of γ is due to the *a priori* estimates. It is a natural and interesting problem to investigate the existence of weak solutions to the Dirichlet problem (1.1)-(1.7) in dimension three. In fact, this is the main aim of the present paper.

Before stating the main result, we first explain the notations and conventions used throughout this paper. For $1 \leq p \leq \infty$ and $k \geq 1$, the standard Lebesgue and Sobolev spaces are defined as follows:

$$L^p = L^p(\Omega), \qquad W^{k,p} = W^{k,p}(\Omega), \qquad H^k = W^{k,2}.$$

To simplify the notation, in what follows, we do not distinguish between function spaces for scalar and vector valued functions; *e.g.* both $L^p(\Omega)$ and $L^p(\Omega;\mathbb{R}^3)$ are denoted $L^p(\Omega)$. Generic constants are denoted by C, their values may vary in the same formula or in the same line.

Next, we give the definition of renormalized weak solutions to the steady MHD equations (1.1)-(1.4) as follows.

Definition 1.1 By a renormalized weak solution of the system (1.1)-(1.7) we mean a triple $(\rho,\mathbf{u},\mathbf{H}) \in L^\gamma(\Omega) \times W_0^{1,2}(\Omega) \times W^{1,2}(\Omega)$ such that:

- $\rho \geq 0$ a.e. in Ω, $\int_\Omega \rho(x)\,dx = M$.
- $\operatorname{div}\mathbf{H} = 0$ in Ω.
- Equations (1.1)-(1.3) hold in the sense of a distribution.
- The mass equation (1.1) holds in the sense of a renormalized solution, *i.e.*,

$$\operatorname{div}\big[b(\rho)\mathbf{u}\big] + \big[b'(\rho)\rho - b(\rho)\big]\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.8}$$

for any $b \in C^1(\mathbb{R})$ such that $b'(z) = 0$ when z is big enough.

Our main result can be stated as follows.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* C^2 *boundary. Assume that* $\mathbf{f} \in L^\infty(\Omega)$ *and* $\mathbf{g} \in H^3(\Omega)$ *with* $\operatorname{div}\mathbf{g} = 0$. *Then for any* $\gamma > 1$, *there exists a renormalized weak solution to the problem* (1.1)-(1.7) *in the sense of Definition* 1.1.

We now make some comments on the key ingredients of the analysis in this paper. The proof of Theorem 1.1 is based on the elaborate a priori uniform estimates of the approximate solutions and the weak convergence method in the framework of Lions [1]. First, inspired by [6, 24], we can construct an approximate scheme to the MHD system (1.1)-(1.7). Then we use a bootstrapping argument to obtain the *a priori* uniform estimates for the approximate solutions $(\rho_\delta,\mathbf{u}_\delta,\mathbf{H}_\delta)$ for any $\gamma > 1$ in the framework of [13]. However, compared with the ones in [1, 13], the main difficulty in the present paper is caused by the magnetic field and its coupling interaction with the fluid variables. To overcome this difficulty, we need some careful analysis to recover all the *a priori* estimates. To this end, we shall consider the momentum equations and the magnetic field equations together. To pass to the limit and obtain the existence of a weak solution, we cannot directly use the arguments in [1], since $\rho_\delta \in L^p(\Omega)$ $(p > \frac{3}{2})$ is required in [1] and this is not the case here. In fact, here we only have $\rho_\delta \in L^{\gamma s}(\Omega)$ with some $s > 1$ being very close to 1 when γ is close to 1. Instead, we use the modified method in [5] to get the existence of a weak solution.

The rest of this paper is organized as follows. In Section 2, we collect some known facts that will be needed in later analysis. In Section 3, we construct a sequence of approximate solutions $(\rho_\delta,\mathbf{u}_\delta,\mathbf{H}_\delta)$. In Section 4, the estimates independent of the parameter δ are obtained. Finally, we give a brief proof of our main result in Section 5.

2 Preliminaries

In this section we shall enumerate some auxiliary lemmas used in this paper.

We begin with an auxiliary function φ. To this end, we denote the distance function $d(x)$ by

$$d(x) \triangleq \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{for } x \in \overline{\Omega}, \\ -\operatorname{dist}(x, \partial\Omega) & \text{for } x \in \mathbb{R}^3 \backslash \Omega. \end{cases} \tag{2.1}$$

For every $c > 0$, the symbols A_c and Ω_c stand for

$$A_c \triangleq \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\Omega) < c\}, \qquad \Omega_c \triangleq A_c \cap \Omega. \tag{2.2}$$

Then we have the following consequence, whose proof can be found in Chapter 14.6 of [25].

Lemma 2.1 *There is $t > 0$ depending only on Ω such that*

$$d \in C^2(\overline{A}_{2t}), \qquad |\nabla d(x)| = 1 \quad \text{in } \overline{A}_{2t}. \tag{2.3}$$

Furthermore, there exists a function $\varphi : \overline{A_{2t} \cup \Omega} \to \mathbb{R}$ satisfying:

- *$\varphi \in C^2(\overline{A_{2t} \cup \Omega})$, $\varphi(x) > 0$ in Ω and $\varphi = d(x)$ in A_{2t}.*
- *There is $k > 0$ such that $\varphi(x) > k$ in $\Omega \backslash \Omega_{2t}$.*

Lemma 2.2 ([5]) *Let $1 < p_1, p_2, p < \infty$, $p \le p_1$, and Ω be a bounded domain in $\mathbb{R}^3$. Suppose that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^{p_1}(\Omega), \tag{2.4}$$

$$g_n \to g \quad \text{strongly in } L^{p_2}(\Omega) \tag{2.5}$$

and

$$f_n g_n \quad \text{are uniformly bounded in } L^p(\Omega).$$

Then there is a subsequence of $f_n g_n$ (still denoted by $f_n g_n$), such that

$$f_n g_n \rightharpoonup fg \quad \text{weakly in } L^p(\Omega).$$

Remark 2.1 If (2.4)-(2.5) hold, combining Lemma 2.2 with Hölder's inequality, we have

$$f_n g_n \rightharpoonup fg \quad \text{weakly in } L^r(\Omega),$$

where $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{r}$.

The following Bogovskii lemma will be needed in Lemma 4.3; its proof can be found in [24].

Lemma 2.3 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $1 < p < \infty$. Then there exists a positive constant $C(p, \Omega)$ such that for any $f \in L^p(\Omega)$ with $\int_\Omega f(x)\,dx = 0$, there is a vector field $\boldsymbol{\phi} \in W^{1,p}(\Omega)$ satisfying*

$$\begin{cases} \operatorname{div} \boldsymbol{\phi} = f & \text{in } \Omega, \\ \boldsymbol{\phi} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

and

$$\|\boldsymbol{\phi}\|_{W^{1,p}} \le C(p,\Omega)\|f\|_{L^p}.$$

Finally, let $W_0^{1,2}(\Omega)$ denote the Sobolev space of elements belonging to $W^{1,2}(\Omega)$ with zero trace at the boundary $\partial\Omega$. Then the following result plays a key role in the proof of Proposition 4.1.

Lemma 2.4 ([13]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. Let $h \in L^2(\Omega)$ satisfy*

$$h \ge 0, \quad \int_\Omega \frac{h(x)}{|x-x_0|}\,dx \le K \quad \textit{for all } x_0 \in \Omega.$$

Then there is $C > 0$ depending only on Ω such that for all $\mathbf{u} \in W_0^{1,2}(\Omega)$,

$$\int_\Omega |\mathbf{u}|^2 h\,dx \le CK\|\mathbf{u}\|_{H^1}^2.$$

3 Approximation

In this section, we briefly explain how to construct an approximative system to the problem (1.1)-(1.7).

