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Existence of weak solutions for a class of quasilinear elliptic systems

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Abstract
Existence of weak solutions for a class of nonlinear elliptic systems is obtained under the certain Landesman-Lazer-type conditions by variational method.

Keywords: elliptic systems; Landesman-Lazer-type condition; (*PS*) condition

1 Introduction and main results

In this paper, we consider the existence of weak solutions for the following gradient elliptic systems:

$$\begin{cases} -\triangle_p u = \lambda_1 a(x)|u|^{p-2}u + \lambda_1 \frac{b(x)}{\beta+1}|u|^\alpha |v|^\beta v + F_u(x,u,v) - h_1(x) & \text{in } \Omega, \\ -\triangle_p v = \lambda_1 c(x)|v|^{p-2}v + \lambda_1 \frac{b(x)}{\alpha+1}|u|^\alpha |v|^\beta u + F_v(x,u,v) - h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N$ ($N \geq 3$) is a bounded smooth domain, $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian, $2 \leq p < N$ and $\alpha \geq 0$, $\beta \geq 0$ satisfy

$$\alpha + \beta + 2 = p.$$

$F \in C^1(\overline{\Omega} \times R^2, R)$ and $F_s(x,s,t)$ designates the partial derivative of F with respect to s and $h_1, h_2 \in L^q(\Omega)$ ($q = p/(p-1)$). The coefficient functions $a, b, c \in C(\Omega) \cap L^\infty(\Omega)$ satisfy one of the following conditions:

(A1) $a^+ \not\equiv 0$, where $a^+(x) := \max\{a(x), 0\}$;

(A2) $c^+ \not\equiv 0$;

(A3) $a = c = 0$ and $b^+ \not\equiv 0$.

Let W be the product space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ equipped with the norm $\|(u,v)\| = (\|u\|^p + \|v\|^p)^{1/p}$ for all $(u,v) \in W$, where $\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$ for any $u \in W_0^{1,p}(\Omega)$. The embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and there exists a positive constant C such that

$$\|u\|_{L^p} \leq C\|u\| \quad \text{for all } u \in W_0^{1,p}(\Omega), \tag{2}$$

where $\|\cdot\|_{L^p}$ denotes the norm of $L^p(\Omega)$.

Consider the following nonlinear eigenvalue problem with weights:

$$\begin{cases} -\triangle_p u = \lambda a(x)|u|^{p-2}u + \lambda \frac{b(x)}{\beta+1}|u|^{\alpha}|v|^{\beta}v & \text{in } \Omega, \\ -\triangle_p v = \lambda c(x)|v|^{p-2}v + \lambda \frac{b(x)}{\alpha+1}|u|^{\alpha}|v|^{\beta}u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

If one of the conditions (A1)-(A3) holds, the first eigenvalue λ_1 of (3) is simple, isolated and positive, and has a unique associated eigenfunction (μ_1, ν_1) with $\|(\mu_1, \nu_1)\| = 1$ and $\mu_1 > 0$, $\nu_1 > 0$ in Ω (the proof is found in [1, 2]).

The Landesman-Lazer-type conditions were introduced by Landesman and Lazer in [3], where they considered the existence of weak solutions for the resonant elliptic problems, and then were widely used and extended (see [1–10] and their references). For nonlinear elliptic systems, let $F_s(x,s,t) = g_1(s)$, $F_t(x,s,t) = g_2(t)$ and by using the some Landesman-Lazer-type conditions, Zographopoulos in [1] proved the existence of weak solutions for problem (1) at resonance with the first eigenvalue λ_1, and by using the Landesman-Lazer-type conditions due to Tang and the G-linking theorem, Ou and Tang in [2] proved the existence of weak solutions for problem (1) at resonance with the higher eigenvalues of problem (3). When $p = 2$, Silva in [10] introduced the new Landesman-Lazer-type conditions and proved the existence of weak solutions for problem (1) by using variational methods, Morse theory and critical groups.

Motivated by [10], we consider the existence of weak solutions for problem (1) under the certain Landesman-Lazer-type conditions. We now give some auxiliary conditions.

