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Discontinuous traveling waves for scalar hyperbolic-parabolic balance law

Tianyuan Xu, Chunhua Jin and Shanming Ji*

*Correspondence:
jishanming7@126.com
School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

Abstract

This paper is concerned with the existence of traveling waves for the scalar hyperbolic-parabolic balance law. Using a phase-plane analysis method, we first prove the existence of an increasing traveling wave solution in $C^1(\mathbb{R})$. Then we construct a family of discontinuous periodic traveling wave entropy solutions.

Keywords: scalar hyperbolic-parabolic balance law; discontinuous traveling waves; entropy solution

1 Introduction

We consider the discontinuous traveling waves for the scalar hyperbolic-parabolic balance law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right) + g(u), \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where $a \in C^1(\mathbb{R})$ with $a(s) \geq 0$ for $s \in \mathbb{R}$.

If $a(u) \equiv 0$, (1.1) reduces to the scalar hyperbolic balance law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = g(u), \quad x \in \mathbb{R}, t > 0, \tag{1.2}$$

which describes the problems under idealizing inviscid assumptions and is extensively studied by numerous authors. For example, in [1], Mascia gave a classification of the possible traveling waves for the case of that f is convex. Lyberopoulos [2] obtained qualitative properties and the long-time behavior of the solutions of (1.2) with periodic initial data. It was Fan and Hale [3] who first studied the discontinuous traveling waves for (1.2) and Sinestrari [4] studied related properties of such traveling waves.

It is well known that the effects of viscosity on the balance law should be included in many practical problems such as fluid flows [5], the model of car traffic flow on a highway [6], ion etching in the semiconductor industry [7], *etc.* In other words, when $a(u) \not\equiv 0$, (1.1) becomes the scalar viscous balance law, to which a lot of important research works have been devoted in the past several decades. When $a(u) \equiv \varepsilon$, Wu and Xing [8] considered the traveling waves of the following scalar viscous balance law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon \frac{\partial^2 u}{\partial x^2} + g(u), \quad x \in \mathbb{R}, t > 0, \tag{1.3}$$

where $\varepsilon > 0$ is the viscosity parameter. Harterich [9] obtained the global attractors of (1.3) depending on the parameter ε. Owing to the parabolicity, such an equation admits only smooth traveling waves with low gradient depending on the parameter ε. It is worthy of noticing that sharp surface and high gradient have been observed in some viscous balance phenomena [10]. Since their discovery, it is necessary for us to derive a nonlinearly viscous balance law model. Inspired by the idea of the localized perturbations in [11], we propose a nonlinear viscosity with degeneracy in this paper.

Now, let us formulate the problem with nonlinear and degenerate viscosity. We consider a typical equation of gas dynamics in one spatial dimension as an illustrative example. In the Oberbeck-Boussinesq approximation, all changes in the fluid properties due to temperature variations are neglected except for the change in density that gives rise to a buoyancy force g. In particular, the momentum conservation equation becomes

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^*}{\partial x} = \frac{\partial}{\partial x}\left(\nu \frac{\partial v}{\partial x}\right) + g, \quad x \in \mathbb{R}, t > 0, \tag{1.4}$$

where $g = \alpha g_0 \theta z$, α is the coefficient of volume expansion of the fluid, g_0 is the acceleration due to gravity, $\theta(x,t) = T(x,t) - T_0$ is the temperature deviation from the mean, z is the unit vector along the vertical direction, ρ_0 is the density at mean temperature T_0, $p^* = p + \rho_0 g_0 z$, p is the pressure, v is the velocity and $\nu = \eta/\rho_0$ is the kinematic viscosity [12]. Based on experimental results that flows (and therefore also the velocity) can be generated by a temperature gradient without any initial pressure gradient, we assume that the thermal buoyancy force is a function depending only on the velocity of the flow [13]. In mathematical form we have $g = g(v)$. Thus the gas dynamics equation considering the thermal buoyancy term can take the form of