First, we consider the following approximative problem:

$$\alpha(\rho - h) + \operatorname{div}(\rho\mathbf{u}) - \varepsilon\Delta\rho = 0, \tag{3.1}$$

$$\alpha(\rho + h)\mathbf{u} - \mu\Delta\mathbf{u} - \tilde{\mu}\nabla\operatorname{div}\mathbf{u} + \frac{1}{2}\left[\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \rho\mathbf{u}\cdot\nabla\mathbf{u}\right]$$
$$\quad + \nabla P_\delta(\rho) - (\nabla\times\mathbf{H})\times\mathbf{H} = \rho\mathbf{f}, \tag{3.2}$$

$$-\nu\Delta\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = \mathbf{0}, \tag{3.3}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{3.4}$$

with the boundary conditions

$$\partial_{\mathbf{n}}\rho = 0, \qquad \mathbf{u} = \mathbf{0} \quad \text{and} \quad \mathbf{H} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{3.5}$$

where $\alpha, \varepsilon, \delta > 0$, $\tilde{\mu} \triangleq \lambda + \mu$, $P_\delta(\rho) \triangleq \rho^\gamma + \delta\rho^4$, and h is a smooth function satisfying $\int_\Omega h\,dx = M$.

Then we follow the same arguments as those in Section 4.3, pp.200-204 of [24].

But in the case of steady compressible MHD flows, the equations for the magnetic field are governed by an elliptic system and we cannot get the bound of $\|\mathbf{H}\|_{L^2}$ and in turn $\|\mathbf{H}\|_{H^1}$ in terms of the velocity coefficient $\mathbf{u}$ directly since here the coefficient $\mathbf{u}$ we consider is an arbitrary data. Instead, by the special structure of the magnetic field in the system, we can consider the existence of the approximate solutions to both the momentum equations and the magnetic field equations together by the Leray-Schauder fixed point theorem.

Making use of the method of weak convergence, we can pass to the limit $\varepsilon \to 0^+$ and $\alpha \to 0^+$ in (3.1) and (3.2) to get the existence of the weak solutions $(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta)$ to the following system:

$$\operatorname{div}(\rho_\delta \mathbf{u}_\delta) = 0, \tag{3.6}$$

$$-\mu\Delta\mathbf{u}_\delta - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u}_\delta + \rho_\delta\mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta + \nabla P_\delta(\rho) - (\nabla\times\mathbf{H}_\delta)\times\mathbf{H}_\delta = \rho_\delta\mathbf{f}, \tag{3.7}$$

$$-\nu\Delta\mathbf{H}_\delta - \nabla\times(\mathbf{u}_\delta\times\mathbf{H}_\delta) = \mathbf{0}, \tag{3.8}$$

$$\operatorname{div}\mathbf{H}_\delta = 0, \tag{3.9}$$

with the boundary conditions (1.6).

More precisely, we have the following.

Lemma 3.1 *Let* $\delta \in (0,1]$. *Then there exists at least a renormalized weak solution* $(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta)$ *to the system* (3.6)-(3.9) *such that for any* $\boldsymbol{\xi} \in W_0^{1,2}(\Omega)$, $\zeta \in C^\infty(\Omega)$, *and* $\psi \in C^1(\Omega)$ *satisfying*

$$\left|\psi(s)\right| + \left|s\psi'(s)\right| \le C\left(1+|s|^4\right), \quad s\in[0,\infty),$$

we have

$$\rho_\delta \in L^8(\Omega), \qquad \mathbf{u}_\delta \in W_0^{1,2}(\Omega), \qquad \mathbf{H}_\delta \in W^{1,2}(\Omega), \qquad \int_\Omega \rho_\delta\,dx = M, \tag{3.10}$$

$$\begin{aligned}&\int_\Omega \left[\mu\nabla\mathbf{u}_\delta : \nabla\boldsymbol{\xi} + (\lambda+\mu)\operatorname{div}\mathbf{u}_\delta\operatorname{div}\boldsymbol{\xi} + \rho_\delta(\mathbf{u}_\delta\cdot\nabla)\mathbf{u}_\delta\cdot\boldsymbol{\xi} - P_\delta\operatorname{div}\boldsymbol{\xi}\right]dx \\ &\quad + \int_\Omega\left[\mathbf{H}_\delta^T\nabla\boldsymbol{\xi}\mathbf{H}_\delta + \frac{1}{2}\nabla\left(|\mathbf{H}_\delta|^2\right)\cdot\boldsymbol{\xi} - \rho_\delta\mathbf{f}\cdot\boldsymbol{\xi}\right]dx = 0,\end{aligned} \tag{3.11}$$

$$\int_\Omega\left[\psi(\rho_\delta)\mathbf{u}_\delta\cdot\nabla\zeta + \zeta\left(\rho_\delta\psi'(\rho_\delta) - \psi(\rho_\delta)\right)\operatorname{div}\mathbf{u}_\delta\right]dx = 0 \tag{3.12}$$

and

$$\int_\Omega\left[\nu|\nabla\mathbf{H}_\delta|^2 + \mu|\nabla\mathbf{u}_\delta|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u}_\delta)^2\right]dx = \int_\Omega \rho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,dx. \tag{3.13}$$

Proof The existence of solutions satisfying (3.10) and (3.12) can be found by the method of [1, 24]. Multiplying (3.7) by $\boldsymbol{\xi}$ and integrating over Ω, we obtain (3.11). It remains to show (3.13). Choosing $\boldsymbol{\xi} = \mathbf{u}_\delta$ in (3.11), we get

$$\begin{aligned}&\int_\Omega\left[\mu|\nabla\mathbf{u}_\delta|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u}_\delta)^2 P_\delta\operatorname{div}\mathbf{u}_\delta\right]dx \\ &\quad + \int_\Omega\left[\mathbf{H}_\delta^T\nabla\mathbf{u}_\delta\mathbf{H}_\delta + \frac{1}{2}\nabla\left(|\mathbf{H}_\delta|^2\right)\cdot\mathbf{u}_\delta - \rho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\right]dx = 0,\end{aligned} \tag{3.14}$$

where we have used

$$\int_\Omega \rho_\delta\mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta\cdot\mathbf{u}_\delta\,dx = \frac{1}{2}\int_\Omega\rho_\delta\mathbf{u}_\delta\cdot\nabla\left(|\mathbf{u}|^2\right)dx = -\frac{1}{2}\int_\Omega\operatorname{div}(\rho_\delta\mathbf{u}_\delta)|\mathbf{u}|^2\,dx = 0.$$

Inserting $\psi(\rho_\delta) \triangleq \frac{1}{\gamma-1}\rho_\delta^\gamma + \frac{\delta}{3}\rho_\delta^4$ and $\zeta\equiv 1$ into (3.12), we deduce that

$$\int_\Omega P_\delta\operatorname{div}\mathbf{u}_\delta\,dx = 0. \tag{3.15}$$

Multiplying (3.8) by $\mathbf{H}_\delta$ and integrating by parts, we derive

$$\int_\Omega \nu|\nabla \mathbf{H}_\delta|^2 = \int_\Omega \left[\mathbf{H}_\delta^T \nabla \mathbf{u}_\delta \mathbf{H}_\delta + \frac{1}{2}\nabla\left(|\mathbf{H}_\delta|^2\right)\cdot \mathbf{u}_\delta\right] dx,$$

which combined with (3.14) and (3.15) gives (3.13) and finishes the proof of Lemma 3.1. □

4 Uniform estimates

In this section, we will establish uniform with respect to δ estimates for the approximate solutions $(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta)$. In what follows, to simplify notations, we omit the subscript δ in $(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta)$.

Following Plotnikov and Weigant [13], we shall perform a bootstrapping argument through the parameters

$$A \triangleq \int_\Omega \rho|\mathbf{u}|^{2(2-\theta)}\varphi^{2\beta}\, dx, \qquad B \triangleq \int_\Omega \rho^\gamma \varphi^{-\beta}\, dx, \tag{4.1}$$

where

$$\theta \triangleq \frac{1}{8}\left(1-\gamma^{-1}\right), \qquad \beta \triangleq \frac{3(1-8\theta^2)}{2(3-8\theta^2)}. \tag{4.2}$$

Then we have the following.

