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Positive periodic solution for high-order p-Laplacian neutral differential equation with singularity

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Abstract

In this paper, we consider the following high-order p-Laplacian neutral differential equation with singularity:

$$(\varphi_p(x(t)-cx(t-\tau))^{(n)})^{(m)}+f(x(t))x'(t)+g(t,x(t-\sigma))=e(t).$$

By applications of coincidence degree theory and some analysis techniques, sufficient conditions for the existence of positive periodic solutions are established.

MSC: 34C25; 34K13; 34K40

Keywords: positive periodic solution; p-Laplacian; high-order; neutral operator; singularity

1 Introduction

In this paper, we consider the following high-order p-Laplacian neutral differential equation with singularity:

$$(\varphi_p(x(t)-cx(t-\tau))^{(n)})^{(m)}+f(x(t))x'(t)+g(t,x(t-\sigma))=e(t), \tag{1.1}$$

where $p\geq 2$, $\varphi_p(x)=|x|^{p-2}x$ for $x\neq 0$ and $\varphi_p(0)=0$; $g:[0,T]\times(0,\infty)\to\mathbb{R}$ is an L^2-Carathéodory function, *i.e.*, it is measurable in the first variable and continuous in the second variable, and for every $0<r<s$ there exists $h_{r,s}\in L^2[0,T]$ such that $|f(t,x(t))|\leq h_{r,s}$ for all $x\in[r,s]$ and a.e. $t\in[0,T]$. $g(t,x)$ being singular at 0 means that $g(t,x)$ becomes unbounded when $x\to 0^+$. τ and σ are constants and $0\leq\tau,\sigma<T$; $e:\mathbb{R}\to\mathbb{R}$ is a continuous periodic function with $e(t+T)\equiv e(t)$ and $\int_0^T e(t)\,dt=0$. T is a positive constant, c is a constant and $|c|\neq 1$; n, m are positive integers.

Generally speaking, differential equations with singularities have been considered from the very beginning of the discipline. The main reason is that singular forces are ubiquitous in applications, gravitational and electromagnetic forces being the most obvious examples. In 1986, Lazer and Solimini [1] discussed the second-order singular equation

$$u''+\frac{1}{u^\alpha}=h(t), \tag{1.2}$$

and they showed that, if $h(t)$ was continuous and T-periodic, then for all $\alpha > 0$ a positive T-periodic solution existed if and only if $h(t)$ had a positive mean value. Afterwards, they studied the singular equation

$$u'' - \frac{1}{u^\alpha} = h(t), \tag{1.3}$$

and they found that if $\alpha \geq 1$, a positive T-periodic solution existed if and only if $h(t)$ had a negative mean value. This last result was best possible in that for any α, $0 < \alpha < 1$, h can be chosen so that h had a negative mean value and the equation had no T-periodic solution.

Lazer and Solimini's work has attracted the attention of many specialists in differential equations. More recently, the method of lower and upper solutions [2–4], the Poincaré-Birkhoff twist theorem [5–7], topological degree theory [8, 9], the Schauder fixed point theorem [10–12], the Leray-Schauder alternative principle [13–15], the Krasnoselskii fixed point theorem in a cone [16, 17], and the fixed point index theory [18] have been employed to investigate the existence of positive periodic solutions of singular second-order, third-order, and fourth-order differential equations.

However, the singular differential equation (1.1), in which there are p-Laplacian and high-order cases, has not attracted much attention in the literature. There are not so many results concerning the existence of a positive periodic solution for (1.1) even when we have a neutral operator. In this paper, we try to fill gap and establish the existence of a positive periodic solution of (1.1) using coincidence degree theory. Our new results generalize in several aspects some recent results contained in [8].

