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Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdélyi-Kober integral boundary conditions on the half-line

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Abstract

This paper investigates the existence of solutions for nonlinear fractional differential equations with *m*-point Erdélyi-Kober fractional integral boundary conditions on an infinite interval via the Leray-Schauder nonlinear alternative and the Banach contraction principle. Some examples illustrating the main results are also presented.

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1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, electrical circuits, biology, control theory, fitting of experimental data, and so on, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. The monographs [1–3] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional order. For more details and examples, see [4–9] and the references therein.

However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivative. Besides these derivatives, the so-called Erdélyi-Kober fractional derivative, as a generalization of the Riemann-Liouville fractional derivative, is often used, too. An Erdélyi-Kober operator is a fractional integration operation introduced by Arthur Erdélyi and Hermann Kober in 1940. These operators have been used by many authors, in particular, to obtain solutions of the single, dual and triple integral equations possessing special functions of mathematical physics as their kernels. For the theory and applications of the Erdélyi-Kober fractional integrals, see, *e.g.*, [1, 2, 10–14] and the references cited therein.

The fractional anomalous diffusion equations have been studied by many researchers [15, 16]. In fact, the mathematical models of nonlinear fractional diffusion equations have been successfully applied to several phenomena, see [17–19] and the references cited therein. In [20] Pagnini investigated the generalized grey Brownian motion (ggBm) which is an anomalous diffusion process derived by the Erdélyi-Kober fractional integral operator. Some relationships between parameters of the Erdélyi-Kober fractional operators and the valuable family of stochastic processes generated by the ggBm were also shown. For more details on fractional diffusion processes in stochastic models, we refer the reader to [21]. In [22], the author gave a theorem about the series representation of the Erdélyi-Kober fractional integral operator which was used to find approximate solutions of linear and nonlinear fractional anomalous diffusions. Numerical analysis and applications to the real experimental data were also discussed.

Boundary value problems on infinite intervals arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena [23]. For boundary value problems of fractional order on infinite intervals, we refer to [24–28].

Zhao and Ge [28] studied the existence of unbounded solutions for the following boundary value problem on the infinite interval:

$$D_0^\alpha u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 1 < \alpha \le 2, t \in [0, \infty), \tag{1.1}$$

$$u(0) = 0, \qquad \lim_{t\to\infty} D_0^{\alpha-1} u(t) = \beta u(\xi), \tag{1.2}$$

where D_0^α denotes the Riemann-Liouville fractional derivative of order α, and $0 < \beta, \xi < \infty$.

Zhang *et al.* [27] studied the existence of nonnegative solutions for the following boundary value problem for fractional differential equations with nonlocal boundary conditions on unbounded domains:

$$D_0^\alpha u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 1 < \alpha \le 2, t \in [0, \infty), \tag{1.3}$$

$$I_0^{\alpha-2} u(0) = 0, \qquad \lim_{t\to\infty} D_0^{\alpha-1} u(t) = \beta I_0^{\alpha-1} u(\eta), \tag{1.4}$$

where D_0^α denotes the Riemann-Liouville fractional derivative of order $\alpha, f \in C([0,\infty) \times \mathbb{R}, \mathbb{R}^+)$ and $0 < \beta, \eta < \infty$. The Leray-Schauder nonlinear alternative is used.

Liang and Zhang [25] used a fixed point theorem for operators on a cone and proved the existence of positive solutions to the following fractional boundary value problem:

$$D_0^\alpha u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 2 < \alpha \le 3, t \in [0, \infty), \tag{1.5}$$

$$u(0) = u'(0) = 0, \qquad \lim_{t\to\infty} D_0^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{1.6}$$

where D_0^α denotes the Riemann-Liouville fractional derivative of order $\alpha, f \in C([0,\infty) \times \mathbb{R}, \mathbb{R}^+)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $\beta_i \ge 0$, $i = 1, 2, \ldots, m-2$, with $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$.