Proposition 4.1 *For A defined by* (4.1), *there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that*

$$A + \|\mathbf{H}\|_{H^1} + \|\mathbf{u}\|_{H^1} + \left\|\rho|\mathbf{u}|^2\right\|_{L^s} + \|P_\delta\|_{L^q} \le C, \tag{4.3}$$

where the quantities s, q are denoted by

$$s \triangleq 1 + 2\theta^2, \qquad q \triangleq 1 + \frac{\beta(s-1)}{\beta + (1-\beta)s}. \tag{4.4}$$

Here, θ and β are as in (4.2).

The proof of Proposition 4.1 will be postponed at the end of this section.

We begin with the following standard energy estimate for $(\mathbf{u}, \mathbf{H})$, which shows that the H^1-norm of $(\mathbf{u}, \mathbf{H})$ can be bounded in terms of A and B.

Lemma 4.1 *Let θ, β be as in* (4.2) *and s be as in* (4.4), *then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that*

$$\|\mathbf{H}\|_{H^1} + \|\mathbf{u}\|_{H^1} \le CA^{\frac{1}{4(2-\theta)}} B^{\frac{1}{2(2-\theta)(2\gamma-1)}}. \tag{4.5}$$

Proof It follows from (3.13) that

$$\int_\Omega \left[\nu|\nabla \mathbf{H}|^2 + \mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u})^2\right] dx = \int_\Omega \rho \mathbf{f}\cdot \mathbf{u}\, dx \le C\|\rho \mathbf{u}\|_{L^1}.$$

Thus, we obtain from (1.6), (1.4), and the Poincaré inequality

$$\|\mathbf{H}\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 \le C\|\rho\mathbf{u}\|_{L^1}. \tag{4.6}$$

Noting that

$$\rho|\mathbf{u}| = \left(\rho|\mathbf{u}|^{2(2-\theta)}\varphi^{2\beta}\right)^{\frac{1}{2(2-\theta)}}\left(\rho^{\gamma}\varphi^{-\beta}\right)^{\frac{1}{(2-\theta)(2\gamma-1)}}\left(\varphi^{-2\beta}\right)^{\frac{\gamma-1}{(2-\theta)(2\gamma-1)}}\rho^{\frac{4\gamma-3-2\theta(2\gamma-1)}{2(2-\theta)(2\gamma-1)}},$$

where all four exponents are positive and their sum is equal to one. Hence, applying Hölder's inequality and recalling (4.1), we arrive at

$$\|\rho\mathbf{u}\|_{L^1} \le A^{\frac{1}{2(2-\theta)}}B^{\frac{1}{(2-\theta)(2\gamma-1)}}\left\|\varphi^{-2\beta}\right\|_{L^1}^{\frac{\gamma-1}{(2-\theta)(2\gamma-1)}}M^{\frac{4\gamma-3-2\theta(2\gamma-1)}{2(2-\theta)(2\gamma-1)}} \le CA^{\frac{1}{2(2-\theta)}}B^{\frac{1}{(2-\theta)(2\gamma-1)}},$$

which combined with (4.6) yields (4.5) and completes the proof of Lemma 4.1. □

From now on, in what follows, the function φ is as in Lemma 2.1.

Lemma 4.2 *Let β be as in* (4.2), *then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that*

$$\begin{aligned}&\left\|P_\delta\varphi^{-\beta}\right\|_{L^1} + \left\|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\right\|_{L^1}\\&\quad\le C\left(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1}^2 + \left\|P_\delta\varphi^{1-\beta}\right\|_{L^1} + \left\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\right\|_{L^1}\right).\end{aligned} \tag{4.7}$$

Proof Motivated by [13], we introduce the vector field

$$\boldsymbol{\xi}(x) = \varphi^{1-\beta}(x)\nabla\varphi(x), \quad x\in\Omega. \tag{4.8}$$

By straightforward computation, we have

$$\nabla\boldsymbol{\xi} = \varphi^{1-\beta}\nabla^2\varphi + (1-\beta)\varphi^{-\beta}\nabla\varphi\otimes\nabla\varphi.$$

Then it follows from (4.2) and Lemma 2.1 that

$$\left|\varphi^{1-\beta}\nabla^2\varphi\right| \le C, \qquad \left|\varphi^{-\beta}\nabla\varphi\otimes\nabla\varphi\right| \le C\varphi^{-\beta}.$$

Hence, we have

$$\boldsymbol{\xi}\in L^\infty(\Omega) \quad\text{and}\quad \boldsymbol{\xi}\in W_0^{1,r}(\Omega) \quad \text{for all } r\in\left[1,\beta^{-1}\right).$$

By the definition of β in (4.2), it is not hard to see that $\beta\in(0,\frac{1}{2})$. Then, substituting (4.8) into (3.11), we thus get

$$\begin{aligned}&\int_\Omega\left[P_\delta\operatorname{div}\boldsymbol{\xi} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\right]dx\\&\quad= \int_\Omega\left[\mu\nabla\mathbf{u}:\nabla\boldsymbol{\xi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\xi} - \rho\mathbf{f}\cdot\boldsymbol{\xi} + \mathbf{H}^T\nabla\boldsymbol{\xi}\mathbf{H} + \frac{1}{2}\nabla\left(|\mathbf{H}|^2\right)\cdot\boldsymbol{\xi}\right]dx\end{aligned}$$

$$\begin{aligned}&\le C\|\mathbf{u}\|_{H^1}+C+C\big\||\mathbf{H}|^2\big\|_{L^2}+C\|\mathbf{H}\|_{L^2}\|\nabla\mathbf{H}\|_{L^2}\\&\le C\big(1+\|\mathbf{u}\|_{H^1}+\|\mathbf{H}\|_{H^1}^2\big).\end{aligned}\tag{4.9}$$

We can bound two terms on the left-hand side of (4.9) as

$$\begin{aligned}\int_\Omega P_\delta\operatorname{div}\boldsymbol{\xi}\,dx&=\int_\Omega\big[(1-\beta)P_\delta\varphi^{-\beta}|\nabla\varphi|^2+P_\delta\varphi^{1-\beta}\Delta\varphi\big]\,dx\\&\ge\int_\Omega\left(\frac{1}{2}\varphi^{-\beta}P_\delta-C\varphi^{1-\beta}P_\delta\right)dx,\end{aligned}\tag{4.10}$$

$$\begin{aligned}&-\int_\Omega\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\,dx\\&\quad=\int_\Omega\big[-\rho\mathbf{u}\cdot\nabla\big(\varphi^{1-\beta}\mathbf{u}\cdot\nabla\varphi\big)+(1-\beta)\varphi^{-\beta}\rho(\mathbf{u}\cdot\nabla\varphi)^2+\varphi^{1-\beta}\rho u^iu^j\partial_{ij}\varphi\big]\,dx\\&\quad\ge\int_\Omega\big[(1-\beta)\varphi^{-\beta}\rho(\mathbf{u}\cdot\nabla\varphi)^2-C\varphi^{1-\beta}\rho|\mathbf{u}|^2\big]\,dx.\end{aligned}\tag{4.11}$$

Hence, inserting (4.10)-(4.11) into (4.9), we deduce the desired result (4.7). □

Lemma 4.3 *Let β, s be as in* (4.2) *and* (4.4), *respectively, then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that*

$$\big\|P_\delta\varphi^{1-\beta}\big\|_{L^s}\le C\big(1+\|\mathbf{u}\|_{H^1}+\|\mathbf{H}\|_{H^1}^2+\big\|P_\delta\varphi^{-\beta}\big\|_{L^1}+\big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^s}\big).\tag{4.12}$$

Proof Choosing a function $h\in L^{\frac{s-1}{s}}(\Omega)$, it follows from Lemma 3.1 that the problem

$$\begin{cases}\operatorname{div}\boldsymbol{\phi}=h-\frac{1}{|\Omega|}\int_\Omega h\,dx & \text{in }\Omega,\\ \boldsymbol{\phi}=\mathbf{0} & \text{on }\partial\Omega,\end{cases}\tag{4.13}$$

has a solution $\boldsymbol{\phi}\in W_0^{1,\frac{s}{s-1}}(\Omega)$ satisfying

$$\|\boldsymbol{\phi}\|_{W^{1,\frac{s}{s-1}}}\le C(s,\Omega)\|h\|_{L^{\frac{s}{s-1}}}.\tag{4.14}$$