In what follows, we need the notations:

$$|u|_\infty = \max_{t\in[0,T]} \bigl|u(t)\bigr|, \quad |u|_0 = \min_{t\in[0,T]} \bigl|u(t)\bigr|, \quad |u|_p = \left(\int_0^T |u|^p\,dt\right)^{\frac{1}{p}}, \quad \bar{h} = \frac{1}{T}\int_0^T h(t)\,dt.$$

2 Preparation

Let $C_T = \{\phi \in C(\mathbb{R},\mathbb{R}) : \phi(t+T) \equiv \phi(t)\}$ with the norm $|\phi|_\infty = \max_{t\in[0,T]} |\phi(t)|$. Define operators A as follows:

$$A : C_T \to C_T, \quad (Ax)(t) = x(t) - cx(t-\tau).$$

Lemma 2.1 (see [19]) *If $|c| \neq 1$, then the operator A has a continuous inverse A^{-1} on C_T, satisfying*:

(1)
$$\bigl[A^{-1}f\bigr](t) = \begin{cases} f(t) + \sum_{j=1}^\infty c^j f(t-j\tau), & \text{for } |c| < 1, \forall f \in C_T, \\ -\frac{f(t+\tau)}{c} - \sum_{j=1}^\infty \frac{1}{c^{j+1}} f(t+(j+1)\tau), & \text{for } |c| > 1, \forall f \in C_T. \end{cases}$$

(2) $|[A^{-1}f](t)| \leq \frac{|f|_\infty}{|1-|c||}$, $\forall f \in C_T$.

(3) $\int_0^T |[A^{-1}f](t)|\,dt \leq \frac{1}{|1-|c||}\int_0^T |f(t)|\,dt$, $\forall f \in C_T$.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y, respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 (Gaines and Mawhin [20]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism*.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to apply coincidence degree theorem, we rewrite (1.1) in the form

$$\begin{cases} (Ax_1)^{(n)}(t) = \varphi_q(x_2(t)), \\ x_2^{(m)}(t) = -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.1), then $x_1(t)$ must be a T-periodic solution to (1.1). Thus, the problem of finding a T-periodic solution for (1.1) reduces to finding one for (2.1).

Now, set $X = \{x = (x_1(t), x_2(t)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$; $Y = \{x = (x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \left\{x \in C^{n+m}(\mathbb{R}, \mathbb{R}^2) : x(t+T) = x(t), t \in \mathbb{R}\right\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)^{(n)}(t) \\ x_2^{(m)}(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \end{pmatrix}. \tag{2.2}$$

Then (2.1) can be converted into the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Setting $L_P = L|_{D(L)\cap \operatorname{Ker} P}$ and writing $L_P^{-1} : \operatorname{Im} L \to D(L)$ to denote the inverse of L_P, then

$$\begin{aligned}
&[L_P^{-1} y](t) = \begin{pmatrix} (A^{-1}Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix}, \\
&[Gy_1](t) = \sum_{i=1}^{n-1} \frac{1}{i!}(Ax_1)^{(i)}(0)t^i + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y_1(s)\,ds, \\
&[Gy_2](t) = \sum_{i=1}^{m-1} \frac{1}{i!}x_2^{(i)}(0)t^i + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} y_2(s)\,ds,
\end{aligned} \tag{2.3}$$

where $(Ax_1)^{(i)}(0)$, $i = 1, 2, \ldots, n-1$ are defined by the following:

$$E_1 Z = B, \quad \text{where } E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ c_{n-3} & c_{n-4} & c_{n-5} & \cdots & 1 & 0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

$Z = ((Ax_1)^{(n-1)}(0), \ldots, (Ax_1)''(0), (Ax_1)'(0))^\top$, $B = (b_1, b_2, \ldots, b_{n-1})^\top$, $b_i = -\frac{1}{i!T}\int_0^T (T-s)^i \times y_1(s)\,ds$, and $c_j = \frac{T^j}{(j+1)!}$, $j = 1, 2, \ldots, n-2$. $x_2^{(i)}(0)$, $i = 1, 2, \ldots, m-1$, are determined by the equation

$$E_2 W = F, \quad \text{where } E_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ c_{m-3} & c_{m-4} & c_{m-5} & \cdots & 1 & 0 \\ c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_1 & 0 \end{pmatrix}_{(m-1)\times(m-1)}.$$

$W = ((x_2)^{(m-1)}(0), \ldots, (x_2)''(0), (x_2)'(0))^\top$, $F = (d_1, d_2, \ldots, d_{n-1})^\top$, $d_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_2(s)\,ds$, and $c_j = \frac{T^j}{(j+1)!}$, $j = 1, 2, \ldots, m-2$.