(F1) There is $h \in C(\Omega, R^+)$ such that

$$\left|F_s(x,s,t)\right| \le h(x) \quad \text{and} \quad \left|F_t(x,s,t)\right| \le h(x), \quad \forall (x,s,t) \in \Omega \times R^2.$$

(F2) There exist functions $f^{++}, f^{--} \in C(\Omega, R)$ such that

$$f^{++}(x) = \lim_{\substack{s\to+\infty \\ t\to+\infty}} F_s(x,s,t), \qquad f^{--}(x) = \lim_{\substack{s\to-\infty \\ t\to-\infty}} F_s(x,s,t).$$

(F3) There exist functions $g^{++}, g^{--} \in C(\Omega, R)$ such that

$$g^{++}(x) = \lim_{\substack{s\to+\infty \\ t\to+\infty}} F_t(x,s,t), \qquad g^{--}(x) = \lim_{\substack{s\to+\infty \\ t\to+\infty}} F_t(x,s,t),$$

where the above limits of conditions (F2) and (F3) are taken uniformly for all $x \in \Omega$. The Landesman-Lazer-type conditions for problem (1) will be assumed either

$$(LL)_1^+ \quad \int_\Omega f^{--}\mu_1 + g^{--}\nu_1\,dx < \int_\Omega h_1\mu_1 + h_2\nu_1\,dx < \int_\Omega f^{++}\mu_1 + g^{++}\nu_1\,dx$$

or

$$(LL)_1^- \quad \int_\Omega f^{--}\mu_1 + g^{--}\nu_1\,dx > \int_\Omega h_1\mu_1 + h_2\nu_1\,dx > \int_\Omega f^{++}\mu_1 + g^{++}\nu_1\,dx.$$

We are ready to introduce the main results of this paper.

Theorem 1 *Assume that* $h_1, h_2 \in L^q(\Omega)$ $(q = p/(p-1))$ *and one of the conditions* (A1)-(A3) *holds. If F satisfies* (F1), (F2), (F3) *and* $(LL)_1^+$, *then problem* (1) *has at least one solution.*

Theorem 2 *Assume that* $h_1, h_2 \in L^q(\Omega)$ $(q = p/(p-1))$ *and one of the conditions* (A1)-(A3) *holds. If F satisfies* (F1), (F2), (F3) *and* $(LL)_1^-$, *then problem* (1) *has at least one solution.*

2 Proofs of theorems

Let $J : W \to R$ be the functional defined by

$$J(u,v) = \phi(u,v) - \lambda_1 \psi(u,v) - \int_\Omega F(x,u,v)\,dx + \int_\Omega h_1(x)u\,dx + \int_\Omega h_2(x)v\,dx, \tag{4}$$

where

$$\phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |\nabla v|^p\,dx, \quad \text{and}$$
$$\psi(u,v) = \frac{1}{p}\int_\Omega a(x)|u|^p\,dx + \frac{1}{p}\int_\Omega c(x)|v|^p\,dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|u|^\alpha|v|^\beta uv\,dx.$$

If one of the conditions (A1)-(A3) holds, by (F1) and $h_1, h_2 \in L^q(\Omega)$, it is not difficult to verify that $J \in C^1(W,R)$, and it is well known that a critical point of the functional J in W corresponds to a weak solution of problem (1). We will prove Theorem 1 by the saddle point theorem due to Rabinowitz (see [11]) and Theorem 2 by Ekeland's variational principle (see [12]).

Proof of Theorem 1 We divide the proof into two steps.

(i) We claim that the functional J satisfies the (PS) condition. Let $(u_n, v_n) \in W$ be a (PS) sequence for the functional J, that is,

$$J(u_n, v_n) \to c \in R \quad \text{and} \quad J'(u_n, v_n) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

We first verify that (u_n, v_n) is bounded in W, and then prove that (u_n, v_n) has a convergent subsequence. Suppose, by contradiction, that $K_n := \|(u_n, v_n)\| = (\|u_n\|^p + \|v_n\|^p)^{1/p} \to \infty$ as $n \to \infty$. Let $\tilde{u}_n = u_n \setminus K_n$, $\tilde{v}_n = v_n \setminus K_n$, then $(\tilde{u}_n, \tilde{v}_n)$ is bounded in W, that is,