By the definition of s in (4.4), we obtain $s\in(1,\frac{33}{32})$, thus $\frac{s}{s-1}>3$. Then Sobolev's embedding theorem gives $W_0^{1,\frac{s}{s-1}}(\Omega)\hookrightarrow C(\overline{\Omega})$. Hence,

$$\|\boldsymbol{\phi}\|_{L^\infty}+\big\|\varphi^{-1}\boldsymbol{\phi}\big\|_{L^{\frac{s}{s-1}}}\le C\|h\|_{L^{\frac{s}{s-1}}}.\tag{4.15}$$

For $x\in\Omega$, setting

$$\boldsymbol{\xi}(x)\triangleq\varphi^{1-\beta}(x)\boldsymbol{\phi}(x).\tag{4.16}$$

Straightforward calculation yields

$$\partial_i\xi^j=(1-\beta)\varphi^{-\beta}\partial_i\varphi\phi^j+\varphi^{1-\beta}\partial_i\phi^j.\tag{4.17}$$

Thus, we derive from (4.14) that

$$\|\boldsymbol{\xi}\|_{W^{1,\frac{s}{s-1}}}\le C\|h\|_{L^{\frac{s}{s-1}}}.\tag{4.18}$$

Inserting (4.16) into (3.11) and using (4.18), we find that

$$\begin{aligned}\int_\Omega P_\delta \operatorname{div}\boldsymbol{\xi} &= \int_\Omega \big[\mu\nabla\mathbf{u}:\nabla\boldsymbol{\xi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\xi} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\big]\,dx \\ &\quad + \int_\Omega \bigg[\mathbf{H}^T\nabla\boldsymbol{\xi}\mathbf{H} + \frac{1}{2}\nabla\big(|\mathbf{H}|^2\big)\cdot\boldsymbol{\xi} - \rho\mathbf{f}\cdot\boldsymbol{\xi}\bigg]\,dx \\ &\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\boldsymbol{\xi}\|_{L^2} + C\big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^s}\big(\big\|\varphi^{-1}\boldsymbol{\phi}\big\|_{L^{\frac{s}{s-1}}} + \|\nabla\boldsymbol{\phi}\|_{L^{\frac{s}{s-1}}}\big) \\ &\quad + C\big\||\mathbf{H}|^2\big\|_{L^2}\|\nabla\boldsymbol{\xi}\|_{L^2} + C\|\mathbf{H}\|_{L^2}\|\nabla\mathbf{H}\|_{L^2}\|\boldsymbol{\xi}\|_{L^\infty} + C\|\rho\|_{L^1}\|\mathbf{f}\|_{L^\infty}\|\boldsymbol{\xi}\|_{L^\infty} \\ &\le C\big(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1}^2 + \big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^s}\big)\|h\|_{L^{\frac{s}{s-1}}},\end{aligned} \tag{4.19}$$

where in the first inequality we have used

$$\begin{aligned}\int_\Omega \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\,dx &= \int_\Omega \rho u^i\partial_i\big(u^j\varphi^{1-\beta}\phi^j\big)\,dx - \int_\Omega \rho u^iu^j\partial_i\big(\varphi^{1-\beta}\phi^j\big)\,dx \\ &= -\int_\Omega (1-\beta)\rho u^iu^j\varphi^{1-\beta}(\partial_i\varphi)\varphi^{-1}\phi^j\,dx - \int_\Omega \rho u^iu^j\varphi^{1-\beta}\partial_i\phi^j\,dx \\ &\le C\big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^s}\big(\big\|\varphi^{-1}\boldsymbol{\phi}\big\|_{L^{\frac{s}{s-1}}} + \|\nabla\boldsymbol{\phi}\|_{L^{\frac{s}{s-1}}}\big).\end{aligned}$$

By virtue of (4.16) and (4.13), we see that

$$\operatorname{div}\boldsymbol{\xi} = (1-\beta)\varphi^{-\beta}\nabla\varphi\cdot\boldsymbol{\phi} + \varphi^{1-\beta}\operatorname{div}\boldsymbol{\phi} = (1-\beta)\varphi^{-\beta}\nabla\varphi\cdot\boldsymbol{\phi} + \varphi^{1-\beta}\bigg(h - \frac{1}{|\Omega|}\int_\Omega h\,dx\bigg),$$

which ensures that

$$\begin{aligned}\int_\Omega P_\delta\operatorname{div}\boldsymbol{\xi}\,dx &= \int_\Omega P_\delta\varphi^{1-\beta}h\,dx - \frac{1}{|\Omega|}\int_\Omega h\,dx\int_\Omega P_\delta\varphi^{1-\beta}\,dx \\ &\quad + (1-\beta)\int_\Omega P_\delta\varphi^{-\beta}\nabla\varphi\cdot\boldsymbol{\phi}\,dx.\end{aligned} \tag{4.20}$$

By Hölder's inequality and the boundedness of φ, we have

$$\bigg|\int_\Omega h\,dx\int_\Omega P_\delta\varphi^{1-\beta}\,dx\bigg| \le C\|h\|_{L^{\frac{s}{s-1}}}\int_\Omega P_\delta\varphi^{-\beta}\,dx. \tag{4.21}$$

For the last term on the right-hand side of (4.20), we obtain from (4.15)

$$\bigg|\int_\Omega P_\delta\varphi^{-\beta}\nabla\varphi\cdot\boldsymbol{\phi}\,dx\bigg| \le \int_\Omega P_\delta\varphi^{-\beta}\,dx\|\nabla\varphi\|_{L^\infty}\|\boldsymbol{\phi}\|_{L^\infty} \le C\|h\|_{L^{\frac{s}{s-1}}}\int_\Omega P_\delta\varphi^{-\beta}\,dx,$$

which combined with (4.20) and (4.21) gives

$$\int_\Omega P_\delta\operatorname{div}\boldsymbol{\xi}\,dx \ge \int_\Omega P_\delta\varphi^{1-\beta}h\,dx - C\big\|P_\delta\varphi^{-\beta}\big\|_{L^1}\|h\|_{L^{\frac{s}{s-1}}}. \tag{4.22}$$

Consequently, we deduce from (4.19) and (4.22) that

$$\int_\Omega P_\delta\varphi^{1-\beta}h\,dx \le C\big(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1}^2 + \big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^s} + \big\|P_\delta\varphi^{-\beta}\big\|_{L^1}\big)\|h\|_{L^{\frac{s}{s-1}}}.$$

Then by the duality argument, we get the desired result (4.12). □

Lemma 4.4 *Let θ, β be as in* (4.2) *and s be as in* (4.4), *then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that*

$$\|P_\delta\varphi^{1-\beta}\|_{L^s} + \|P_\delta\varphi^{-\beta}\|_{L^1} + \|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\|_{L^1} \le C\bigl(1 + A^{\frac{1+\theta}{2}}\bigr), \tag{4.23}$$

$$\|\mathbf{H}\|_{H^1} + \|\mathbf{u}\|_{H^1} \le C\bigl(1 + A^{\frac{1-2\theta}{4}}\bigr), \tag{4.24}$$

$$\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\|_{L^s} \le C\bigl(1 + A^{\frac{1+\theta}{2}}\bigr). \tag{4.25}$$

Proof It follows from (4.7) and (4.12) that

$$\begin{aligned}&\|P_\delta\varphi^{1-\beta}\|_{L^s} + \|P_\delta\varphi^{-\beta}\|_{L^1} + \|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\|_{L^1}\\ &\quad\le C\bigl(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1}^2 + \|P_\delta\varphi^{1-\beta}\|_{L^1} + \|\rho|\mathbf{u}|^2\varphi^{1-\beta}\|_{L^s}\bigr).\end{aligned} \tag{4.26}$$