From (2.2) and (2.3), it is clear that QN and $K(I-Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

3 Existence of positive periodic solutions for (1.1)

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H$_1$) There exist constants $0 < D_1 < D_2$ such that if x is a positive continuous T-periodic function satisfying

$$\int_0^T g\bigl(t, x(t)\bigr)\,dt = 0,$$

then

$$D_1 \leq x(\tau) \leq D_2$$

for some $\tau \in [0, T]$.

(H_2) $\bar{g}(x) < 0$ for all $x \in (0, D_1)$, and $\bar{g}(x) > 0$ for all $x > D_2$.

(H_3) Assume that

$$\psi(t) = \lim_{x\to+\infty} \sup \frac{g(t,x)}{x^{p-1}},$$

exist uniformly a.e. $t \in [0, T]$, *i.e.*, for any $\varepsilon > 0$ there is $g_\varepsilon \in L^2(0, T)$ such that

$$g(t,x) \leq \big(\psi(t) + \varepsilon\big)x^{p-1} + g_\varepsilon(t)$$

for all $x > 0$ and a.e. $t \in [0, T]$. Moreover, $\psi \in C(\mathbb{R}, \mathbb{R})$ and $\psi(t + T) = \psi(t)$.

(H_4) $g(t,x) = g_0(x) + g_1(t,x)$, where $g_0 \in C((0,\infty); \mathbb{R})$ and $g_1 : [0, T] \times [0,\infty) \to \mathbb{R}$ is an L^2-Carathéodory function.

(H_5) $\int_0^1 g_0(x)\, dx = -\infty$.

(H_6) There exist two positive constants a, b such that

$$\big|f\big(x(t)\big)\big| \leq a|x_1|^{p-2} + b, \quad \forall x \in \mathbb{R}.$$

Theorem 3.1 *Assume that conditions* (H_1)-(H_6) *hold. Suppose one of the following conditions is satisfied*:

(i) $p > 2$ *and* $\frac{(a+|\psi|_\infty T)T^p}{2^p|1-|c||^{p-1}}(\frac{T}{2\pi})^{(n-1)(p-1)+(m-2)} < 1$;

(ii) $p = 2$ *and* $\frac{(a+|\psi|_\infty T)T^p}{2^p|1-|c||^{p-1}}(\frac{T}{2\pi})^{(n-1)(p-1)+(m-2)} + \frac{bT^2(\frac{T}{2\pi})^{n+m-3}}{4|1-|c||} < 1$.

Then (1.1) *has at least one positive* T*-periodic solution*.

Proof Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)^{(n)}(t) = \lambda\varphi_q(x_2(t)), \\ x_2^{(m)}(t) = -\lambda f(x_1(t))x_1'(t) - \lambda g(t, x_1(t-\sigma)) + \lambda e(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \lambda^{1-p}\varphi_p[(Ax_1)^{(n)}(t)]$ into the second equation of (3.1)

$$\big(\varphi_p(Ax_1)^{(n)}(t)\big)^{(m)} + \lambda^p f\big(x_1(t)\big)x_1'(t) + \lambda^p g\big(t, x_1(t-\sigma)\big) = \lambda^p e(t). \tag{3.2}$$

Integrating of both sides of (3.2) from 0 to T, we have

$$\int_0^T g\big(t, x_1(t-\sigma)\big)\, dt = 0. \tag{3.3}$$

In view of (H_1), there exist positive constants D_1, D_2, and $\xi \in [0,T]$ such that

$$D_1 \le x_1(\xi) \le D_2.$$

Then we have

$$|x_1(t)| = \left| x_1(\xi) + \int_\xi^t x_1'(s)\,ds \right| \le D_2 + \int_\xi^t |x_1'(s)|\,ds, \quad t \in [\xi, \xi + T],$$

and

$$|x_1(t)| = |x_1(t-T)| = \left| x_1(\xi) - \int_{t-T}^\xi x_1'(s)\,ds \right| \le D_2 + \int_{t-T}^\xi |x_1'(s)|\,ds, \quad t \in [\xi, \xi + T].$$

Combing the above two inequalities, we obtain

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[\xi,\xi+T]} |x_1(t)| \\ &\le \max_{t\in[\xi,\xi+T]} \left\{ D_2 + \frac{1}{2}\left(\int_\xi^t |x_1'(s)|\,ds + \int_{t-T}^\xi |x_1'(s)|\,ds \right) \right\} \\ &\le D_2 + \frac{1}{2}\int_0^T |x_1'(s)|\,ds. \end{aligned} \tag{3.4}$$