$$\|\tilde{u}_n\|^p + \|\tilde{v}_n\|^p = 1 \quad \text{for all } n.$$

Hence there is a subsequence of $(\tilde{u}_n, \tilde{v}_n)$, still denoted by $(\tilde{u}_n, \tilde{v}_n)$, and $(\tilde{u}, \tilde{v}) \in W$ such that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in W, $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})$ strongly in $L^p(\Omega) \times L^p(\Omega)$ and $(\tilde{u}_n(x), \tilde{v}_n(x)) \to (\tilde{u}(x), \tilde{v}(x))$ for a.e. $x \in \Omega$. From (F1), (2) and Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_\Omega F(x,u,v)\,dx\right| &\le \int_\Omega \bigl|F(x,u,v)\bigr|\,dx \\ &= \int_\Omega \bigl|F(x,u,v) - F(x,0,0) + F(x,0,0)\bigr|\,dx \\ &\le \int_\Omega \left|\int_0^1 \bigl(F_s(x,\tau u,\tau v)u + F_t(x,\tau u,\tau v)v\bigr)\,d\tau\right|\,dx + \int_\Omega \bigl|F(x,0,0)\bigr|\,dx\end{aligned}$$

$$\begin{aligned}&\leq \int_\Omega h(x)\big(|u|+|v|\big)\,dx + C_0\\&\leq C\|h\|_{L^q}\big(\|u\|+\|v\|\big)+C_0\end{aligned} \tag{6}$$

for all $(u,v)\in W$, where $C_0=\int_\Omega |F(x,0,0)|\,dx$, hence we get

$$\frac{1}{\|u_n\|^p+\|v_n\|^p}\int_\Omega F(x,u_n,v_n)\,dx\to 0 \quad \text{as } n\to\infty, \tag{7}$$

and from $h_1,h_2\in L^q(\Omega)$ $(q=p/(p-1))$ and Hölder's inequality, it follows that

$$\frac{1}{\|u_n\|^p+\|v_n\|^p}\int_\Omega (h_1u_n+h_2v_n)\,dx\to 0 \quad \text{as } n\to\infty. \tag{8}$$

From $(\tilde{u}_n,\tilde{v}_n)\to(\tilde{u},\tilde{v})$ strongly in $L^p(\Omega)\times L^p(\Omega)$, we have $|\tilde{u}_n|^p\to|\tilde{u}|^p$ and $|\tilde{v}_n|^p\to|\tilde{v}|^p$ strongly in $L^1(\Omega)\times L^1(\Omega)$. Hence, it follows that

$$\left|\int_\Omega a(x)|\tilde{u}_n|^p\,dx-\int_\Omega a(x)|\tilde{u}|^p\,dx\right|\leq\|a\|_{L^\infty}\int_\Omega\big||\tilde{u}_n|^p-|\tilde{u}|^p\big|\,dx\to 0 \tag{9}$$

as $n\to\infty$.

From $(\tilde{u}_n(x),\tilde{v}_n(x))\to(\tilde{u}(x),\tilde{v}(x))$ for a.e. $x\in\Omega$ and

$$\begin{aligned}&\int_\Omega\big||\tilde{u}_n|^\alpha\tilde{u}_n\big|^{\frac{p}{\alpha+1}}\,dx=\|\tilde{u}_n\|_{L^p}^p\to\|\tilde{u}\|_{L^p}^p=\int_\Omega\big||\tilde{u}|^\alpha\tilde{u}\big|^{\frac{p}{\alpha+1}}\,dx,\\&\int_\Omega\big||\tilde{v}_n|^\beta\tilde{v}_n\big|^{\frac{p}{\beta+1}}\,dx=\|\tilde{v}_n\|_{L^p}^p\to\|\tilde{v}\|_{L^p}^p=\int_\Omega\big||\tilde{v}|^\beta\tilde{v}\big|^{\frac{p}{\beta+1}}\,dx\end{aligned}$$

as $n\to\infty$, it follows that $|\tilde{u}_n|^\alpha\tilde{u}_n\to|\tilde{u}|^\alpha\tilde{u}$ strongly in $L^{\frac{p}{\alpha+1}}(\Omega)$ and $|\tilde{v}_n|^\beta\tilde{v}_n\to|\tilde{v}|^\beta\tilde{v}$ strongly in $L^{\frac{p}{\beta+1}}(\Omega)$. Hence from Hölder's inequality we obtain

