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Existence of solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}$

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Abstract

In this paper, we study the existence of nontrivial solution for a class of quasilinear Schrödinger equations in $\mathbb{R}$ with the nonlinearity asymptotically linear and, furthermore, the potential indefinite in sign. The tool used in this paper is the direct variation method.

MSC: 34J10; 35J20; 35J60

Keywords: quasilinear Schrödinger equation; indefinite potential; variational method

1 Introduction and main result

In this paper, we consider the following quasilinear Schrödinger equations:

$$-u'' + V(x)u - \left(|u|^2\right)'' u = f(x,u), \quad x \in \mathbb{R}. \tag{1.1}$$

Problem (1.1) arises the theory of superfluids or in dissipative quantum mechanics, plasma physics, fluid mechanics, and the theory of Heisenberg ferromagnets; for more details, the reader may refer to [1–3] and the references therein.

This problem has been studied by many people in the recent years [3–6], in which the existence of solutions has been established. Besides, Alves *et al.* [7] consider the existence and concentration of positive solutions as $\varepsilon \to 0$ for a related equation with ε^2. At the same time, many authors considered the corresponding high dimensional equations; see [1, 2, 8–11] and the references therein. However, in all the above papers, we notice that the potential was assumed positive definite. In this paper, we will investigate the nontrivial solution for equation (1.1) with the potential indefinite in sign; the tool used in our paper, which is different from the literature mentioned above, is the direct variation method. It is noticed that the ideas in this article come from the paper of Chen and Wang [12], where a Schrödinger-Poisson system was considered.

To stated our main result, one needs to describe the eigenvalue of Schrödinger operator $-\Delta + V$:

Consider the increasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ of the minmax value defined by

$$\lambda_n = \inf_{O\in\Theta_n} \sup_{u\in O, u\neq 0} \frac{\int_{\mathbb{R}}(|u''|^2 + V(x)u^2)\,dx}{\int_{\mathbb{R}} u^2\,dx},$$

where Θ_n is the family of n-dimensional subspaces of $C_0^\infty(\mathbb{R})$. Set

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n.$$

It is well known that λ_∞ is the bottom of the essential spectrum of $-\Delta + V$ if it is finite. So, λ_n is an eigenvalue of $-\Delta + V$ of finite multiplicity whenever $\lambda_n < \lambda_\infty$ (see [13, 14]). It is easy to see that λ_n is well defined and is finite if V is bounded from below.

In this paper, we assume that:

(V) $V \in C(\mathbb{R})$ bounded from below and there exists an integer $k \geq 1$ such that $\lambda_k < 0 < \lambda_{k+1}$.

(f_1) $f \in C^1(\mathbb{R}\times\mathbb{R})$ and there exist constants $p > 2$ and $c > 0$ such that

$$\bigl|f(x,t)\bigr| \leq c\bigl(1 + |t|^{p-1}\bigr), \quad \forall x \in \mathbb{R}, t \in \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

(f_3) There exists $0 < h < \lambda_\infty$ such that $F(x,t) = \int_0^t f(x,s)\,ds \leq \frac{1}{2}ht^2$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

Our main result of this paper is as follows.

Theorem 1.1 *Assume* (V), (f_1)-(f_3) *are satisfied, then problem* (1.1) *has at least one nontrivial solution.*

For equation (1.1), one may face many difficulties. First of all, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact. To overcome this difficulty, one can restrict the functional I to a subspace of $H^1(\mathbb{R})$, which embeds compactly into $L^2(\mathbb{R})$ with certain qualifications or consists of radially symmetric functions. For this point, Shen and Han [15] considered equation (1.1) with the nonlinearity satisfying some conditions and the potential V satisfying (V) and

$$\mu\bigl(V^{-1}(-\infty, M]\bigr) < \infty \tag{1.2}$$

for all $M > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}$. Due to the condition (1.2), the space

$$X = \left\{ u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2\,dx < +\infty \right\}$$

embeds compactly into $L^2(\mathbb{R})$ (see [16]), which is crucial in their paper. However, in our paper the condition like type (1.2) does not need to be assumed.

Second, the potential V in this paper is indefinite in sign, so under our conditions (f_1)-(f_3), the mountain pass lemma cannot be valid in the same way as in [17, 18]. Furthermore, the competing effect of $\int_{\mathbb{R}} |u'|^2u^2\,dx$ with the nonlinear term gives rise to much greater difficulty. For results as regards the potential change sign, we also mention [19] and the references in therein.