Since

$$(Ax_1)^{(n)}(t) = \big(x_1(t) - cx_1(t-\sigma)\big)^{(n)} = x_1^{(n)}(t) - cx_1^{(n)}(t-\sigma) = Ax_1^{(n)}(t),$$

from Lemma 2.1 and the first equation of (3.1), we have

$$\begin{aligned} |x_1^{(n)}|_\infty &= \max_{t\in[0,T]} |A^{-1}Ax_1^{(n)}(t)| \\ &\le \frac{\max_{t\in[0,T]} |(Ax_1)^{(n)}(t)|}{|1-|c||} \\ &\le \frac{\varphi_q(|x_2|_\infty)}{|1-|c||}. \end{aligned} \tag{3.5}$$

On the other hand, from $x_2^{(m-2)}(0) = x_2^{(m-2)}(T)$, there exists a point $t_1 \in [0,T]$ such that $x_2^{(m-1)}(t_1) = 0$, which together with the integration of the second equation of (3.1) on the interval $[0,T]$ yields

$$\begin{aligned} 2|x_2^{(m-1)}(t)| &\le 2\left(x_2^{(m-1)}(t_1) + \frac{1}{2}\int_0^T |x_2^{(m)}(t)|\,dt \right) \\ &\le \lambda \int_0^T \big| -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \big|\,dt \\ &\le \int_0^T |f(x_1(t))||x_1'(t)|\,dt + \int_0^T |g(t, x_1(t-\sigma))|\,dt + \int_0^T |e(t)|\,dt. \end{aligned} \tag{3.6}$$

Write

$$I_+ = \{t \in [0,T] : g(t, x_1(t-\sigma)) \geq 0\}; \qquad I_- = \{t \in [0,T] : g(t, x_1(t-\sigma)) \leq 0\}.$$

Then we get from (H_3) and (3.3) that

$$\begin{aligned}
\int_0^T |g(t, x_1(t-\sigma))| \, dt &= \int_{I_+} g(t, x_1(t-\sigma)) \, dt - \int_{I_-} g(t, x_1(t-\sigma)) \, dt \\
&= 2 \int_{I_+} g(t, x_1(t-\sigma)) \, dt \\
&\leq 2 \int_{I_+} \big((\psi(t)+\varepsilon) x_1^{p-1}(t-\sigma) + g_\varepsilon(t)\big) \, dt \\
&\leq 2\big(|\psi|_\infty + \varepsilon\big) \int_0^T |x_1(t)|^{p-1} \, dt + 2 \int_0^T |g_\varepsilon(t)| \, dt.
\end{aligned} \tag{3.7}$$

Substituting (3.4) and (3.7) into (3.6), and from (H_5), we have

$$\begin{aligned}
2|x_2^{(m-1)}(t)| &\leq a \int_0^T |x_1(t)|^{p-2} |x_1'(t)| \, dt + b \int_0^T |x_1'(t)| \, dt \\
&\quad + 2\big(|\psi|_\infty + \varepsilon\big) \int_0^T |x_1(t)|^{p-1} \, dt + 2 \int_0^T |g_\varepsilon(t)| \, dt + \int_0^T |e(t)| \, dt \\
&\leq a \left(D_2 + \frac{1}{2} \int_0^T |x_1'(t)| \, dt\right)^{p-2} \int_0^T |x_1'(t)| \, dt + b \int_0^T |x_1'(t)| \, dt \\
&\quad + 2\big(|\psi|_\infty + \varepsilon\big) T \left(D_2 + \frac{1}{2} \int_0^T |x_1'(t)| \, dt\right)^{p-1} + 2T^{\frac{1}{2}} |g_\varepsilon|_2 + |e|_\infty T.
\end{aligned} \tag{3.8}$$