$$\begin{aligned}&\left|\int_\Omega b(x)\big(|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\tilde{u}_n\tilde{v}_n-|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\big)\,dx\right|\\&\quad\leq\|b\|_{L^\infty}\int_\Omega\big||\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\tilde{u}_n\tilde{v}_n-|\tilde{u}_n|^\alpha|\tilde{v}|^\beta\tilde{u}_n\tilde{v}\big|\,dx\\&\qquad+\|b\|_{L^\infty}\int_\Omega\big||\tilde{u}_n|^\alpha|\tilde{v}|^\beta\tilde{u}_n\tilde{v}-|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\big|\,dx\\&\quad\leq\|b\|_{L^\infty}\int_\Omega|\tilde{u}_n|^{\alpha+1}\cdot\big||\tilde{v}_n|^\beta\tilde{v}_n-|\tilde{v}|^\beta\tilde{v}\big|\,dx\\&\qquad+\|b\|_{L^\infty}\int_\Omega\big||\tilde{u}_n|^\alpha\tilde{u}_n-|\tilde{u}|^\alpha\tilde{u}\big|\cdot|\tilde{v}|^{\beta+1}\,dx\\&\quad\leq\|b\|_{L^\infty}\|\tilde{u}_n\|_{L^p}^{\alpha+1}\big\||\tilde{v}_n|^\beta\tilde{v}_n-|\tilde{v}|^\beta\tilde{v}\big\|_{L^{\frac{p}{\beta+1}}}\\&\qquad+\|b\|_{L^\infty}\|\tilde{v}_n\|_{L^p}^{\beta+1}\big\||\tilde{u}_n|^\alpha\tilde{u}_n-|\tilde{u}|^\alpha\tilde{u}\big\|_{L^{\frac{p}{\alpha+1}}}\\&\quad\to 0\quad\text{as } n\to\infty.\end{aligned} \tag{10}$$

From (5) it follows that

$$\limsup_{n\to\infty}\frac{J(u_n,v_n)}{K_n^p}\leq 0.$$

Combining the above inequality with (7), (8), (9) (10) and $\alpha + \beta + 2 = p$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\int_\Omega |\nabla\tilde{u}_n|^p\,dx + \int_\Omega |\nabla\tilde{v}_n|^p\,dx\right) \\
&\quad\le \lambda_1\left(\int_\Omega a(x)|\tilde{u}|^p\,dx + \int_\Omega c(x)|\tilde{v}|^p\,dx + \frac{p}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\,dx\right).
\end{aligned}$$

Hence, using the weak lower semicontinuity of the norm and the Poincaré inequality, we obtain

$$\begin{aligned}
&\lambda_1\left(\int_\Omega a(x)|\tilde{u}|^p\,dx + \int_\Omega c(x)|\tilde{v}|^p\,dx + \frac{p}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\,dx\right) \\
&\quad\le \int_\Omega |\nabla\tilde{u}|^p\,dx + \int_\Omega |\nabla\tilde{v}|^p\,dx \\
&\quad\le \liminf_{n\to\infty}\left(\int_\Omega |\nabla\tilde{u}_n|^p\,dx + \int_\Omega |\nabla\tilde{v}_n|^p\,dx\right) \\
&\quad\le \limsup_{n\to\infty}\left(\int_\Omega |\nabla\tilde{u}_n|^p\,dx + \int_\Omega |\nabla\tilde{v}_n|^p\,dx\right) \\
&\quad\le \lambda_1\left(\int_\Omega a(x)|\tilde{u}|^p\,dx + \int_\Omega c(x)|\tilde{v}|^p\,dx + \frac{p}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\,dx\right),
\end{aligned}$$

which implies that the following equality holds:

$$\begin{aligned}
&\int_\Omega |\nabla\tilde{u}|^p\,dx + \int_\Omega |\nabla\tilde{v}|^p\,dx \\
&\quad= \lambda_1\left(\int_\Omega a(x)|\tilde{u}|^p\,dx + \int_\Omega c(x)|\tilde{v}|^p\,dx + \frac{p}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\,dx\right).
\end{aligned}$$

By the uniform convexity of W, we have that $(\tilde{u}_n, \tilde{v}_n)$ converges strongly to $(\tilde{u}, \tilde{v})$ in W, and from the definition of (μ_1, ν_1), it follows that $(\tilde{u}, \tilde{v}) = \pm(\mu_1, \nu_1)$.