By Hölder's and Young's inequalities, we get

$$\|P_\delta\varphi^{1-\beta}\|_{L^1} \le C(\Omega)\|P_\delta\varphi^{1-\beta}\|_{L^s}^{\frac{3s}{4s-1}} \le \frac{1}{2}\|P_\delta\varphi^{1-\beta}\|_{L^s} + C, \tag{4.27}$$

$$\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\|_{L^s} \le CA^{\frac{1+\theta}{2}}B^{\frac{\theta}{(2-\theta)(2\gamma-1)}}. \tag{4.28}$$

Substituting (4.5), (4.27), (4.28), and (4.26), we arrive at, for every $\varepsilon > 0$,

$$\begin{aligned}&\|P_\delta\varphi^{1-\beta}\|_{L^s} + \|P_\delta\varphi^{-\beta}\|_{L^1} + \|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\|_{L^1}\\ &\quad\le C\bigl(1 + A^{\frac{1}{2(2-\theta)}}B^{\frac{1}{(2-\theta)(2\gamma-1)}} + A^{\frac{1}{2-\theta}}B^{\frac{\theta}{(2-\theta)(2\gamma-1)}}\bigr)\\ &\quad\le C\bigl(1 + \varepsilon B + C(\varepsilon)A^{\frac{1}{2}\frac{2\gamma-1}{(2-\theta)(2\gamma-1)-1}} + \varepsilon B + C(\varepsilon)A^{\frac{1+\theta}{2}}\bigr).\end{aligned} \tag{4.29}$$

Recalling (4.2), after a straightforward computation, we find that

$$\frac{1}{2}\frac{2\gamma-1}{(2-\theta)(2\gamma-1)-1} < \frac{1}{2}.$$

Consequently, we obtain from (4.29)

$$\|P_\delta\varphi^{1-\beta}\|_{L^s} + \|P_\delta\varphi^{-\beta}\|_{L^1} + \|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\|_{L^1} \le C\varepsilon B + C(\varepsilon)\bigl(1 + A^{\frac{1+\theta}{2}}\bigr). \tag{4.30}$$

Since $P_\delta \ge \rho^\gamma$, we have

$$B = \|\rho^\gamma\varphi^{-\beta}\|_{L^1} \le \|P_\delta\varphi^{-\beta}\|_{L^1}. \tag{4.31}$$

Inserting this into (4.30) and choosing ε sufficiently small, we deduce the estimate (4.23).

We now turn to showing that (4.24) and (4.25) hold. First of all, (4.23) and (4.31) imply

$$B \le C\bigl(1 + A^{\frac{1+\theta}{2}}\bigr), \tag{4.32}$$

which together with (4.5) yields

$$\|\mathbf{H}\|_{H^1} + \|\mathbf{u}\|_{H^1} \le CA^{\frac{1}{4(2-\theta)}}\bigl(1 + A^{\frac{1+\theta}{2}}\bigr)^{\frac{1}{2(2-\theta)(2\gamma-1)}} \le C\bigl(1 + A^{\frac{2\gamma+\theta}{4(2-\theta)(2\gamma-1)}}\bigr). \tag{4.33}$$

We infer from the definition of θ in (4.2) that

$$\frac{2\gamma+\theta}{4(2-\theta)(2\gamma-1)} \le \frac{1-2\theta}{4},$$

which along with (4.33) gives (4.24). It remains to prove (4.25). Substituting (4.32) into (4.28), we derive

$$\left\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\right\|_{L^s} \le CA^{\frac{1}{2-\theta}}\left(1+A^{\frac{1+\theta}{2}}\right)^{\frac{\theta}{(2-\theta)(2\gamma-1)}} \le C\left(1+A^{\frac{4\gamma-2+\theta+\theta^2}{2(2-\theta)(2\gamma-1)}}\right). \tag{4.34}$$

By the definition of θ in (4.2), it is not hard to show that

$$\frac{4\gamma-2+\theta+\theta^2}{2(2-\theta)(2\gamma-1)} \le \frac{1+\theta}{2},$$

which combined with (4.34) leads to (4.25). □

In terms of A, we can derive the following weighted estimates for P_δ and $\rho|\mathbf{u}|^2$.

Lemma 4.5 *Let β, θ be as in* (4.2), *then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that for every $\alpha \in (0,1)$ and $x_0 \in \Omega$, we have*

$$\int_\Omega \frac{(P_\delta+\rho|\mathbf{u}|^2)\varphi^{\frac{3}{2}-\beta}(x)}{|x-x_0|^\alpha}\,dx \le C\left(1+A^{\frac{1+\theta}{2}}\right). \tag{4.35}$$

Proof Fix $\alpha \in (0,1)$ and $x_0 \in \Omega$, we denote by

$$\boldsymbol{\xi}(x) \triangleq \frac{\varphi^{\frac{3}{2}-\beta}(x)(x-x_0)}{|x-x_0|^\alpha}. \tag{4.36}$$

Direct calculus gives

$$\nabla\boldsymbol{\xi} = \frac{\varphi^{\frac{3}{2}-\beta}(x)}{|x-x_0|^\alpha}\left(\mathbb{I} - \frac{\alpha}{|x-x_0|^2}(x-x_0)\otimes(x-x_0) + \frac{3-2\beta}{2\varphi}(x-x_0)\otimes\nabla\varphi\right).$$

Combining this with $\beta \in (0,\frac{1}{2})$, we get

$$|\nabla\boldsymbol{\xi}| \le C|x-x_0|^{-\alpha}.$$

Hence, $\boldsymbol{\xi} \in W_0^{1,r}(\Omega)$ for all $r \in [1,\frac{3}{\alpha})$. In particular,

$$\|\nabla\boldsymbol{\xi}\|_{L^2} \le C, \qquad \|\boldsymbol{\xi}\|_{L^\infty} \le C.$$

Thus, substituting $\boldsymbol{\xi}$ into (3.11) yields

$$\begin{aligned}
&\int_\Omega \left(P_\delta \operatorname{div}\boldsymbol{\xi} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\right)dx \\
&\quad = \int_\Omega \left[\mu\nabla\mathbf{u} : \nabla\boldsymbol{\xi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\xi}\right]dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \left[\mathbf{H}^T\nabla\boldsymbol{\xi}\mathbf{H}+\frac{1}{2}\nabla\big(|\mathbf{H}|^2\big)\cdot\boldsymbol{\xi}-\rho\mathbf{f}\cdot\boldsymbol{\xi}\right]dx\\
&\le C\big(1+\|\mathbf{u}\|_{H^1}+\|\mathbf{H}\|_{H^1}^2\big).
\end{aligned}
\tag{4.37}
$$

The left-hand side of (4.37) can be estimated as follows.

On the one hand, we have

$$
\operatorname{div}\boldsymbol{\xi}=\frac{(3-\alpha)\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha}-\frac{(3-2\beta)\varphi^{\frac{1}{2}-\beta}}{2|x-x_0|^\alpha}(x-x_0)\cdot\nabla\varphi\ge\frac{(3-\alpha)\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha}-C.
\tag{4.38}
$$

On the other hand, direct computation shows that

$$
\begin{aligned}
-\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}={}&-\rho\mathbf{u}\cdot\nabla\left[\frac{\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha}(x-x_0)\cdot\nabla\mathbf{u}\right]\\
&+\frac{\varphi^{\frac{3}{2}-\beta}\rho}{|x-x_0|^\alpha}\left(|\mathbf{u}|^2-\frac{\alpha}{|x-x_0|^2}\big((x-x_0)\cdot\mathbf{u}\big)^2\right)\\
&+\frac{(3-2\beta)\varphi^{\frac{1}{2}-\beta}\rho}{2|x-x_0|^\alpha}\big((x-x_0)\cdot\mathbf{u}\big)(\nabla\varphi\cdot\mathbf{u}).
\end{aligned}
$$