For a given constant $\delta > 0$, which is only dependent on $k > 0$, we have

$$(1+x)^k \leq 1 + (1+k)x \quad \text{for } x \in [0, \delta].$$

From (3.8), we have

$$\begin{aligned}
2|x_2^{(m-1)}(t)| &\leq \frac{a}{2^{p-2}} \left(\frac{2D_2}{\int_0^T |x_1'(t)| \, dt} + 1\right)^{p-2} \left(\int_0^T |x_1'(t)| \, dt\right)^{p-1} \\
&\quad + \frac{(|\psi|_\infty + \varepsilon) T}{2^{p-2}} \left(\frac{2D_2}{\int_0^T |x_1'(t)| \, dt} + 1\right)^{p-1} \\
&\quad \cdot \left(\int_0^T |x_1'(t)| \, dt\right)^{p-1} + b \int_0^T |x_1'(t)| \, dt + 2T^{\frac{1}{2}} |g_\varepsilon|_2 + |e|_\infty T \\
&\leq \left[\frac{a}{2^{p-2}} \left(1 + \frac{2D_2(p-1)}{\int_0^T |x_1'(t)| \, dt}\right) + \frac{(|\psi|_\infty + \varepsilon) T}{2^{p-2}} \left(1 + \frac{2D_2 p}{\int_0^T |x_1'(t)| \, dt}\right)\right] \\
&\quad \cdot \left(\int_0^T |x_1'(t)| \, dt\right)^{p-1} + b \int_0^T |x_1'(t)| \, dt + 2T^{\frac{1}{2}} |g_\varepsilon|_2 + |e|_\infty T \\
&= \left(\frac{a}{2^{p-2}} + \frac{(|\psi|_\infty + \varepsilon) T}{2^{p-2}}\right) \left(\int_0^T |x_1'(t)| \, dt\right)^{p-1}
\end{aligned}$$

$$+\frac{aD_2(p-1)+\big(|\psi|_\infty+\varepsilon\big)TD_2p}{2^{p-3}}$$
$$\cdot\left(\int_0^T\left|x_1'(t)\right|dt\right)^{p-2}+b\int_0^T\left|x_1'(t)\right|dt+2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T. \tag{3.9}$$

From the Wirtinger inequality (see [21], Lemma 2.4), we get

$$\begin{aligned}\int_0^T\left|x_1'(s)\right|ds&\le T^{\frac{1}{2}}\left(\int_0^T\left|x_1'(s)\right|^2ds\right)^{\frac{1}{2}}\\&\le T^{\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{n-1}\left(\int_0^T\left|x_1^{(n)}(s)\right|^2ds\right)^{\frac{1}{2}}\\&\le T\left(\frac{T}{2\pi}\right)^{n-1}\left|x_1^{(n)}\right|_\infty.\end{aligned}\tag{3.10}$$

Substituting (3.10) into (3.9), we have

$$\begin{aligned}2\left|x_2^{(m-1)}(t)\right|\le{}&\frac{(a+(|\psi|_\infty+\varepsilon)T)T^{p-1}}{2^{p-2}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)}\left|x_1^{(n)}\right|_\infty^{p-1}\\&+\frac{(aD_2(p-1)+(|\psi|_\infty+\varepsilon)TD_2p)T^{p-2}}{2^{p-3}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)}\left|x_1^{(n)}\right|_\infty^{p-2}\\&+bT\left(\frac{T}{2\pi}\right)^{n-1}\left|x_1^{(n)}\right|_\infty+2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T.\end{aligned}\tag{3.11}$$

Substituting (3.5) into (3.11), we have

$$\begin{aligned}2\left|x_2^{(m-1)}(t)\right|\le{}&\frac{(a+(|\psi|_\infty+\varepsilon)T)T^{p-1}}{2^{p-2}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)}\frac{(\varphi_q(|x_2|_\infty))^{p-1}}{|1-|c||^{p-1}}\\&+\frac{(aD_2(p-1)+(|\psi|_\infty+\varepsilon)TD_2p)T^{p-2}}{2^{p-3}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)}\frac{(\varphi_q(|x_2|_\infty))^{p-2}}{|1-|c||^{p-2}}\\&+bT\left(\frac{T}{2\pi}\right)^{n-1}\frac{\varphi_q(|x_2|_\infty)}{|1-|c||}+2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T\\={}&\frac{(a+(|\psi|_\infty+\varepsilon)T)T^{p-1}}{2^{p-2}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)}\frac{|x_2|_\infty}{|1-|c||^{p-1}}\\&+\frac{(aD_2(p-1)+(|\psi|_\infty+\varepsilon)TD_2p)T^{p-2}}{2^{p-3}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)}\frac{|x_2|_\infty^{2-q}}{|1-|c||^{p-2}}\\&+bT\left(\frac{T}{2\pi}\right)^{n-1}\frac{|x_2|_\infty^{q-1}}{|1-|c||}+2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T.\end{aligned}\tag{3.12}$$