In the following, we assume that $(\tilde{u}, \tilde{v}) = (\mu_1, \nu_1)$, and the case where $(\tilde{u}, \tilde{v}) = -(\mu_1, \nu_1)$ may be treated similarly. Noting that $\alpha + \beta + 2 = p$, it follows that

$$\begin{aligned}
&\frac{p}{K_n(\alpha+1)(\beta+1)}\int_\Omega b(x)|u_n|^\alpha|v_n|^\beta u_n v_n\,dx \\
&\quad= \frac{1}{\beta+1}\int_\Omega b(x)|u_n|^\alpha|v_n|^\beta\tilde{u}_n v_n\,dx + \frac{1}{\alpha+1}\int_\Omega b(x)|u_n|^\alpha|v_n|^\beta u_n\tilde{v}_n\,dx.
\end{aligned}$$

Hence from (4) and the above equality, we have

$$\begin{aligned}
&\frac{pJ(u_n, v_n)}{K_n} - \langle J'(u_n, v_n), (\tilde{u}_n, \tilde{v}_n)\rangle \\
&\quad= \int_\Omega \bigl(F_s(x, u_n, v_n)\tilde{u}_n + F_t(x, u_n, v_n)\tilde{v}_n\bigr)\,dx - \frac{p}{K_n}\int_\Omega F(x, u_n, v_n)\,dx \\
&\qquad+ (p-1)\int_\Omega (h_1\tilde{u}_n + h_2\tilde{v}_n)\,dx. \qquad (11)
\end{aligned}$$

From $h_1, h_2 \in L^q(\Omega)$, we observe

$$\int_\Omega h_1\tilde{u}_n + h_2\tilde{v}_n\,dx \to \int_\Omega h_1\mu_1 + h_2\nu_1\,dx \quad \text{as } n \to \infty. \tag{12}$$

From (F2) and (F3), we have

$$\int_\Omega F_s(x,u_n,v_n)\tilde{u}_n + F_t(x,u_n,v_n)\tilde{v}_n\,dx \to \int_\Omega \bigl(f^{++}\mu_1 + g^{++}\nu_1\bigr)\,dx \quad \text{as } n \to \infty. \tag{13}$$

Finally, from the Lebesgue dominated convergence theorem, (F2) and (F3), we have

$$\begin{aligned}
&\frac{1}{K_n}\int_\Omega F(x,u_n,v_n)\,dx \\
&\quad = \frac{1}{K_n}\int_\Omega\int_0^1 \bigl(F_s(x,\tau u_n,\tau v_n)u_n + F_t(x,\tau u_n,\tau v_n)v_n\bigr)\,d\tau\,dx + \frac{C_0}{K_n} \\
&\quad = \int_\Omega\int_0^1 \bigl(F_s(x,\tau u_n,\tau v_n)\tilde{u}_n + F_t(x,\tau u_n,\tau v_n)\tilde{v}_n\bigr)\,d\tau\,dx + \frac{C_0}{K_n} \\
&\quad \to \int_\Omega \bigl(f^{++}\mu_1 + g^{++}\nu_1\bigr)\,dx \quad \text{as } n \to \infty.
\end{aligned} \tag{14}$$

Therefore, taking the limit in (11) and from (5), (12), (13) and (14), we get

$$\int_\Omega (h_1\mu_1 + h_2\nu_1)\,dx = \int_\Omega \bigl(f^{++}\mu_1 + g^{++}\nu_1\bigr)\,dx,$$

which is a contradiction with the condition $(LL)_1^+$. Hence, (u_n, v_n) is bounded in W, and there is a subsequence of (u_n, v_n) without any loss of generality still denoted by (u_n, v_n), and $(u,v) \in W$ such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in W, $(u_n, v_n) \to (u,v)$ strongly in $L^p(\Omega) \times L^p(\Omega)$. Consequently, from (5), one has

$$\lim_{n\to\infty}\bigl\langle J'(u_n,v_n),(u_n-u,0)\bigr\rangle = 0. \tag{15}$$

From (F1) and Hölder's inequality, it follows that

$$\left|\int_\Omega F_s(x,u_n,v_n)(u_n-u)\,dx\right| \le \|h\|_{L^q}\|u_n-u\|_{L^p} \to 0$$

as $n \to \infty$. Similarly, we obtain

$$\left|\int_\Omega h_1(x)(u_n-u)\,dx\right| \le \|h_1\|_{L^q}\|u_n-u\|_{L^p} \to 0$$