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = \frac{\partial}{\partial x}\left(\mathbf{a}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}\right) + \mathbf{g}(\mathbf{u}), \quad x \in \mathbb{R}, t > 0, \tag{1.5}$$

where $\mathbf{u}$ is a vector of densities of conserved quantities in $\mathbb{R}^n$, including mass, momentum, and energy in the case of gas dynamics, $\mathbf{f}$ a vector of corresponding fluxes in $\mathbb{R}^n$, the thermal buoyancy term $\mathbf{g}$ in $\mathbb{R}^n$, and $\mathbf{a}(\mathbf{u})$ a matrix of transport coefficients in $\mathbb{R}^{n\times n}$ [5, 14].

To propose the basic assumption, we illustrate peculiar properties of the degenerate nonlinear viscosity term. It is well understood that the viscosity of gas is small when temperature is low, *vice versa*. At the same time, the thermal buoyancy force can be ignored for temperatures below a certain critical temperature. So we can assume that $\mathbf{a}(\mathbf{u}) \equiv 0$ around the zero of the thermal buoyancy term $\mathbf{g}$. Considering the scalar hyperbolic-parabolic balance law (1.1), namely the case $n = 1$ of (1.5), we present the basic assumption on the viscosity term

$$\operatorname{supp} a = \bigcup_{k=1}^{2K} [a_k, b_k], \tag{1.6}$$

where $[a_k, b_k] \subset (v_k, v_{k+1})$ for any $1 \le k \le 2K$. The notations v_k and K will be explained in Section 2.

This paper is organized as follows. In Section 2, after introducing some definitions and notations, we present some auxiliary lemmas and state the main results. Subsequently, in Section 3, we prove the existence of discontinuous traveling waves.

2 Preliminaries and main results

In this section, we present some closely related results and definitions of entropy solutions. Sinestrari [4] studied the discontinuous traveling waves of the scalar hyperbolic balance law (1.2) with the initial value

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{2.1}$$

being periodic, under the conditions:

(A1) $f \in C^2(\mathbb{R}), f'$ is strictly increasing;
(A2) $g \in C^1(\mathbb{R}), g' \leq k_g$ for some constant k_g;
(A3) $u_0 \in BV_L(\mathbb{R})$, where BV_L is the space of functions which are locally of bounded variation and are L-periodic for some given constant $L > 0$;
(A4) The zeros of g are simple, *i.e.*, $g'(v) \neq 0$ for any v such that $g(v) = 0$.

Moreover, g has at least one zero v such that $g'(v) > 0$ and there exists $M_g > 0$ such that

$$vg(v) < 0, \qquad |v| > M_g.$$

The zeros of g are labeled in an increasing order by $v_1, v_2, \ldots, v_{2K+1}$, for some positive integer K, with $g'(v_i) < 0$ if i is odd, $g'(v_i) > 0$ if i is even.

To any zero v_{2k} of g there can be associated a continuous traveling wave solution and a family of discontinuous periodic traveling waves of (1.2) with speed $f'(v_{2k})$.

It is well known that the problem (1.2)-(2.1) does not have global classical solutions even if u_0 is smooth [4]. On the other hand, discontinuous solutions in the distributional sense may not be unique. This leads to the definition of entropy solutions as follows.

Definition 2.1 ([4]) A function $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+) \cap C(\mathbb{R}^+, L^1_{\text{loc}}(\mathbb{R}))$ is an entropy solution of the problem (1.2)-(2.1) if it satisfies (1.2)-(2.1) in the sense of distributions, $u(\cdot,t) \in BV_{\text{loc}}(\mathbb{R})$ for every t and the entropy condition

$$u(x^+,t) \leq u(x^-,t), \quad x \in \mathbb{R}, t > 0 \tag{2.2}$$

holds, where $u(x^+,t)$ and $u(x^-,t)$ denote the rightward and leftward one-sided limits of $u(\cdot,t)$.

A function u is a solution of (1.2) in the sense of distributions with the entropy condition (2.2) if and only if u satisfies Definition 2.2, which is derived by a vanishing viscosity method [15].