Then we obtain from integration by parts and Hölder's inequality

$$
\begin{aligned}
&-\int_\Omega\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\xi}\,dx\\
&\ge\int_\Omega\left(\frac{(1-\alpha)\varphi^{\frac{3}{2}-\beta}\rho}{|x-x_0|^\alpha}|\mathbf{u}|^2-C\varphi^{\frac{1}{2}-\beta}\rho|\mathbf{u}||\nabla\varphi\cdot\mathbf{u}|\right)dx\\
&\ge\int_\Omega\left(\frac{(1-\alpha)\varphi^{\frac{3}{2}-\beta}\rho}{|x-x_0|^\alpha}|\mathbf{u}|^2-C\varphi^{1-\beta}\rho|\mathbf{u}|^2-C\varphi^{-\beta}\rho(\nabla\varphi\cdot\mathbf{u})^2\right)dx.
\end{aligned}
\tag{4.39}
$$

Inserting (4.38) and (4.39) into (4.37), we derive that

$$
\begin{aligned}
&\int_\Omega\frac{(P_\delta+\rho|\mathbf{u}|^2)\varphi^{\frac{3}{2}-\beta}(x)}{|x-x_0|^\alpha}\,dx\\
&\quad\le C\big(1+\|\mathbf{u}\|_{H^1}+\|\mathbf{H}\|_{H^1}^2+\big\|\rho(\mathbf{u}\cdot\nabla\varphi)^2\varphi^{-\beta}\big\|_{L^1}+\big\|P_\delta\varphi^{-\beta}\big\|_{L^1}+\big\|\rho|\mathbf{u}|^2\varphi^{1-\beta}\big\|_{L^1}\big),
\end{aligned}
$$

which together with (4.23)-(4.25) implies that

$$
\int_\Omega\frac{(P_\delta+\rho|\mathbf{u}|^2)\varphi^{\frac{3}{2}-\beta}(x)}{|x-x_0|^\alpha}\,dx\le C\big(1+A^{\frac{1+\theta}{2}}+A^{\frac{1-2\theta}{4}}\big)\le C\big(1+A^{\frac{1+\theta}{2}}\big).
$$

The proof of Lemma 4.5 is finished. □

Lemma 4.6 *Let θ, β be as in* (4.2), *then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, and $\|\mathbf{f}\|_{L^\infty}$ such that for every $x_0\in\Omega$, we have*

$$
\int_\Omega\frac{\rho|\mathbf{u}|^{2(1-\theta)}\varphi^{2\beta}(x)}{|x-x_0|}\,dx\le C\big(1+A^{\frac{1+\theta}{2}}\big).
\tag{4.40}
$$

Proof Note that

$$\frac{\rho|\mathbf{u}|^{2(1-\theta)}\varphi^{2\beta}(x)}{|x-x_0|} = \left(\frac{\rho^\gamma\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha}\right)^{\frac{\theta}{\gamma}}\left(\frac{\rho|\mathbf{u}|^2\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha}\right)^{1-\theta}\left(\frac{1}{|x-x_0|^2}\right)^{\theta-\frac{\theta}{\gamma}}.$$

Hence, due to $\rho^\gamma \le P_\delta$, we obtain from Young's inequality

$$\frac{\rho|\mathbf{u}|^{2(1-\theta)}\varphi^{2\beta}(x)}{|x-x_0|} \le C\left(\frac{P_\delta\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha} + \frac{\rho|\mathbf{u}|^2\varphi^{\frac{3}{2}-\beta}}{|x-x_0|^\alpha} + \frac{1}{|x-x_0|^2}\right).$$

Integrating both sides of this inequality over Ω and using (4.35), we get (4.40) and finish the proof of Lemma 4.6. □

In order to show the boundedness of $\|\rho|\mathbf{u}|^2\|_{L^s}$, we have some delicate analysis. In view of Lemma 2.1, there is $t > 0$ such that $\varphi(x) = d(x)$ in A_{2t}. Introduce the vector field

$$\mathbf{n}(x) = \nabla\varphi(x), \quad \mathbf{n} \in C^1(\overline{A}_{2t}), \qquad \left|\mathbf{n}(x)\right| = 1. \tag{4.41}$$

Fix an arbitrary $\alpha \in (0,1)$ and $x_0 \in A_t$. Define the vector field

$$\boldsymbol{\xi}(x) = \left[\frac{\varphi(x)-\varphi(x_0)}{\Delta_-(x,x_0)^\alpha} + \frac{\varphi(x)+\varphi(x_0)}{\Delta_+(x,x_0)^\alpha}\right]\mathbf{n}(x), \tag{4.42}$$

where

$$\Delta_\pm(x,x_0) \triangleq \left|\varphi(x)\pm\varphi(x_0)\right| + |x-x_0|^2.$$

Then we have the following properties of $\boldsymbol{\xi}$, whose proof can be found in Appendix A of [13].

Lemma 4.7 *There is a positive constant C depending only on α and Ω such that for every $x, x_0 \in A_t$ and for every $\mathbf{u} \in \mathbb{R}^3$, we have*

$$\left|\boldsymbol{\xi}(x)\right| \le C, \qquad \left|\nabla\boldsymbol{\xi}(x)\right| \le C\left(\frac{1}{\Delta_-(x,x_0)^\alpha} + \frac{1}{\Delta_+(x,x_0)^\alpha} + 1\right), \tag{4.43}$$

$$\partial_j\xi^i(x)u^iu^j \ge \frac{1-\alpha}{2}\left(\frac{1}{\Delta_-(x,x_0)^\alpha} + \frac{1}{\Delta_+(x,x_0)^\alpha}\right)\left|\mathbf{u}\cdot\mathbf{n}(x)\right|^2 - C|\mathbf{u}|^2, \tag{4.44}$$

$$\operatorname{div}\boldsymbol{\xi} \ge \frac{1-\alpha}{2}\left(\frac{1}{\Delta_-(x,x_0)^\alpha} + \frac{1}{\Delta_+(x,x_0)^\alpha}\right) - C, \tag{4.45}$$

$$\|\nabla\boldsymbol{\xi}\|_{L^2(\Omega_t)} \le C. \tag{4.46}$$

Lemma 4.8 *Let $\alpha \in (0,1)$ and $x_0 \in \Omega$. Assume that $\zeta \in C^\infty(\overline{\Omega})$ satisfies*

$$\zeta \ge 0 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{in } \Omega\backslash\Omega_{t/2},$$

then there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, $\|\mathbf{f}\|_{L^\infty}$, α, and ζ such that

$$\int_\Omega \frac{\zeta P_\delta(\rho)(x)}{|x-x_0|^\alpha}\,dx \le C\left(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \left\|\rho|\mathbf{u}|^2\right\|_{L^1}\right). \tag{4.47}$$

Proof If $x_0 \in \Omega_t$. Let $\boldsymbol{\xi}$ be as in (4.42). Noting that

$$|\boldsymbol{\xi}| \le 4(\operatorname{diam}\Omega)^{1-\alpha},$$

which combined with (4.46) yields

$$\|\zeta\boldsymbol{\xi}\|_{H^1} \le C.$$

Hence, replacing $\boldsymbol{\xi}$ by $\zeta\boldsymbol{\xi}$ in (3.11) implies

$$\begin{aligned}
&\int_\Omega \big(\zeta\rho u^i u^j \partial_i \xi^j + \zeta P_\delta \operatorname{div}\boldsymbol{\xi}\big)\,dx \\
&\quad = \int_\Omega \big[\mu\nabla\mathbf{u} : \nabla(\zeta\boldsymbol{\xi}) + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}(\zeta\boldsymbol{\xi})\big]\,dx \\
&\qquad + \int_\Omega \left[\mathbf{H}^T\nabla(\zeta\boldsymbol{\xi})\mathbf{H} + \frac{1}{2}\nabla\big(|\mathbf{H}|^2\big)\cdot\zeta\boldsymbol{\xi} - \rho\mathbf{f}\cdot\zeta\boldsymbol{\xi}\right]dx \\
&\qquad - \int_\Omega \big[P_\delta\nabla\zeta\cdot\boldsymbol{\xi} + \rho(\nabla\zeta\cdot\mathbf{u})(\mathbf{u}\cdot\boldsymbol{\xi})\big]\,dx.
\end{aligned} \tag{4.48}$$