Since $\int_0^T(\varphi_q(x_2(t)))\,dt=\int_0^T(Ax_1(t))^{(n)}(t)\,dt=0$, there exists a point $t_2\in[0,T]$ such that $x_2(t_2)=0$. From the Wirtinger inequality, we can easily get

$$\begin{aligned}|x_2|_\infty&\le\frac{1}{2}\int_0^T\left|x_2'(t)\right|dt\\&\le\frac{\sqrt{T}}{2}\left(\int_0^T\left|x_2'(t)\right|^2\right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned} &\le \frac{\sqrt{T}}{2}\left(\frac{T}{2\pi}\right)^{m-2}\left(\int_0^T \left|x_2^{(m-1)}(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-2}\left|x_2^{(m-1)}\right|_\infty. \end{aligned} \tag{3.13}$$

Combination of (3.13) and (3.12) implies

$$\begin{aligned} |x_2|_\infty &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-2}\left|x_2^{(m-1)}\right|_\infty \\ &\le \frac{T}{4}\left(\frac{T}{2\pi}\right)^{m-2}\left[\frac{(a+(|\psi|_\infty+\varepsilon)T)T^{p-1}}{2^{p-2}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)}\frac{|x_2|_\infty}{|1-|c||^{p-1}}\right. \\ &\quad + \frac{(aD_2(p-1)+(|\psi|_\infty+\varepsilon)TD_2p)T^{p-2}}{2^{p-3}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)}\frac{|x_2|_\infty^{2-q}}{|1-|c||^{p-2}} \\ &\quad \left. + bT\left(\frac{T}{2\pi}\right)^{n-1}\frac{|x_2|_\infty^{q-1}}{|1-|c||} + 2T^{\frac{1}{2}}|g_\varepsilon|_2 + |e|_\infty T\right]. \end{aligned}$$

So, we have

$$\begin{aligned} |x_2|_\infty &\le \frac{(a+(|\psi|_\infty+\varepsilon)T)T^p}{2^p|1-|c||^{p-1}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)}|x_2|_\infty \\ &\quad + \frac{(aD_2(p-1)+\left(|\psi|_\infty+\varepsilon\right)TD_2p)T^{p-1}}{2^{p-1}|1-|c||^{p-2}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)+(m-2)}|x_2|_\infty^{2-q} \\ &\quad + \frac{bT^2}{4}\left(\frac{T}{2\pi}\right)^{n+m-3}\frac{|x_2|_\infty^{q-1}}{|1-|c||} + \frac{T}{4}\left(\frac{T}{2\pi}\right)^{m-2}\left(2T^{\frac{1}{2}}|g_\varepsilon|_2 + |e|_\infty T\right). \end{aligned}$$

Case (i): If $p > 2$, we can get $1 < q < 2$. Since ε sufficiently small, we know that

$$\frac{(a+|\psi|_\infty T)T^p}{2^p|1-|c||^{p-1}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)} < 1,$$

there exists a positive constant M_1 such that

$$|x_2|_\infty \le M_1. \tag{3.14}$$

Case (ii): If $p = 2$, we can get $q = 2$. Since ε is sufficiently small, we know that

$$\frac{(a+|\psi|_\infty T)T^p}{2^p|1-|c||^{p-1}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)} + \frac{bT^2(\frac{T}{2\pi})^{n+m-3}}{4|1-|c||} < 1,$$

there exists a positive constant M_1 such that

$$|x_2|_\infty \le M_1.$$

On the other hand, from (3.5), we have

$$\left|x_1^{(n)}\right|_\infty \le \frac{\varphi_q(|x_2|_\infty)}{|1-|c||} \le \frac{M_1^{q-1}}{|1-|c||} := M_n'. \tag{3.15}$$

Since $x_1(0) = x_1(T)$, there exists a point $t_3 \in [0, T]$ such that $x_1'(t_3) = 0$. From the Wirtinger inequality, we can easily get

$$\begin{aligned}
|x_1'|_\infty &\le \frac{1}{2}\int_0^T |x_1''(t)|\,dt \le \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x_1''(t)|^2\,dt\right)^{\frac{1}{2}} \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(n-2)} |x_1^{(n)}|_\infty \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-2} M_n' := M_2.
\end{aligned} \tag{3.16}$$