Definition 2.2 A function $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ is an entropy solution of equation (1.2) if for any $k \in \mathbb{R}$ and any smooth function $w(x,t) \geq 0$ with compact support, the following inequality holds:

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \big(|u-k|w_t + \operatorname{sign}(u-k)\big(f(u) - f(k)\big)w_x + \operatorname{sign}(u-k)g(u)w\big)\,dx\,dt \geq 0. \tag{2.3}$$

With this notion of entropy solutions and under the assumptions (A1)-(A3), the problem (1.2)-(2.1) is well posed, as is shown by [15] and references therein.

We are concerned with the discontinuous traveling waves for the scalar hyperbolic-parabolic balance law (1.1). For the sake of convenience, we define

$$A(s) = \int_0^s a(\tau)\,d\tau.$$

Using the vanishing viscosity method, we give the following definition of entropy solutions.

Definition 2.3 A function $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ is said to be an entropy solution of equation (1.1), if for any $k \in \mathbb{R}$ and any smooth function $w(x,t) \geq 0$ with compact support, the following inequality holds:

$$\begin{aligned} \iint_{\mathbb{R}\times\mathbb{R}^+} &\bigl(|u-k|w_t + \operatorname{sign}(u-k)\bigl(f(u)-f(k)\bigr)w_x \\ &+ \operatorname{sign}(u-k)g(u)w + \operatorname{sign}(u-k)\bigl(A(u)-A(k)\bigr)w_{xx}\bigr)\,dx\,dt \geq 0. \end{aligned} \tag{2.4}$$

It was Vol'pert and Hudjaev [16] who first treated the solvability of the initial value problem for the hyperbolic-parabolic balance law (1.1). The uniqueness of entropy solution of equation (1.1) subject to given initial value was proved by Wu and Yin [17] in an equivalent form of discontinuity condition.

Now we show that for piecewise continuous functions, the definition of entropy solutions can be valid by satisfying the assumptions in the following lemma.

Lemma 2.1 *Suppose that a function $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ is piecewise C^2 continuous with the discontinuous lines $x = x(t)$ being C^1 regular and the following conditions are fulfilled*:

(i) *in any piecewise continuous domain, u satisfies* (1.1) *in the classical sense*;

(ii) *along any discontinuous line $x = x(t)$ the following Rankine-Hugoniot condition holds*:

$$\bigl(u(x^+,t) - u(x^-,t)\bigr)\frac{dx}{dt} = f\bigl(u(x^+,t)\bigr) - f\bigl(u(x^-,t)\bigr); \tag{2.5}$$

(iii) *along any discontinuous line $x = x(t)$ the entropy condition* (2.2) *is valid and there exists a $1 \leq k \leq 2K$ such that $u(x^+,t), u(x^-,t) \in (b_{k-1}, a_k)$.*

Then u is an entropy solution of equation (1.1).

Proof Since $a(s) \equiv 0$ for $s \in (b_{k-1}, a_k)$ and $u(x^+,t), u(x^-,t) \in (b_{k-1}, a_k)$, the second derivative term $A(u)_{xx} = (a(u)u_x)_x$ is strongly degenerate. The rest of the proof is similar to the proof of the equivalence between the inequality (2.3) in Definition 2.2 and the definition of entropy solutions for the first order equation (1.2) in Definition 2.1. □

We state our main result here and leave its proof to the next section.

Theorem 2.1 *Assume that the assumptions* (A1), (A2), (A4), *and* (1.6) *hold. We further assume that $a(s)$ satisfies the following inequality*:

$$a(s) \leq \frac{(f'(a_{2k}) - f'(b_{2k-1}))^2}{2(1 + \max_{\tau\in(\nu_{2k-1},\nu_{2k+1})}|g(\tau)|)} \min\bigl\{|s - a_{2k}|, |s - b_{2k-1}|\bigr\}, \tag{2.6}$$

for $s \in (a_{2k-1}, b_{2k-1}) \cup (a_{2k}, b_{2k})$, *and for any* $1 \le k \le K$. *Then for any positive integer* k *with* $1 \le k \le K$, *equation* (1.1) *admits infinitely many discontinuous traveling wave entropy solutions with speed* $f'(v_{2k})$.