By Hölder's inequality, it follows that

$$\begin{aligned}
&\int_\Omega \big[\mu\nabla\mathbf{u} : \nabla(\zeta\boldsymbol{\xi}) + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}(\zeta\boldsymbol{\xi})\big]\,dx \le C\|\mathbf{u}\|_{H^1}, \\
&\int_\Omega \left[\mathbf{H}^T\nabla(\zeta\boldsymbol{\xi})\mathbf{H} + \frac{1}{2}\nabla\big(|\mathbf{H}|^2\big)\cdot\zeta\boldsymbol{\xi} - \rho\mathbf{f}\cdot\zeta\boldsymbol{\xi}\right]dx \le C\big(\|\mathbf{H}\|_{H^1}+1\big), \\
&\int_\Omega \big[P_\delta\nabla\zeta\cdot\boldsymbol{\xi} + \rho(\nabla\zeta\cdot\mathbf{u})(\mathbf{u}\cdot\boldsymbol{\xi})\big]\,dx \le C\big(\|P_\delta\|_{L^1} + \big\|\rho|\mathbf{u}|^2\big\|_{L^1}\big).
\end{aligned}$$

Substituting the above estimates into (4.48), we obtain, for all $x_0 \in A_t$,

$$\int_\Omega \big(\zeta\rho u^i u^j \partial_i \xi^j + \zeta P_\delta \operatorname{div}\boldsymbol{\xi}\big)\,dx \le C\big(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \big\|\rho|\mathbf{u}|^2\big\|_{L^1}\big). \tag{4.49}$$

It follows from (4.44) that, for $x \in A_t$,

$$\rho u^i u^j \partial_j \xi^i(x) \ge \frac{1-\alpha}{2}\left(\frac{1}{\Delta_-(x,x_0)^\alpha} + \frac{1}{\Delta_+(x,x_0)^\alpha}\right)\rho\big|\mathbf{u}\cdot\mathbf{n}(x)\big|^2 - C\rho|\mathbf{u}|^2,$$

which together with (4.45) and (4.49) leads to

$$\begin{aligned}
&\int_\Omega \zeta P_\delta\left(\frac{1}{\Delta_-(x,x_0)^\alpha} + \frac{1}{\Delta_+(x,x_0)^\alpha}\right)dx \\
&\quad \le C\big(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \big\|\rho|\mathbf{u}|^2\big\|_{L^1}\big).
\end{aligned} \tag{4.50}$$

Since $|\varphi(x) - \varphi(x_0)| \le |x - x_0|$, we have $\Delta_-(x,x_0) \le C|x-x_0|$. Combining this with (4.50) gives, for all $x_0 \in A_t$,

$$\int_\Omega \frac{\zeta P_\delta(\rho)(x)}{|x-x_0|^\alpha}\,dx \le C\big(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \big\|\rho|\mathbf{u}|^2\big\|_{L^1}\big). \tag{4.51}$$

If $x_0 \in \Omega \backslash A_t$. Since ζ vanishes in $\Omega \backslash A_{t/2}$, the inequality $2|x - x_0| \geq t$ holds for all $x \in \operatorname{supp} \zeta$ and $x_0 \in \Omega \backslash A_t$, and hence for all $x_0 \in \Omega \backslash A_t$,

$$\int_\Omega \frac{\zeta P_\delta(\rho)(x)}{|x - x_0|^\alpha}\,dx \leq C\|P_\delta\|_{L^1},$$

which combined with (4.51) yields the desired estimate (4.47). □

Lemma 4.9 *For every nonnegative function $\eta \in C_0^\infty(\Omega)$ and every $x_0 \in \Omega$, there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, $\|\mathbf{f}\|_{L^\infty}$, and η such that*

$$\int_\Omega \frac{\eta P_\delta(\rho)}{|x - x_0|}\,dx \leq C\bigl(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \bigl\|\rho|\mathbf{u}|^2\bigr\|_{L^1}\bigr). \tag{4.52}$$

Proof Fix an arbitrary $x_0 \in \Omega$, letting

$$\boldsymbol{\phi}(x) \triangleq \frac{x - x_0}{|x - x_0|}.$$

Obviously, $\nabla\boldsymbol{\phi} \leq \frac{C}{|x-x_0|}$. Thus $\eta\boldsymbol{\phi} \in W_0^{1,2}(\Omega)$ and

$$\|\eta\boldsymbol{\phi}\|_{H^1} \leq C. \tag{4.53}$$

Integral identity (3.11) with $\boldsymbol{\xi}$ replaced by $\eta\boldsymbol{\phi}$ implies

$$\begin{aligned}
&\int_\Omega \bigl(\eta\rho u^i u^j \partial_i \xi^j + \eta P_\delta \operatorname{div}\boldsymbol{\phi}\bigr)\,dx \\
&\quad = \int_\Omega \bigl[\mu\nabla\mathbf{u} : \nabla(\eta\boldsymbol{\phi}) + (\lambda + \mu)\operatorname{div}\mathbf{u}\operatorname{div}(\eta\boldsymbol{\phi})\bigr]\,dx \\
&\qquad + \int_\Omega \biggl[\mathbf{H}^T\nabla(\eta\boldsymbol{\phi})\mathbf{H} + \frac{1}{2}\nabla\bigl(|\mathbf{H}|^2\bigr)\cdot\eta\boldsymbol{\phi} - \rho\mathbf{f}\cdot\eta\boldsymbol{\phi}\biggr]\,dx \\
&\qquad - \int_\Omega \bigl[P_\delta\nabla\eta\cdot\boldsymbol{\phi} + \rho(\nabla\eta\cdot\mathbf{u})(\mathbf{u}\cdot\boldsymbol{\phi})\bigr]\,dx,
\end{aligned}$$

which combined with (4.53) and $|\boldsymbol{\phi}| = 1$ leads to

$$\int_\Omega \bigl(\eta\rho u^i u^j \partial_i \phi^j + \eta P_\delta \operatorname{div}\boldsymbol{\phi}\bigr)\,dx \leq C\bigl(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \bigl\|\rho|\mathbf{u}|^2\bigr\|_{L^1}\bigr). \tag{4.54}$$

On the other hand, direct computations give

$$u^i u^j \partial_i \phi^j = \frac{(\mathbb{I} - \boldsymbol{\phi}\otimes\boldsymbol{\phi})\mathbf{u}\cdot\mathbf{u}}{|x - x_0|} \geq 0, \qquad \operatorname{div}\boldsymbol{\phi} = \frac{2}{|x - x_0|}. \tag{4.55}$$

Inserting (4.55) into (4.54), we deduce the desired result (4.52). □

Lemma 4.10 *Let $\alpha \in (0,1)$. Then for every $x_0 \in \Omega$, there exists a positive constant C depending only on γ, λ, μ, ν, M, Ω, $\|\mathbf{f}\|_{L^\infty}$, and α such that*

$$\int_\Omega \frac{P_\delta^\alpha(\rho)}{|x - x_0|}\,dx \leq C\bigl(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \bigl\|\rho|\mathbf{u}|^2\bigr\|_{L^1}\bigr). \tag{4.56}$$

Proof Choose a nonnegative function $\eta \in C^\infty(\Omega)$ such that η equals 1 in a neighborhood of $\partial\Omega$ and η vanishes in $\Omega\backslash\Omega_{t/2}$. In particular, we have $1-\eta \in C_0^\infty(\Omega)$. We obtain from Lemmas 4.8 and 4.9

$$\begin{aligned}\int_\Omega \frac{P_\delta(\rho)}{|x-x_0|^\alpha}\,dx &\le \int_\Omega \frac{\eta P_\delta(\rho)}{|x-x_0|^\alpha}\,dx + C\int_\Omega \frac{(1-\eta)P_\delta(\rho)}{|x-x_0|^\alpha}\,dx \\ &\le C\bigl(1+\|\mathbf{u}\|_{H^1}+\|\mathbf{H}\|_{H^1}+\|P_\delta\|_{L^1}+\bigl\|\rho|\mathbf{u}|^2\bigr\|_{L^1}\bigr). \end{aligned} \tag{4.57}$$