Motivated by the above papers, in this article, we study a new class of boundary value problems on fractional differential equations with m-point Erdélyi-Kober fractional integral boundary conditions on an infinite interval of the form

$$D_0^\alpha u(t) + f\big(t, u(t)\big) = 0, \quad 1 < \alpha \leq 2, t \in (0, \infty), \tag{1.7}$$

$$u(0) = 0, \qquad D_0^{\alpha-1} u(\infty) = \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} u(\xi_i), \tag{1.8}$$

where D_0^α denotes the Riemann-Liouville fractional derivative of order α, $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-2$, $\beta_i \in \mathbb{R}$, and $\xi_i \in (0, \infty)$, $i = 1, 2, \ldots, m-2$, are given constants. We prove the existence and uniqueness of an unbounded solution of the boundary value problem (1.7)-(1.8) by using the Leray-Schauder nonlinear alternative and the Banach contraction principle.

This paper is organized as follows. In Section 2, we prepare some material needed to prove our main results. In Section 3, we obtain the existence and uniqueness results, while in Section 4 we give some examples to illustrate our results.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1] and present preliminary results needed in our proofs later.

Definition 2.1 The Riemann-Liouville fractional derivative of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D_0^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\, ds, \quad q > 0, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The Riemann-Liouville fractional integral of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds, \quad q > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.3 The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}}\, ds$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$.

Remark 2.4 For $\eta = 1$, the above operator is reduced to the Kober operator

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}}\, ds, \quad \gamma, \delta > 0,$$

that was introduced for the first time by Kober in [12]. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight

$$I_1^{0,\delta}f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}}\,ds, \quad \delta > 0.$$

From the definition of the Riemann-Liouville fractional derivative and integral, we can obtain the following lemmas.

Lemma 2.5 (See [1]) *Let $q > 0$ and $y \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $D_0^q y(t) = 0$ has a unique solution*

$$y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 < q < n$.

Lemma 2.6 (See [1]) *Let $q > 0$. Then, for $y \in C(0,T) \cap L(0,T)$, it holds*

$$I^q D_0^q y(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 < q < n$.

The following lemmas will be used in the proof of our main results.

Lemma 2.7 *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \tag{2.1}$$

Proof Recall the beta function and its property

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du \quad \text{and} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y > 0$. From Definition 2.3, we have

$$\begin{aligned} I_\eta^{\gamma,\delta} t^q &= \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} \cdot s^q}{(t^\eta - s^\eta)^{1-\delta}}\,ds \\ &= \frac{t^q}{\Gamma(\delta)} \int_0^1 u^{\gamma+\frac{q}{\eta}}(1-u)^{\delta-1}\,du \\ &= \frac{t^q}{\Gamma(\delta)} B\left(\gamma + \frac{q}{\eta} + 1, \delta\right) \\ &= \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \end{aligned}$$

The proof is complete. □

Lemma 2.8 *Let $h \in C[0,\infty)$ with $0 < \int_0^\infty h(s)\,ds < \infty$, and*

$$\Lambda = \Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} \frac{\Gamma(\gamma_i + (\alpha-1)/\eta_i + 1)}{\Gamma(\gamma_i + (\alpha-1)/\eta_i + \delta_i + 1)} \neq 0. \tag{2.2}$$

Then the unique solution of the following linear Riemann-Liouville fractional differential equation

$$D_0^\alpha u(t) + h(t) = 0, \quad t \in (0,\infty), 1 < \alpha \leq 2, \tag{2.3}$$

subject to the Erdélyi-Kober fractional integral boundary condition

$$u(0) = 0, \qquad D_0^{\alpha-1} u(\infty) = \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} u(\xi_i), \tag{2.4}$$

is given by

$$u(t) = \frac{t^{\alpha-1}}{\Lambda} \int_0^\infty h(s)\,ds - \frac{t^{\alpha-1}}{\Lambda} \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha h(\xi_i) - I^\alpha h(t). \tag{2.5}$$

Proof Applying the Riemann-Liouville fractional integral of order α to both sides of (2.3), we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - I^\alpha h(t), \tag{2.6}$$

where $c_1, c_2 \in \mathbb{R}$.