3 Proof of the main results

For any given k with $1 \le k \le K$, in order to find the discontinuous traveling waves with speed $f'(v_{2k})$, we first prove that the degenerate equation (1.1) has a continuous traveling wave solution with speed $f'(v_{2k})$ and range (v_{2k-1}, v_{2k+1}). Let the traveling wave

$$u(x,t) = \phi\bigl(x - f'(v_{2k})t\bigr) = \phi(\xi), \quad \xi = x - f'(v_{2k})t.$$

We have

$$\bigl(f'(\phi) - f'(v_{2k})\bigr)\phi'(\xi) = \bigl(a(\phi)\phi'(\xi)\bigr)' + g(\phi), \quad \xi \in \mathbb{R}. \tag{3.1}$$

Since (3.1) is autonomous, without loss of generality, we assume that $\phi(0) = v_{2k}$. Note that

$$v_{2k-1} < a_{2k-1} < b_{2k-1} < v_{2k} < a_{2k} < b_{2k} < v_{2k+1}$$

and $a(s) \equiv 0$ for $s \in (b_{2k-1}, a_{2k})$. There exist $\xi^- < 0 < \xi^+$, such that

$$\int_{v_{2k}}^{\phi(\xi)} \frac{f'(v) - f'(v_{2k})}{g(v)}\,dv = \xi, \quad \xi \in \bigl[\xi^-, \xi^+\bigr],$$

and $\phi(\xi^-) = b_{2k-1}$, $\phi(\xi^+) = a_{2k}$.

Now we need to extend the solution ϕ of equation (3.1) to the domain $(-\infty, \xi^-)$ and $(\xi^+, +\infty)$. We begin with the right-hand extension to the interval $(\xi^+, +\infty)$. We have

$$\phi\bigl(\xi^+\bigr) = a_{2k}, \qquad \phi'\bigl(\xi^+\bigr) = \frac{g(a_{2k})}{f'(a_{2k}) - f'(v_{2k})} > 0.$$

Lemma 3.1 *Assume that*

$$a(s) \le \frac{(f'(a_{2k}) - f'(v_{2k}))^2}{2(1 + \max_{\tau \in (v_{2k}, v_{2k+1})} |g(\tau)|)}(s - a_{2k}), \quad s \in (a_{2k}, b_{2k}). \tag{3.2}$$

Then equation (3.1) *admits a continuous solution* $\phi(\xi) \in C^1([0, +\infty))$, *such that* $\phi(0) = v_{2k}$, ϕ *is piecewise* C^2 *continuous and strictly increasing, and the range of* ϕ *is* $[v_{2k}, v_{2k+1})$.

Proof We only need to consider solving (3.1) in the interval $(\xi^+, +\infty)$. Since $\phi(\xi^+) = a_{2k}$, $\phi'(\xi^+) > 0$, there exists a right neighborhood (ξ^+, ξ_1) of ξ^+ such that for any $\xi \in (\xi^+, \xi_1)$, we have $\phi(\xi) \in (a_{2k}, b_{2k})$, and if $\xi_1 < +\infty$, then $\phi(\xi_1) = a_{2k}$ or $\phi(\xi_1) = b_{2k}$. That is, (ξ^+, ξ_1) is the maximal rightward interval of ξ such that $\phi(\xi) \in (a_{2k}, b_{2k})$. Define

$$\psi(\xi) = a(\phi)\phi'(\xi).$$

Noticing that $a(\phi) > 0$ for $\xi \in (\xi^+, \xi_1)$, we can convert the second-order degenerate differential equation (3.1) into the following singular planar dynamic system:

$$\begin{cases} \phi' = \frac{1}{a(\phi)}\psi, \\ \psi' = \frac{f'(\phi)-f'(\nu_{2k})}{a(\phi)}\psi - g(\phi). \end{cases} \tag{3.3}$$

For the sake of convenience, we let (Φ, Ψ) designate the right-side vector field of the above dynamic system.