and

$$\begin{aligned}
\left|\int_\Omega a(x)|u_n|^{p-2}u_n(u_n-u)\,dx\right| &\le \|a\|_{L^\infty}\|u_n\|_{L^p}^{p-1}\|u_n-u\|_{L^p} \\
&\le C^{p-1}\|a\|_{L^\infty}\|u_n\|^{p-1}\|u_n-u\|_{L^p} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Combining the above three inequalities and (15), we get

$$\int_\Omega \bigl(|\nabla u_n|^{p-2}\nabla u_n, \nabla(u_n - u)\bigr)\,dx \to 0$$

as $n \to \infty$. Similarly, we also obtain

$$\lim_{n\to\infty} \int_\Omega \bigl(|\nabla u|^{p-2}\nabla u, \nabla(u_n - u)\bigr)\,dx = 0,$$

hence

$$\lim_{n\to\infty} \int_\Omega \bigl(\bigl(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\bigr), \nabla(u_n - u)\bigr)\,dx = 0.$$

From Clarkson's inequality, that is, there is $C_p > 0$ such that for all $\mu, \nu \in R^N$ and $p \geq 2$,

$$|\mu - \nu|^p \leq C_p\bigl(|\mu|^{p-2}\mu - |\nu|^{p-2}\nu\bigr)(\mu - \nu),$$

it follows that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p\,dx = 0,$$

this is, $u_n \to u$ in $W_0^{1,p}(\Omega)$. Similarly, we have $v_n \to v$ in $W_0^{1,p}(\Omega)$, hence $(u_n, v_n) \to (u, v)$ strongly in W.

(ii) We claim that the functional J satisfies the geometries of the saddle point theorem with respect to (E_1, E_2), where $E_1 = \operatorname{span}\{(\mu_1, \nu_1)\}$, $E_2 = \{(\phi, \psi) \in W : \int_\Omega(\mu_1^{p-1}\phi + \nu_1^{p-1}\psi)\,dx = 0\}$ and $W = E_1 \oplus E_2$.

By the definition of (μ_1, ν_1), for all $t \in R$, we have

$$\begin{aligned}
\lambda_1\biggl(&\int_\Omega a(x)|t\mu_1|^p\,dx + \int_\Omega c(x)|t\nu_1|^p\,dx \\
&+ \frac{p}{(\alpha+1)(\beta+1)} \int_\Omega b(x)|t\mu_1|^\alpha|t\nu_1|^\beta t\mu_1 t\nu_1\,dx\biggr) \\
&= \int_\Omega \bigl|\nabla(t\mu_1)\bigr|^p\,dx + \int_\Omega \bigl|\nabla(t\nu_1)\bigr|^p\,dx.
\end{aligned} \tag{16}$$

Moreover, we have

$$\begin{aligned}
&\int_\Omega F(x, t\mu_1, t\nu_1)\,dx \\
&\quad= \int_\Omega \bigl(F(x, t\mu_1, t\nu_1) - F(x,0,0)\bigr)\,dx + \int_\Omega F(x,0,0)\,dx \\
&\quad= \int_\Omega \int_0^1 \bigl(F_s(x, \tau t\mu_1, \tau t\nu_1)t\mu_1 + F_t(x, \tau t\mu_1, \tau t\nu_1)t\nu_1\bigr)\,d\tau\,dx + \int_\Omega F(x,0,0)\,dx \\
&\quad= t\int_\Omega \int_0^1 \bigl(F_s(x, \tau t\mu_1, \tau t\nu_1)\mu_1 + F_t(x, \tau t\mu_1, \tau t\nu_1)\nu_1\bigr)\,d\tau\,dx + \int_\Omega F(x,0,0)\,dx.
\end{aligned} \tag{17}$$

From the Lebesgue dominated convergence theorem, (F1), (F2) and (F3), we obtain

$$\begin{aligned}&\lim_{t\to+\infty}\int_\Omega\int_0^1\bigl(F_s(x,\tau t\mu_1,\tau t\nu_1)\mu_1+F_t(x,\tau t\mu_1,\tau t\nu_1)\nu_1\bigr)\,d\tau\,dx\\&\quad=\int_\Omega\bigl(f^{++}\mu_1+g^{++}\nu_1\bigr)\,dx.\end{aligned}\tag{18}$$

Hence, from (4), $(LL)_1^+$, (16), (17) and (18), it follows that

$$\begin{aligned}J(t\mu_1,t\nu_1)&=t\int_\Omega(h_1\mu_1+h_2\nu_1)\,dx-\int_\Omega F(x,t\mu_1,t\nu_1)\,dx\\&\to-\infty\quad\text{as } t\to\infty.\end{aligned}$$