The first condition of (2.4) implies $c_2 = 0$. Therefore, we have

$$u(t) = c_1 t^{\alpha-1} - I^\alpha h(t). \tag{2.7}$$

The second condition of (2.4) leads to

$$c_1 = \frac{1}{\Lambda}\left(\int_0^\infty h(s)\,ds - \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha h(\xi_i)\right), \tag{2.8}$$

where Λ is defined by (2.2). Thus, the unique solution of fractional boundary value problem (2.3)-(2.4) is as the following integral equation:

$$u(t) = \frac{t^{\alpha-1}}{\Lambda} \int_0^\infty h(s)\,ds - \frac{t^{\alpha-1}}{\Lambda} \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha h(\xi_i) - I^\alpha h(t).$$

Now, by uniqueness of constants c_1, c_2, we conclude that (2.5) is the unique solution of the boundary value problem (2.3)-(2.4). The proof is completed. □

In this paper, we will use the following space E, which is defined by

$$E = \left\{ u \in C[0,\infty) : \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}} < \infty \right\}$$

and is equipped with the norm

$$\|u\|_E = \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}}.$$

It is known from [24] that E with the above norm is a Banach space.

Using Lemma 2.8 with $h(t) = f(t, u(t))$, we define the operator $T : E \to E$ by

$$\begin{aligned} Tu(t) = {}& \frac{t^{\alpha-1}}{\Lambda} \int_0^\infty f\bigl(s, u(s)\bigr)\,ds - \frac{t^{\alpha-1}}{\Lambda} \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i, u(\xi_i)\bigr) \\ & - I^\alpha f\bigl(t, u(t)\bigr). \end{aligned} \tag{2.9}$$

Notice that problem (1.7)-(1.8) has a solution if and only if the operator equation $u = Tu$ has a fixed point, where T is given by (2.9).

We recall the following well-known fixed point theorem which we use in the next section.

Theorem 2.9 (Nonlinear alternative for single-valued maps) ([29]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Lemma 2.10 ([26]) *Let $V = \{u \in E : \|u\|_E < l, l > 0\}$, $V_1 = \{u(t)/(1 + t^{\alpha-1}) : u \in V\}$. If V_1 is equicontinuous on any compact intervals of $[0,\infty)$ and equiconvergent at infinity, then V is relatively compact on E.*

Remark 2.11 V_1 is called equiconvergent at infinity if and only if for all $\epsilon > 0$ there exists $\nu(\epsilon) > 0$ such that for all $u \in V_1$, $t_1, t_2 \geq \nu$, it holds

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \epsilon.$$

Throughout this paper, we assume that the following conditions hold:

(A_1) Let $|f(t,(1+t^{\alpha-1})u)| \leq \varphi_1(t)\omega_1(|u|)$ on $[0,\infty)\times\mathbb{R}$ with $\omega_1 \in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi_1 \in L^1[0,\infty)$.

(A_2) There exists a positive function $\varphi_2(t)$ with $\varphi_2 \in L^1[0,\infty)$ such that

$$\bigl|f\bigl(t,\bigl(1+t^{\alpha-1}\bigr)u\bigr) - f\bigl(t,\bigl(1+t^{\alpha-1}\bigr)v\bigr)\bigr| \leq \varphi_2(t)|u-v| \tag{2.10}$$

for each $t \in [0,\infty)$ and $u, v \in E$.

Remark 2.12 Condition (A_2) means a kind of sublinearity. Such a condition is known as Krasnosel'ski's condition.

For convenience, we set

$$\begin{aligned}\Omega_1 &= \left(\frac{1}{|\Lambda|} + \frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \varphi_1(s)\,ds \\ &\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\,ds\,dr. \end{aligned}\tag{2.11}$$

Lemma 2.13 *Let* (A_1) *hold. Then the operator* $T: E \to E$ *is completely continuous.*

Proof We divide the proof into four steps.

Step 1: *We show that* T *is uniformly bounded on* E.