We apply the phase-plane arguments to this problem in the domain

$$G_0 = \{(\phi, \psi); \phi \in (a_{2k}, b_{2k}), \psi > 0\}.$$

We need to show that there exists a trajectory connecting the two singular points $(a_{2k}, 0)$ and $(b_{2k}, 0)$. Define the curve

$$\Gamma_0 = \left\{(\phi, \psi); \psi = \frac{g(\phi)a(\phi)}{f'(\phi) - f'(\nu_{2k})}, \phi \in (a_{2k}, b_{2k})\right\}.$$

We can verify that Γ_0 connects the singular points $(a_{2k}, 0)$ and $(b_{2k}, 0)$, and divides the domain G_0 into two parts, with the following assertions holding:

(i) along the curve Γ_0, $\Phi > 0$, $\Psi = 0$;

(ii) in the domain $G_1 = \{(\phi, \psi); \psi > \frac{g(\phi)a(\phi)}{f'(\phi)-f'(\nu_{2k})}, \phi \in (a_{2k}, b_{2k})\}$, $\Phi > 0$, $\Psi > 0$;

(iii) while in the domain $G_2 = G_0 \backslash \overline{G}_1$, $\Phi > 0$, $\Psi < 0$.

In order to prove the existence of a trajectory that goes out from the singular points $(a_{2k}, 0)$, we construct the following curve:

$$\Gamma_1 = \{(\phi, \psi); \psi = c(\phi - a_{2k})^\alpha, \phi \in (a_{2k}, b_{2k})\},$$

and let $G_1' = \{(\phi, \psi); \psi > c(\phi - a_{2k})^\alpha, \phi \in (a_{2k}, b_{2k})\}$, $G_1'' = G_1 \backslash \overline{G}_1'$, where $c > 0$, $\alpha > 0$ are constants that will be determined below, such that Γ_1 has the following properties:

(i) the curve Γ_1 lies in the domain G_1;

(ii) any trajectory intersecting with the curve Γ_1 all runs through Γ_1 from G_1'' into G_1'.

The above two properties are equivalent to

$$c(\phi - a_{2k})^\alpha > \frac{g(\phi)a(\phi)}{f'(\phi) - f'(\nu_{2k})}, \qquad \frac{d}{d\phi}\left(c(\phi - a_{2k})^\alpha\right) < \frac{\Psi}{\Phi},$$
$$\forall \phi \in (a_{2k}, b_{2k}), \psi = c(\phi - a_{2k})^\alpha.$$

That is,

$$f'(\phi) - f'(\nu_{2k}) > c\alpha(\phi - a_{2k})^{\alpha-1} + \frac{g(\phi)a(\phi)}{c(\phi - a_{2k})^\alpha}, \quad \forall \phi \in (a_{2k}, b_{2k}). \tag{3.4}$$

Take $\alpha = 1$ and $c = (f'(a_{2k}) - f'(\nu_{2k}))/2$. According to the assumption (3.2), we see that (3.4) is true.

Let Γ_ε and Γ_δ be the trajectories that arrive at the point (b_{2k}, ε) and $(b_{2k} - \delta, 0)$, respectively, where $\varepsilon > 0$, $0 < \delta < b_{2k} - a_{2k}$. Since $\Phi > 0$, $\Psi < 0$ in the domain G_2, Γ_ε cannot intersect with the segment $L = \{(\phi, 0); \phi \in (a_{2k}, b_{2k})\}$ and Γ_δ cannot intersect with L except the point $(b_{2k} - \delta, 0)$. On the other hand, according to the properties of the curve Γ_1, all the trajectories Γ_ε and Γ_δ do not intersect with the curve Γ_1. Thus Γ_ε and Γ_δ are all starting from the singular point $(a_{2k}, 0)$. By the continuity of system (3.3), there exists a trajectory Γ that connects $(a_{2k}, 0)$ and $(b_{2k}, 0)$. It follows that equation (3.1) admits a solution $\phi(\xi)$ such that ϕ is defined on the interval (ξ^+, ξ_1), ϕ is strictly increasing and $\phi(\xi_1) = b_{2k}$. Combining with (3.1), we see that $\phi'(\xi_1) = \frac{g(b_{2k})}{f'(b_{2k})-f'(v_{2k})} > 0$, which implies that ξ_1 is a finite real number.