2 Proof of main result

In order to prove the main results, first we introduce the space

$$E := \left\{ u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2\,dx < \infty \right\},$$

which is a linear subspace of $H^1(\mathbb{R})$.

Let $\widetilde{E}^-$ and $\widetilde{E}^+$ be the negative space and positive space of the quadratic form

$$\int_{\mathbb{R}} \left(\left|u'\right|^2 + V(x)u^2\right) dx.$$

From (V), we deduce that $E = \widetilde{E}^- \oplus \widetilde{E}^+$ and $\widetilde{E}^-$ is a finite dimensional space which is spanned by the eigenfunctions with corresponding eigenvalues $\lambda_1 \le \cdots \le \lambda_k$. For any $u, v \in E$, we define

$$(u,v) = \int_{\mathbb{R}} \left(\left(\widetilde{u}^+\right)'\left(\widetilde{v}^+\right)' + V(x)\widetilde{u}^+\widetilde{v}^+\right) dx - \int_{\mathbb{R}} \left(\left(\widetilde{u}^-\right)'\left(\widetilde{v}^-\right)' + V(x)\widetilde{u}^-\widetilde{v}^-\right) dx,$$

where $u = \widetilde{u}^- + \widetilde{u}^+, v = \widetilde{v}^- + \widetilde{v}^+ \in E = \widetilde{E}^- \oplus \widetilde{E}^+$. Then $(\cdot,\cdot)$ is an inner product on E. Therefore, E is a Hilbert space with the norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. It is easy to see that

$$\int_{\mathbb{R}} \left(\left|u'\right|^2 + V(x)u^2\right) dx = \left\|\widetilde{u}^+\right\|^2 - \left\|\widetilde{u}^-\right\|^2.$$

For any $r \in [2,\infty]$, the embedding $E \hookrightarrow L^r(\mathbb{R})$ is continuous.
Let $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \left(\left|u'\right|^2 + V(x)u^2\right) dx + \int_{\mathbb{R}} \left|u'\right|^2 u^2 - \int_{\mathbb{R}} F(x,u)\, dx.$$

Then under conditions (f_1) and (f_2), $I \in C^1(E,\mathbb{R})$ and for all $u, v \in E$

$$\left\langle I'(u), v\right\rangle = \int_{\mathbb{R}} \left(u'v' + V(x)uv\right) dx + 2\int_{\mathbb{R}} \left(\left|u'\right|^2 uv + u^2u'v'\right) dx - \int_{\mathbb{R}} f(x,u)v\, dx.$$

It is well known that the critical points of I are the solutions of problem (1.1).

Proof of Theorem 1.1

Step 1: To proof I is coercive. Suppose it is not true, then there exist $M > 0$ and $\|u_n\| \to \infty$ such that $I(u_n) \le M$.

Since $h < \lambda_\infty$, we can choose $h < h^\star < \lambda_\infty$ such that $h^\star \ne \lambda_i, 1 \le i < \infty$. Let E^- be the space spanned by the eigenfunctions with corresponding eigenvalue less than $h^\star$. By the choice of $h^\star$, E^- is finite dimensional space. Let $E^+ = (E^-)^\perp$, then $E = E^- \oplus E^+$. For $u \in E$, we have a unique decomposition $u = u^+ + u^-$ with $u^+ \in E^+$ and $u^- \in E^-$. By the choice of $h^\star$, there exists an equivalent norm of E, still denoted by $\|\cdot\|$, such that

$$\int_{\mathbb{R}} \left(\left|u_n'\right|^2 + V(x)u_n^2 - h^\star u_n^2\right) dx = \left\|u_n^+\right\|^2 - \left\|u_n^-\right\|^2.$$

So, from (f_3), we have

$$\begin{aligned} I(u_n) &= \frac{1}{2}\int_{\mathbb{R}} \left(\left|u_n'\right|^2 + V(x)u_n^2\right) + \int_{\mathbb{R}} \left|u_n'\right|^2 u_n^2\, dx - \int_{\mathbb{R}} F(x,u_n)\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(\left|u_n'\right|^2 + V(x)u_n^2 - h^\star u_n^2\right) dx + \int_{\mathbb{R}} \left|u_n'\right|^2 u_n^2\, dx \\ &\quad + \int_{\mathbb{R}} \left(\frac{1}{2}h^\star u_n^2 - F(x,u_n)\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} |u_n'|^2 u_n^2\,dx + \int_{\mathbb{R}} \left(\frac{1}{2}h^\star u_n^2 - F(x,u_n)\right) dx \\
&\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2.
\end{aligned} \tag{2.1}
$$

Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Multiplying both sides of (2.1) by $\|u_n\|^{-2}$, since $\|u_n\| \to \infty$ and $I(u_n) \leq M$, we get

$$
\|v_n^+\|^2 \leq \|v_n^-\|^2 + o_n(1). \tag{2.2}
$$

Up to a subsequence, we assume that $v_n \rightharpoonup v$ in E and $v_n \to v$ a.e. in $\mathbb{R}$. If $v = 0$, for the finite dimension of E^-, one has $v_n^- \to 0$ in E. By (2.2), $v_n \to 0$ in E. It is impossible, since $\|v_n\| = 1$. Hence, $v \neq 0$. Since $\int_{\mathbb{R}} |u'|^2 u^2\,dx$ is weak sequential lower semi-continuous [3] (see also [7]), we have

$$
\liminf_{n\to\infty} \|u_n\|^{-4} \int_{\mathbb{R}} |u_n'|^2 u_n^2\,dx = \liminf_{n\to\infty} \int_{\mathbb{R}} |v_n'|^2 v_n^2\,dx \geq \int_{\mathbb{R}} |v'|^2 v^2\,dx > 0. \tag{2.3}
$$

By (f_3), one gets

$$
I(u_n) \geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} |u_n'|^2 u_n^2\,dx.
$$

Multiplying both sides of the above inequality by $\|u_n\|^{-4}$, by (2.3) and $\|u_n\|^{-4} I(u_n) \to 0$ as $n \to \infty$, we have

$$
0 \geq \int_{\mathbb{R}} |v'|^2 v^2\,dx > 0.
$$

This is a contradiction. So, I is coercive.

Step 2: We will show that I is weakly sequentially lower semi-continuous. Let $u_n \rightharpoonup u$ in E. Then $\liminf_{n\to\infty} \|u_n^+\| \geq \|u^+\|$ and $u_n^- \to u^-$ in E^- since the dimension of E^- is finite. So, it follows from (f_3) and $\int_{\mathbb{R}} |u'|^2 u^2\,dx$ being weakly sequentially lower semi-continuous that

$$
\begin{aligned}
\liminf_{n\to\infty} I(u_n) &= \liminf_{n\to\infty} \left(\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} |u_n'|^2 u_n^2\,dx + \int_{\mathbb{R}} \left(\frac{1}{2}h^\star u_n^2 - F(x,u_n)\right) dx\right) \\
&\geq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} |u'|^2 u^2\,dx + \int_{\mathbb{R}} \left(\frac{1}{2}h^\star u^2 - F(x,u)\right) dx \\
&= I(u).
\end{aligned} \tag{2.4}
$$

Hence, I is weakly sequentially lower semi-continuous.

Step 3: We will show that I is bounded from below and $\inf_E I < 0$. It is easy to see that I maps every bounded set in E into bounded set in $\mathbb{R}$. Hence by the Step 1, we see that I is bounded from below. Next, we will show that $\inf_E I < 0$. In fact, since (V), for any $e \in E^- \setminus \{0\}$ we get

$$
\int_{\mathbb{R}} \left(|e'|^2 + V(x)e^2\right) dx < 0.
$$

From (f_2), we have $\int_{\mathbb{R}} F(x, te)\,dx = o(t^2)$. Furthermore,

$$\int_{\mathbb{R}} \left|(te)'\right|^2 (te)^2\,dx = t^4 \int_{\mathbb{R}} \left|e'\right|^2 e^2\,dx = O\left(t^4\right).$$

Therefore, we can choose some $t > 0$ small enough such that $I(te) < 0$. So, we get $\inf_E I < 0$.

By using direct variation method, Steps 1 and 2 show that I has a global minimizer. From Step 3, the global minimizer of I is not zero. Therefore, we get a nontrivial solution of problem (1.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by D-BW; D-BW prepared the manuscript initially and KY performed a part of steps of the proofs in this research. All authors read and approved the final manuscript.

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