Let Φ be any bounded subset of E, then there exists a constant $L_1 > 0$ such that $\|u\|_E \le L_1$ for all $u \in \Phi$. It follows that

$$\begin{aligned} &\|Tu\|_E \\ &= \sup_{t\in[0,\infty)}\Bigg|\frac{t^{\alpha-1}}{\Lambda(1+t^{\alpha-1})}\int_0^\infty f\big(s,u(s)\big)\,ds \\ &\quad - \frac{t^{\alpha-1}}{\Lambda(1+t^{\alpha-1})}\sum_{i=1}^{m-2}\frac{\beta_i\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f\big(s,u(s)\big)\,ds\,dr \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} f\big(s,u(s)\big)\,ds\Bigg| \\ &\le \frac{1}{|\Lambda|}\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds \\ &\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds\,dr \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds \\ &\le \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \varphi_1(s)\omega_1\left(\frac{|u(s)|}{1+s^{\alpha-1}}\right)ds \\ &\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\omega_1\left(\frac{|u(s)|}{1+s^{\alpha-1}}\right)ds\,dr \\ &\le \omega_1(L_1)\left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \varphi_1(s)\,ds \\ &\quad + \frac{\omega_1(L_1)}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\,ds\,dr \\ &= \omega_1(L_1)\Omega_1 < \infty \quad \text{for } u \in \Phi. \end{aligned}$$

Therefore $T\Phi$ is uniformly bounded.

Step 2: *We show that* T *is equicontinuous on any compact interval of* $[0,\infty)$.

For any $S > 0$, $t_1, t_2 \in [0, S]$ and $u \in \Phi$, without loss of generality, we assume that $t_1 < t_2$. It follows that

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
&= \left|\frac{t_2^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds - \frac{t_2^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr)\right. \\
&\quad - \frac{I^\alpha f(t_2,u(t_2))}{(1+t_2^{\alpha-1})} - \frac{t_1^{\alpha-1}}{\Lambda(1+t_1^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds \\
&\quad \left. + \frac{t_1^{\alpha-1}}{\Lambda(1+t_1^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr) + \frac{I^\alpha f(t_1,u(t_1))}{(1+t_1^{\alpha-1})}\right| \\
&= \left|\frac{t_2^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds - \frac{t_2^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr)\right. \\
&\quad - \frac{I^\alpha f(t_2,u(t_2))}{(1+t_2^{\alpha-1})} - \frac{t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds \\
&\quad + \frac{t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr) + \frac{I^\alpha f(t_1,u(t_1))}{(1+t_2^{\alpha-1})} \\
&\quad + \frac{t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds - \frac{t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr) \\
&\quad - \frac{I^\alpha f(t_1,u(t_1))}{(1+t_2^{\alpha-1})} - \frac{t_1^{\alpha-1}}{\Lambda(1+t_1^{\alpha-1})}\int_0^\infty f\bigl(s,u(s)\bigr)\,ds \\
&\quad \left. + \frac{t_1^{\alpha-1}}{\Lambda(1+t_1^{\alpha-1})}\sum_{i=1}^{m-2}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i,u(\xi_i)\bigr) + \frac{I^\alpha f(t_1,u(t_1))}{(1+t_1^{\alpha-1})}\right| \\
&\le \left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\right| \int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds + \left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\right| \sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \\
&\quad \times \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds\,dr + \frac{1}{\Gamma(\alpha)(1+t_2^{\alpha-1})} \\
&\quad \times \left|\int_0^{t_1}(t_1-s)^{\alpha-1} f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)ds - \int_0^{t_2}(t_2-s)^{\alpha-1} f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)ds\right| \\
&\quad + \left|\frac{1}{1+t_2^{\alpha-1}} - \frac{1}{1+t_1^{\alpha-1}}\right|\left(\frac{t_1^{\alpha-1}}{|\Lambda|}\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds\right. \\
&\quad + \frac{t_1^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds\,dr \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|ds\right) \\
&\le \omega_1(L_1)\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\right|\int_0^\infty \varphi_1(s)\,ds
\end{aligned}$$