Now we solve equation (3.1) in the interval $(\xi_1, +\infty)$. Since $a(s) \equiv 0$ for any $s \in (b_{2k}, v_{2k+1})$, we find that $\phi(\xi)$ satisfies

$$\int_{b_{2k}}^{\phi(\xi)} \frac{f'(v) - f'(v_{2k})}{g(v)}\, dv = \xi - \xi_1, \quad \xi > \xi_1.$$

We note that v_{2k+1} is an odd zero of g. It follows that the above integral has a non-integrable singularity at v_{2k+1}. Thus ϕ is strictly increasing and $\lim_{\xi\to+\infty} \phi(\xi) = v_{2k+1}$. □

The left-hand extension of ϕ to the interval $(-\infty, \xi^-)$ is similar.

Lemma 3.2 *Assume that*

$$a(s) \le \frac{(f'(v_{2k}) - f'(b_{2k-1}))^2}{2(1 + \max_{\tau\in(v_{2k-1},v_{2k})} |g(\tau)|)}(b_{2k-1} - s), \quad s \in (a_{2k-1}, b_{2k-1}). \tag{3.5}$$

Then equation (3.1) *admits a continuous solution* $\phi(\xi) \in C^1((-\infty, 0])$, *such that* $\phi(0) = v_{2k}$, ϕ *is piecewise* C^2 *continuous and strictly increasing, and the range of* ϕ *is* $(v_{2k-1}, v_{2k}]$.

Proof The proof of this lemma is similar to that of Lemma 3.1. □

Utilizing the above two lemmas, we can prove that the degenerate parabolic equation (1.1) admits a strictly increasing traveling wave solution whose range is (v_{2k-1}, v_{2k+1}).

Lemma 3.3 *Assume that the conditions* (3.2) *and* (3.5) *hold, then equation* (1.1) *admits a continuous traveling wave* $\phi(\xi) \in C^1(\mathbb{R})$ *such that* ϕ *is piecewise* C^2 *continuous and strictly increasing,* $\phi(0) = v_{2k}$, *and the range of* ϕ *is* (v_{2k-1}, v_{2k+1}).

Proof This is a simple conclusion of Lemma 3.1 and Lemma 3.2. □

Concerned with the Rankine-Hugoniot condition (2.5), we give the following property of convex functions.

Lemma 3.4 *Assume that*

$$\frac{f(a_{2k}) - f(b_{2k-1})}{a_{2k} - b_{2k-1}} \le f'(v_{2k}), \tag{3.6}$$

then for any $\varphi^+ \in (v_{2k}, a_{2k})$, *there exists a unique* $\varphi^- \in (b_{2k-1}, v_{2k})$, *such that*

$$\frac{f(\varphi^+) - f(\varphi^-)}{\varphi^+ - \varphi^-} = f'(v_{2k}).$$

Similarly, assume that

$$\frac{f(a_{2k})-f(b_{2k-1})}{a_{2k}-b_{2k-1}} \geq f'(v_{2k}), \tag{3.7}$$

then for any $\varphi^- \in (b_{2k-1}, v_{2k})$, *there exists a unique* $\varphi^+ \in (v_{2k}, a_{2k})$, *such that*

$$\frac{f(\varphi^+)-f(\varphi^-)}{\varphi^+-\varphi^-} = f'(v_{2k}).$$

Proof According to the strict convexity of f, we conclude the above assertions. □

Using the continuous traveling wave of equation (1.1), we can construct a family of discontinuous periodic traveling waves. Suppose that the conditions (3.2) and (3.5) are fulfilled. We note that either the condition (3.6) or the condition (3.7) is true. Without loss of generality, suppose that (3.6) holds. Lemma 3.3 implies that equation (1.1) admits a continuous traveling wave, denoted by $\phi_{2k}(\xi)$ with $\xi = x - f'(v_{2k})t$ and the speed of this traveling wave is $f'(v_{2k})$.