Similarly, if t tends to $-\infty$, the same result is obtained with f^{++} and g^{++} exchanged with f^{--} and g^{--} respectively. Hence, in both cases we have

$$\lim_{|t|\to\infty}J(t\mu_1,t\nu_1)=-\infty.\tag{19}$$

On the other hand, from the definition of λ_1, there is $\bar\lambda>\lambda_1$ such that

$$\begin{aligned}&\int_\Omega|\nabla u|^p\,dx+\int_\Omega|\nabla v|^p\,dx\\&\quad\geq\bar\lambda\biggl(\int_\Omega a(x)|u|^p\,dx+\int_\Omega c(x)|v|^p\,dx+\frac{p}{(\alpha+1)(\beta+1)}\int_\Omega b(x)|u|^\alpha|v|^\beta uv\,dx\biggr)\end{aligned}$$

for all $(u,v)\in E_2$. From (2), (4), (6), the above inequality and Hölder's inequality, we obtain

$$\begin{aligned}J(u,v)&\geq\frac{\bar\lambda-\lambda_1}{p\bar\lambda}\bigl(\|u\|^p+\|v\|^p\bigr)-C\|h\|_{L^q}\bigl(\|u\|+\|v\|\bigr)\\&\quad-\bigl(\|h_1\|_{L^q}\|u\|_{L^p}+\|h_2\|_{L^q}\|v\|_{L^p}\bigr)-C_0\\&\geq\frac{\bar\lambda-\lambda_1}{p\bar\lambda}\bigl(\|u\|^p+\|v\|^p\bigr)-C_1\bigl(\|u\|+\|v\|\bigr)-C_0\end{aligned}\tag{20}$$

for all $(u,v)\in E_2$, where $C_1=C(\|h\|_{L^q}+\min\{\|h_1\|_{L^q},\|h_2\|_{L^q}\})$.

Thus, from (19) and (20), there is $\delta\in R$ and $R_0>0$ such that if $|t|=R_0$ we obtain

$$J(t\mu_1,t\nu_1)<\delta<\min_{(u,v)\in E_2}J(u,v).$$

From the saddle point theorem, Theorem 1 is proved. □

Proof of Theorem 2 (i) Similar to (i) of the proof of Theorem 1, we can prove that from $(LL)_1^-$, the functional J satisfies the (PS) condition.

(ii) Now we will prove that the functional J is coercive, that is,

$$J(u,v)\to+\infty\quad\text{as }\bigl\|(u,v)\bigr\|\to\infty.$$

If the claim does not hold, there is a constant c and a sequence (u_n, v_n) with $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$ such that $J(u_n, v_n) \le c$. Let $K_n := (\|u_n\|^p + \|v_n\|^p)^{1/p}$, hence we have $K_n \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \frac{J(u_n, v_n)}{K_n} \le 0.$$

Define $\tilde{u}_n = u_n \setminus K_n$, $\tilde{v}_n = v_n \setminus K_n$, similar to the proof of the (PS) condition of Theorem 1 again, we obtain that $(\tilde{u}_n, \tilde{v}_n)$ converges strongly to $\pm(\mu_1, \nu_1)$ as $n \to \infty$.

Assume that $(\tilde{u}_n, \tilde{v}_n)$ converges strongly to (μ_1, ν_1) as $n \to \infty$ (the case $(\tilde{u}_n, \tilde{v}_n)$ converges strongly to $-(\mu_1, \nu_1)$ as $n \to \infty$ may be treated similarly), from (14) we have

$$\begin{aligned} 0 &\ge \limsup_{n\to\infty} \frac{J(u_n, v_n)}{K_n} \\ &\ge \lim_{n\to\infty} \left(\int_\Omega h_1 \tilde{u}_n + h_2 \tilde{v}_n \, dx - \frac{1}{K_n} \int_\Omega F(x, u_n, v_n) \, dx \right) \\ &= \int_\Omega (h_1 \mu_1 + h_2 \nu_1) \, dx - \int_\Omega \left(f^{++} \mu_1 + g^{++} \nu_1 \right) dx, \end{aligned}$$

which is a contradiction with $(LL)_1^-$. By Ekeland's variational principle, Theorem 2 is proved. □

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors typed, read and approved the final manuscript.

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