$$\begin{aligned}
&+\omega_1(L_1)\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Lambda(1+t_2^{\alpha-1})}\right|\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r\frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\,ds\,dr\\
&+\frac{\omega_1(L_1)}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\varphi_1(s)\,ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi_1(s)\,ds\right)\\
&+\omega_1(L_1)\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}\right|\left(\frac{t_1^{\alpha-1}}{|\Lambda|}\int_0^{\infty}\varphi_1(s)\,ds\right.\\
&+\frac{t_1^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r\frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\,ds\,dr\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_1(s)\,ds\right),
\end{aligned}$$

which is independent of u and tends to zero as $t_1\to t_2$. Thus $T\Phi$ is equicontinuous on $[0,\infty)$.

Step 3: *We show that T is equiconvergent at ∞.*

For any $u\in\Phi$, we have

$$\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds\le\omega_1(L_1)\int_0^{\infty}\varphi_1(s)\,ds<\infty,$$

and

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| = \lim_{t\to\infty}\left|\frac{1}{1+t^{\alpha-1}}\right[&\frac{t^{\alpha-1}}{\Lambda}\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds\\
&\left.\left.-\frac{t^{\alpha-1}}{\Lambda}\sum_{i=1}^{m-2}\beta_iI_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha}f\bigl(\xi_i,u(\xi_i)\bigr)-I^{\alpha}f\bigl(t,u(t)\bigr)\right]\right|.
\end{aligned}$$

We now consider

$$\lim_{t\to\infty}\frac{1}{1+t^{\alpha-1}}\cdot\frac{t^{\alpha-1}}{\Lambda}\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds=\frac{1}{\Lambda}\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds$$

and

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{1+t^{\alpha-1}}I^{\alpha}f\bigl(t,u(t)\bigr) &= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}f\bigl(s,u(s)\bigr)\,ds\\
&\le\lim_{t\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t f\bigl(s,u(s)\bigr)\,ds=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \le{}&\left|\left(\frac{1}{\Lambda}-\frac{1}{\Gamma(\alpha)}\right)\int_0^{\infty}f\bigl(s,u(s)\bigr)\,ds\right.\\
&\left.-\frac{1}{\Lambda}\sum_{i=1}^{m-2}\frac{\beta_i\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r\frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}f\bigl(s,u(s)\bigr)\,ds\,dr\right|\\
<{}&\infty.
\end{aligned}$$

Hence, $T\Phi$ is equiconvergent at infinity.

Step 4: *We show that* T *is continuous.*

Taking $u_n, u \in E$ such that $\|u_n\|_E < \infty$, $\|u\|_E < \infty$ and $u_n \to u$ as $n \to \infty$, then by (A_1) we have

$$\int_0^\infty f(s,u_n(s))\,ds \le \omega_1(\|u_n\|_E)\int_0^\infty \varphi_1(s)\,ds < \infty,$$

and

$$\begin{aligned}&\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f(s,u_n(s))\,ds\,dr\\ &\quad\le \omega_1(\|u_n\|_E)\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_1(s)\,ds\,dr < \infty \quad \text{for } i=1,2,\ldots,m-2.\end{aligned}$$

Hence the Lebesgue dominated convergence and the continuity of f guarantee that

$$\int_0^\infty f(s,u_n(s))\,ds \to \int_0^\infty f(s,u(s))\,ds, \quad \text{as } n\to\infty,$$

and

$$\begin{aligned}&\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f(s,u_n(s))\,ds\,dr \to \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f(s,u(s))\,ds\,dr,\\ &\quad \text{as } n\to\infty, \text{ for } i=1,2,\ldots,m-2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|Tu_n - Tu\|_E &\le \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\left|\int_0^\infty f(s,u_n(s))\,ds - \int_0^\infty f(s,u(s))\,ds\right|\\ &\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\left|\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f(s,u_n(s))\,ds\,dr\right.\\ &\quad \left. - \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} f(s,u(s))\,ds\,dr\right| \to 0, \quad \text{as } n\to\infty.\end{aligned}$$

So, T is continuous.