For any given positive integer N, and any given real number sets $\{\varepsilon_i\}_{i=1}^N$, $\{\eta_i\}_{i=1}^N$, such that $\varepsilon_i \geq 0$, $\eta_i \in (v_{2k}, a_{2k})$, $i = 1, 2, \ldots, N$, there exists a unique set $\{\mu_i\}_{i=1}^N \subset (b_{2k-1}, v_{2k})$ such that

$$\frac{f'(\eta_i)-f'(\mu_i)}{\eta_i-\mu_i} = f'(v_{2k}), \quad i = 1, 2, \ldots, N, \tag{3.8}$$

according to Lemma 3.4.

Since $\phi_{2k}(\xi)$ is strictly increasing with range (v_{2k-1}, v_{2k+1}), we see that there exists a unique $\{\xi_i^+\}_{i=1}^N \subset \mathbb{R}^+$ and $\{\xi_i^-\}_{i=1}^N \subset \mathbb{R}^-$, such that

$$\phi_{2k}(\xi_i^+) = \eta_i, \qquad \phi_{2k}(\xi_i^-) = \mu_i, \quad 1 \leq i \leq N.$$

Fix any $\xi_0 \in \mathbb{R}$ and define

$$\psi_{2k}(\xi) = \begin{cases} v_{2k}, & \xi \in [\xi_0, \xi_0 + \varepsilon_1), \\ \phi_{2k}(\xi - (\xi_0 + \varepsilon_1)), & \xi \in [\xi_0 + \varepsilon_1, \xi_0 + \varepsilon_1 + \xi_1^+), \\ \phi_{2k}(\xi - (\xi_0 + T_1)), & \xi \in [\xi_0 + T_1 + \xi_1^-, \xi_0 + T_1), \\ v_{2k}, & \xi \in [\xi_0 + T_1, \xi_0 + T_1 + \varepsilon_2), \\ \phi_{2k}(\xi - (\xi_0 + T_1 + \varepsilon_2)), & \xi \in [\xi_0 + T_1 + \varepsilon_2, \xi_0 + T_1 + \varepsilon_2 + \xi_2^+), \\ \phi_{2k}(\xi - (\xi_0 + T_1 + T_2)), & \xi \in [\xi_0 + T_1 + T_2 + \xi_2^-, \xi_0 + T_1 + T_2), \\ v_{2k}, & \xi \in [\xi_0 + T_1 + T_2, \xi_0 + T_1 + T_2 + \varepsilon_3), \\ \ldots, & \\ \phi_{2k}(\xi - (\xi_0 + T)), & \xi \in [\xi_0 + T + \xi_N^-, \xi_0 + T), \end{cases} \tag{3.9}$$

where $T_i = \varepsilon_i + \xi_i^+ + |\xi_i^-|$, $1 \leq i \leq N$, and $T = \sum_{i=1}^N T_i$. Extend $\psi_{2k}(\xi)$ periodically for $\xi \in \mathbb{R}$ and denote this extension still by $\psi_{2k}(\xi)$.

Lemma 3.5 *Assume that the conditions* (3.2) *and* (3.5) *are fulfilled. The periodic function* $u(x,t) = \psi_{2k}(\xi)$ *with* $\xi = x - f'(v_{2k})t$ *defined by* (3.9) *is a discontinuous traveling wave entropy solution with speed* $f'(v_{2k})$ *for equation* (1.1).