By virtue of Young's inequality, we have

$$\frac{P_\delta(\rho)^\alpha}{|x-x_0|} = \left(\frac{P_\delta(\rho)}{|x-x_0|^\alpha}\right)^\alpha\left(\frac{1}{|x-x_0|^{1+\alpha}}\right)^{1-\alpha} \le \frac{CP_\delta(\rho)}{|x-x_0|^\alpha} + \frac{C}{|x-x_0|^{1+\alpha}}.$$

Integrating both sides over Ω and noting that $1+\alpha<2$, we get

$$\int_\Omega \frac{P_\delta(\rho)^\alpha}{|x-x_0|}\,dx \le C\int_\Omega \frac{P_\delta(\rho)}{|x-x_0|^\alpha}\,dx + C,$$

which together with (4.57) implies (4.56) and completes the proof of Lemma 4.10. □

With Lemmas 4.1-4.10 at hand, we are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1 It follows from (4.40) and Lemma 2.4 that

$$\begin{aligned} A &\triangleq \int_\Omega \rho|\mathbf{u}|^{2(2-\theta)}\varphi^{2\beta}\,dx \\ &\le C\|\mathbf{u}\|_{H^1}^2 \sup_{x_0\in\Omega}\int_\Omega \frac{\rho|\mathbf{u}|^{2(1-\theta)}\varphi^{2\beta}(x)}{|x-x_0|}\,dx \\ &\le C\|\mathbf{u}\|_{H^1}^2\bigl(1+A^{\frac{1+\theta}{2}}\bigr), \end{aligned} \tag{4.58}$$

which combined with (4.24) implies that

$$A \le C\bigl(1+A^{\frac{1}{2}-\theta}\bigr)\bigl(1+A^{\frac{1+\theta}{2}}\bigr) \le C\bigl(1+A^{1-\frac{\theta}{2}}\bigr). \tag{4.59}$$

Since $1-\frac{\theta}{2}\in(0,1)$, we easily obtain from (4.59)

$$A \le C, \tag{4.60}$$

which together with (4.24) yields

$$\|\mathbf{H}\|_{H^1} + \|\mathbf{u}\|_{H^1} \le C. \tag{4.61}$$

Note that

$$P_\delta^q = \bigl(P_\delta^s\varphi^{s(1-\beta)}\bigr)^{\frac{\beta}{\beta+(1-\beta)s}}\bigl(P_\delta\varphi^{-\beta}\bigr)^{\frac{(1-\beta)s}{\beta+(1-\beta)s}}.$$

Making use of Young's inequality, we derive from (4.23) and (4.60) that

$$\int_\Omega P_\delta^q\,dx \le C\int_\Omega\bigl(\bigl(P_\delta\varphi^{(1-\beta)}\bigr)^s + P_\delta\varphi^{-\beta}\bigr)\,dx \le C\bigl(1+A^{\frac{1+\theta}{2}}\bigr) \le C. \tag{4.62}$$

It remains to show that

$$\left\|\rho|\mathbf{u}|^2\right\|_{L^s} \le C. \tag{4.63}$$

Set $\alpha \triangleq \frac{2s}{\gamma(3-s)}$. It follows from (4.4) and (4.2) that $\alpha \in (0,1)$. Thus, we obtain from Lemma 2.4, (4.56), (4.61), and (4.62)

$$\begin{aligned}\int_\Omega P_\delta^\alpha(\rho)|\mathbf{u}|^2\,dx &\le C\bigl(1 + \|\mathbf{u}\|_{H^1} + \|\mathbf{H}\|_{H^1} + \|P_\delta\|_{L^1} + \left\|\rho|\mathbf{u}|^2\right\|_{L^1}\bigr)\|\mathbf{u}\|_{H^1}^2 \\ &\le C\bigl(1 + \left\|\rho|\mathbf{u}|^2\right\|_{L^1}\bigr).\end{aligned} \tag{4.64}$$

On the other hand, we have

$$\rho^{\frac{2s}{3-s}} = \rho^{\alpha\gamma} \le CP_\delta^\alpha(\rho).$$

Hence

$$\int_\Omega \rho^{\frac{2s}{3-s}}|\mathbf{u}|^2\,dx \le C\bigl(1 + \left\|\rho|\mathbf{u}|^2\right\|_{L^1}\bigr). \tag{4.65}$$

Notice that

$$\rho^s|\mathbf{u}|^{2s} = \bigl(\rho^{\frac{2s}{3-s}}|\mathbf{u}|^2\bigr)^{\frac{3-s}{2}}|\mathbf{u}|^{6\cdot\frac{s-1}{2}}.$$

Then Hölder's inequality yields

$$\int_\Omega \rho^s|\mathbf{u}|^{2s}\,dx \le \left(\int_\Omega \rho^{\frac{2s}{3-s}}|\mathbf{u}|^2\,dx\right)^{\frac{3-s}{2}}\left(\int_\Omega |\mathbf{u}|^6\,dx\right)^{\frac{s-1}{2}}. \tag{4.66}$$

It follows from Sobolev's embedding theorem that

$$\|\mathbf{u}\|_{L^6} \le \|\mathbf{u}\|_{H^1} \le C,$$

which combined with (4.65)-(4.66) implies

$$\int_\Omega \rho^s|\mathbf{u}|^{2s}\,dx \le C\bigl(\left\|\rho|\mathbf{u}|^2\right\|_{L^1}^{\frac{3-s}{2}} + 1\bigr). \tag{4.67}$$

Since $\frac{3-s}{2} < s$, (4.63) easily follows from (4.67).

This completes the proof of Proposition 4.1. □

5 Proof of Theorem 1.1

According to (4.3) and the compact embedding $W^{1,2}(\Omega) \to L^p(\Omega)$ for $p \in [1,6)$, we can choose a subsequence such that

$$\begin{aligned}&\delta\rho_\delta^4 \to 0 \quad \text{in } \mathcal{D}'(\Omega),\\ &\mathbf{u}_\delta \rightharpoonup \mathbf{u}, \qquad \mathbf{H}_\delta \rightharpoonup \mathbf{H} \quad \text{weakly in } W^{1,2}(\Omega),\end{aligned}$$

$$\mathbf{u}_\delta \to \mathbf{u}, \qquad \mathbf{H}_\delta \to \mathbf{H} \quad \text{strongly in } L^p(\Omega) \text{ for } p \in [1,6),$$
$$\rho_\delta \rightharpoonup \rho \quad \text{weakly in } L^{\gamma s}(\Omega),$$
$$\rho_\delta^\gamma \rightharpoonup \overline{\rho}^\gamma \quad \text{weakly in } L^s(\Omega).$$

Combining this with Lemma 2.2 and (4.3), we obtain

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \quad \text{and} \quad \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^s(\Omega).$$

Then, passing to the limit $\delta \to 0$ in the approximate equations (3.6)-(3.9), we get

$$\operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega),$$
$$-\mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \overline{\rho}^\gamma - (\nabla \times \mathbf{H}) \times \mathbf{H} = \rho \mathbf{f} \quad \text{in } \mathcal{D}'(\Omega),$$
$$-\nu \Delta \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = \mathbf{0},$$
$$\operatorname{div} \mathbf{H} = 0,$$
$$\operatorname{div}\big(\overline{b(\rho_\delta)}\mathbf{u}_\delta\big) + \overline{\big(\rho_\delta b'(\rho_\delta) - b(\rho_\delta)\big) \operatorname{div} \mathbf{u}_\delta} = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Hence, to complete the proof of Theorem 1.1, we have to show the strong convergence of ρ_δ to ρ in $L^1(\Omega)$. This task can be fulfilled following Section 4.11, pp.239-245 in [24] and we will not give the details here.

The proof of Theorem 1.1 is finished.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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