Using Lemma 2.10, we obtain that $T: E \to E$ is completely continuous. The proof is completed. □

3 Main results

Theorem 3.1 *Assume that* (A_1) *holds. If there exists* $\kappa > 0$ *such that*

$$\frac{\kappa}{\omega_1(\kappa)\Omega_1} > 1, \tag{3.1}$$

where Ω_1 *is defined in* (2.11), *then the boundary value problem* (1.7)-(1.8) *has at least one solution on* $[0,\infty)$.

Proof Consider the operator $T: E \to E$ defined by (2.9). This operator is continuous and completely continuous by Lemma 2.13. We will show that there exists an open set $U \subseteq E$

with $u \neq \lambda T(u)$ for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u = \lambda T(u)$ for $\lambda \in (0,1)$. Then we have

$$\begin{aligned}
\|u\|_E &= \sup_{t\in[0,\infty)} \left| \frac{\lambda(Tu)(t)}{1+t^{\alpha-1}} \right| \\
&\leq \sup_{t\in[0,\infty)} \left| \frac{1}{1+t^{\alpha-1}} \left[\frac{t^{\alpha-1}}{\Lambda} \int_0^\infty f\bigl(s,u(s)\bigr)\,ds \right.\right. \\
&\quad \left.\left. - \frac{t^{\alpha-1}}{\Lambda} \sum_{i=1}^{m-2} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f\bigl(\xi_i, u(\xi_i)\bigr) - I^\alpha f\bigl(t,u(t)\bigr) \right] \right| \\
&\leq \omega_1\bigl(\|u\|_E\bigr) \left(\frac{1}{|\Lambda|} + \frac{1}{\Gamma(\alpha)} \right) \int_0^\infty \varphi_1(s)\,ds \\
&\quad + \frac{\omega_1(\|u\|_E)}{|\Lambda|} \sum_{i=1}^{m-2} \frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} \varphi_1(s)\,ds\,dr \\
&= \omega_1\bigl(\|u\|_E\bigr)\Omega_1.
\end{aligned}$$

This implies that

$$\frac{\|u\|_E}{\omega_1(\|u\|_E)\Omega_1} \leq 1.$$

In view of (3.1), there exists κ such that $\|u\|_E \neq \kappa$. We define $U = \{u \in E : \|u\|_E < \kappa\}$. Note that the operator $T : \overline{U} \to E$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda Tu$ for some $\lambda \in (0,1)$. Consequently, by Lemma 2.9, the boundary value problem (1.7)-(1.8) has at least one solution on $[0,\infty)$. The proof is completed. □

In the next theorem we prove an existence and uniqueness result for the boundary value problem (1.7)-(1.8) by using the Banach fixed point theorem. To simplify its proof, we set

$$\rho_1 = \left(\frac{1}{|\Lambda|} + \frac{1}{\Gamma(\alpha)} \right) \int_0^\infty \bigl|f(s,0)\bigr|\,ds < \infty, \tag{3.2}$$

$$\rho_2 = \frac{1}{|\Lambda|} \sum_{i=1}^{m-2} \frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} \bigl|f(s,0)\bigr|\,ds\,dr, \tag{3.3}$$

$$\begin{aligned}
\Omega_2 &= \left(\frac{1}{|\Lambda|} + \frac{1}{\Gamma(\alpha)} \right) \int_0^\infty \varphi_2(s)\,ds \\
&\quad + \frac{1}{|\Lambda|} \sum_{i=1}^{m-2} \frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} \varphi_2(s)\,ds\,dr.
\end{aligned} \tag{3.4}$$

Theorem 3.2 *Assume that* (A_2) *holds. If*

$$\Omega_2 < 1, \tag{3.5}$$

then the boundary value problem (1.7)-(1.8) *has a unique solution on* $[0,\infty)$.