Proof By the construction of $\psi_{2k}(\xi)$, we see that the two single-side limits of $\psi_{2k}(\xi)$ at the discontinuous point $\xi_0 + jT + \sum_{i=1}^{m} T_i + \xi_m^-$ are $\phi_{2k}(\xi_m^+) = \eta_m$ and $\phi_{2k}(\xi_m^-) = \mu_m$, for $j \in \mathbb{Z}$, $1 \leq m \leq N$. Since $a(s) \equiv 0$ for $s \in (b_{2k-1}, a_{2k})$ and the equality (3.8), we prove that ψ_{2k} satisfies the Rankine-Hugoniot condition (2.5) at any discontinuous points. The entropy condition (2.2) is fulfilled as $\mu_m < \eta_m$. Clearly, $\mu_m, \eta_m \in (b_{2k-1}, a_{2k})$. Lemma 2.1 implies that $u(x,t)$ is an entropy solution of (1.1). □

Proof of Theorem 2.1 Under the assumptions of Theorem 2.1, we see that the conditions (3.2) and (3.5) are fulfilled. According to the construction of $\psi_{2k}(\xi)$ and Lemma 3.5, we conclude that equation (1.1) admits infinitely many discontinuous traveling wave entropy solutions. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Mascia, C: Continuity in finite time of entropy solutions for nonconvex conservation laws with reaction term. Commun. Partial Differ. Equ. **23**, 913-931 (1998)
2. Lyberopoulos, AN: A Poincaré-Bendixson theorem for scalar balance law. Proc. R. Soc. Edinb., Sect. A **124**, 589-607 (1994)
3. Fan, H, Hale, JK: Large-time behavior in inhomogeneous conservation laws. Arch. Ration. Mech. Anal. **125**, 201-216 (1993)
4. Sinestrari, C: Instability of discontinuous traveling waves for hyperbolic balance laws. J. Differ. Equ. **134**, 269-285 (1997)
5. Zumbrun, K, Texier, B: Galloping instability of viscous shock waves. Physica D **237**, 1553-1601 (2008)
6. Wu, C, Zhang, P, Wong, SC, Qiao, D, Dai, S: Solitary wave solution to Aw-Rascle viscous model of traffic flow. Appl. Math. Mech. **34**, 523-528 (2013)
7. Haragus, M, Scheel, A: Corner defects in almost planar interface propagation. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **23**, 283-329 (2006)
8. Wu, Y, Xing, X: The stability of travelling fronts for general scalar viscous balance law. J. Math. Anal. Appl. **305**, 698-711 (2005)
9. Harterich, J: Attractors of viscous balance laws: uniform estimates for the dimension. J. Differ. Equ. **142**, 142-188 (1997)
10. Barbera, E, Curro, C, Valenti, G: On discontinuous travelling wave solutions for a class of hyperbolic reaction-diffusion models. Physica D **308**, 116-126 (2015)
11. Johnson, MA, Noble, P, Rodrigues, LM, Zumbrun, K: Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations. Invent. Math. **197**, 115-213 (2014)
12. Gray, DD, Giorgini, A: The validity of the Boussinesq approximation for liquids and gases. Int. J. Heat Mass Transf. **19**, 545-551 (1976)
13. Valougeorgis, D, Thomas, JR: Exact numerical results for Poiseuille and thermal creep flow in a cylindrical tube. Phys. Fluids **29**, 423-427 (1986)
14. Liu, TP: Nonlinear stability of shock waves for viscous conservation laws. Bull. Am. Math. Soc. **12**, 233-236 (1985)
15. Kruzhkov, SN: First order quasilinear equations in several independent variables. Mat. Sb. **81**, 228-255 (1970) (English transl.: Math. USSR Sb. **10**, 217-243 (1970))
16. Vol'pert, AI, Hudjaev, SI: Cauchy's problem for second order quasilinear degenerate parabolic equations. Mat. Sb. **78**, 389-411 (1969)
17. Wu, ZQ, Yin, JX: Some properties of functions in BV_x and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations. Northeast. Math. J. **5**, 395-422 (1989)