Proof We define the operator T as in (2.9). Then we obtain

$$\begin{aligned}
&\frac{|(Tu)(t)|}{1+t^{\alpha-1}} \\
&\le \frac{1}{|\Lambda|}\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right| ds \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right| ds\,dr \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right| ds \\
&\le \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \left(\varphi_2(s)\left|\frac{u(s)}{1+s^{\alpha-1}}\right| + \left|f(s,0)\right|\right) ds \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \\
&\quad \times \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left(\varphi_2(s)\left|\frac{u(s)}{1+s^{\alpha-1}}\right| + \left|f(s,0)\right|\right) ds\,dr \\
&\le \|u\|_E\Bigg[\left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \varphi_2(s)\,ds + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \\
&\quad \times \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_2(s)\,ds\,dr\Bigg] + \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \left|f(s,0)\right| ds \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\left|f(s,0)\right| ds\,dr \\
&= \Omega_2\|u\|_E + \rho_1 + \rho_2,
\end{aligned}$$

which leads to

$$\|Tu\|_E \le \Omega_2\|u\|_E + \rho_1 + \rho_2 < +\infty \quad \text{for } u \in E,$$

where ρ_1, ρ_2, Ω_2 are defined by (3.2), (3.3) and (3.4), respectively. This shows that T maps E into itself.

On the other hand, for any $u, v \in E$ and any $t \in [0,\infty)$, we get

$$\begin{aligned}
&\frac{|(Tu)(t)-(Tv)(t)|}{1+t^{\alpha-1}} \\
&\le \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right) - f\left(s,\frac{(1+s^{\alpha-1})v(s)}{1+s^{\alpha-1}}\right)\right| ds \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}} \\
&\quad \times \left|f\left(s,\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right) - f\left(s,\frac{(1+s^{\alpha-1})v(s)}{1+s^{\alpha-1}}\right)\right| ds\,dr \\
&\le \left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty \varphi_2(s)\left|\frac{u(s)}{1+s^{\alpha-1}} - \frac{v(s)}{1+s^{\alpha-1}}\right| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)} \\
&\times\int_0^{\xi_i}\int_0^r\frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_2(s)\left|\frac{u(s)}{1+s^{\alpha-1}}-\frac{v(s)}{1+s^{\alpha-1}}\right|ds\,dr \\
\le{}&\|u-v\|_E\Bigg[\left(\frac{1}{|\Lambda|}+\frac{1}{\Gamma(\alpha)}\right)\int_0^\infty\varphi_2(s)\,ds \\
&+\frac{1}{|\Lambda|}\sum_{i=1}^{m-2}\frac{|\beta_i|\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\alpha)\Gamma(\delta_i)}\int_0^{\xi_i}\int_0^r\frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{\alpha-1}}{(\xi_i^{\eta_i}-r^{\eta_i})^{1-\delta_i}}\varphi_2(s)\,ds\,dr\Bigg] \\
={}&\Omega_2\|u-v\|_E.
\end{aligned}
$$

Therefore

$$
\|Tu-Tv\|_E\le\Omega_2\|u-v\|_E\quad\text{for } u,v\in E.
$$

As $\Omega_2<1$, therefore T is a contraction. Hence, by the Banach fixed point theorem, we get that T has a fixed point which is a unique solution of problem (1.7)-(1.8). The proof is completed. □

4 Examples

In this section we present examples illustrating the obtained results.

Example 4.1 Consider the following nonlinear Riemann-Liouville fractional differential equation with Erdélyi-Kober fractional integral condition:

$$
\begin{cases}
D_0^{\frac{4}{3}}u(t)+\frac{u^2(t)e^{-2t}}{100(1+t^{1/3})^2}=0,\quad t\in(0,\infty),\\
u(0)=0,\\
D_0^{\frac{1}{3}}u(\infty)=2I_3^{2,4}u(3)-\sqrt{\pi}I_{\frac{1}{2}}^{-\frac{3}{4},e}u(\frac{25}{4})+e^2I_5^{\frac{2}{3},6}u(10)-\frac{9}{7}I_{\frac{5}{3}}^{3,\pi}u(2).
\end{cases}
\tag{4.1}
$$

Here $\alpha=4/3$, $m=6$, $\beta_1=2$, $\beta_2=-\sqrt{\pi}$, $\beta_3=e^2$, $\beta_4=-9/7$, $\gamma_1=2$, $\gamma_2=-3/4$, $\gamma_3=2/3$, $\gamma_4=3$, $\delta_1=4$, $\delta_2=e$, $\delta_3=6$, $\delta_4=\pi$, $\eta_1=3$, $\eta_2=1/2$, $\eta_3=5$, $\eta_4=5/3$, $\xi_1=3$, $\xi_2=25/4$, $\xi_3=10$, $\xi_4=2$ and $f(t,u)=u^2e^{-2t}/100(1+t^{1/3})^2$. Clearly,

$$
\left|f\left(t,\left(1+t^{1/3}\right)u\right)\right|=\frac{e^{-2t}|u|^2}{100}.
$$

Choosing $\varphi_1(t)=e^{-2t}$, $\omega_1(|u|)=|u|^2/100$, then $\varphi_1\in L^1[0,\infty)$ and $\omega_1\in C([0,\infty),[0,\infty))$ is nondecreasing. We can show that

$$
\Lambda\approx 1.780857113,\qquad \Omega_1\approx 0.9547223123,
$$

and

$$
\frac{\kappa}{(\frac{\kappa^2}{100})(0.9547223123)}>1,
$$

which implies that $0 < \kappa < 104.7424981$. Hence, by Theorem 3.1, the boundary value problem (4.1) has at least one solution on $[0, \infty)$.

Example 4.2 Consider the following nonlinear Riemann-Liouville fractional differential equation with Erdélyi-Kober fractional integral condition:

$$\begin{cases} D_0^{\frac{9}{5}} u(t) + t^2 e^{-3t} \cos(\frac{u(t)}{1+t^{4/5}}) = 0, \quad t \in (0, \infty), \\ u(0) = 0, \\ D_0^{\frac{4}{5}} u(\infty) = 3I_2^{-1,4} u(\frac{2}{7}) + \frac{3}{5} I_1^{4,2} u(6) - \frac{22}{7} I_\pi^{-\frac{2}{3}, \frac{33}{5}} u(\frac{9}{7}) \\ \qquad + 11 I_{\frac{7}{3}}^{2,e} u(\pi^3) - \frac{13}{2} I_4^{\frac{3}{2},6} u(12). \end{cases} \tag{4.2}$$

Here $\alpha = 9/5$, $m = 7$, $\beta_1 = 3$, $\beta_2 = 3/5$, $\beta_3 = -22/7$, $\beta_4 = 11$, $\beta_5 = -13/2$, $\gamma_1 = -1$, $\gamma_2 = 4$, $\gamma_3 = -2/3$, $\gamma_4 = 2$, $\gamma_5 = 3/2$, $\delta_1 = 4$, $\delta_2 = 2$, $\delta_3 = 33/5$, $\delta_4 = e$, $\delta_5 = 6$, $\eta_1 = 2$, $\eta_2 = 1$, $\eta_3 = \pi$, $\eta_4 = 7/3$, $\eta_5 = 4$, $\xi_1 = 2/7$, $\xi_2 = 6$, $\xi_3 = 9/7$, $\xi_4 = \pi^3$, $\xi_5 = 12$ and $f(t,u) = t^2 e^{-3t} \cos(u/(1 + t^{4/5}))$. Since

$$\left| f\left(t, \left(1 + t^{\frac{4}{5}}\right)u\right) - f\left(t, \left(1 + t^{\frac{4}{5}}\right)v\right)\right| \le t^2 e^{-3t} |u - v|,$$

then (A_2) is satisfied with $\varphi_2(t) = t^2 e^{-3t}$. We can show that

$$\Lambda \approx -2.978499743, \qquad \Omega_2 \approx 0.1989863061 < 1.$$

Hence, by Theorem 3.2, the boundary value problem (4.2) has a unique solution on $[0, \infty)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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