Hence, from (3.4), we have

$$|x_1|_\infty \le D + \frac{1}{2}\int_0^T |x_1'(t)|\,dt \le D + \frac{TM_2}{2} := M_3. \tag{3.17}$$

From (3.7), (3.16), and (3.17) we have

$$\begin{aligned}
|x_2^{(m-1)}|_\infty &\le \frac{1}{2}\max\left|\int_0^T x_2^{(m)}(t)\,dt\right| \\
&\le \frac{\lambda}{2}\int_0^T \left|-f\bigl(x_1(t)\bigr)x_1'(t) - g\bigl(t, x_1(t-\sigma)\bigr) + e(t)\right|\,dt \\
&\le \frac{\lambda}{2}\bigl(|f|_{M_3} TM_2 + 2\bigl(|\psi|_\infty + \varepsilon\bigr)TM_3^{p-1} + 2\sqrt{T}|g_\varepsilon|_2 + T|e|_\infty\bigr) := \lambda M_{m-2},
\end{aligned}$$

where $|f|_{M_3} = \max_{0<x_1(t)\le M_3} |f(x_1(t))|$. Since $x_2(0) = x_2(T)$, there exists a point $t_4 \in [0, T]$ such that $x_2'(t_4) = 0$. From the Wirtinger inequality, we can easily get

$$\begin{aligned}
|x_2'|_\infty &\le \frac{1}{2}\int_0^T |x_2''(t)|\,dt \\
&\le \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x_2''(t)|^2\,dt\right)^{\frac{1}{2}} \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(m-3)} |x_2^{(m-1)}|_\infty \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(m-3)} \lambda M_{m-2} := \lambda M_2.
\end{aligned}$$

Next, it follows (3.2) that

$$\begin{aligned}
&\bigl(\varphi_p(Ax_1)^{(n)}(t+\sigma)\bigr)^{(m)} + \lambda^p\bigl(f\bigl(x_1(t+\sigma)\bigr)x_1'(t+\sigma) + \lambda^p g\bigl(t+\sigma, x_1(t)\bigr)\bigr) \\
&\quad = \lambda^p e(t+\sigma).
\end{aligned} \tag{3.18}$$

Namely,

$$\begin{aligned}
&\bigl(\varphi_p(Ax_1)^{(n)}(t+\sigma)\bigr)^{(m)} + \lambda^p f\bigl(x_1(t+\sigma)\bigr)x_1'(t+\sigma) + \lambda^p\bigl(g_0\bigl(x_1(t)\bigr)\bigr) + g_1\bigl(t+\sigma, x_1(t)\bigr) \\
&\quad = \lambda^p e(t+\sigma).
\end{aligned} \tag{3.19}$$

Multiplying both sides of (3.19) by $x_1'(t)$, we get

$$\begin{aligned}&\big(\varphi_p(Ax_1)^{(n)}(t+\sigma)\big)^{(m)}x_1'(t)+\lambda^p f\big(x_1(t+\sigma)\big)x_1'(t+\sigma)x_1'(t)\\&\quad+\lambda^p g_0\big(x_1(t)\big)x_1'(t)+\lambda^p g_1\big(t+\sigma,x_1(t)\big)x_1'(t)=\lambda^p e(t+\sigma)x_1'(t).\end{aligned}\tag{3.20}$$

Let $\tau\in[0,T]$, for any $\tau\le t\le T$, we integrate (3.20) on $[\tau,t]$ and get

$$\begin{aligned}&\lambda^p\int_{x_1(\tau)}^{x_1(t)}g_0(u)\,du\\&\quad=\lambda^p\int_\tau^t g_0\big(x_1(s)\big)x_1'(s)\,ds\\&\quad=-\int_\tau^t\big(\varphi_p(Ax_1)^{(n)}(s+\sigma)\big)^{(m)}x_1'(s)\,ds-\lambda^p\int_\tau^t f\big(x_1(s+\sigma)\big)x_1'(s+\sigma)x_1'(s)\,ds\\&\qquad-\lambda^p\int_\tau^t g_1\big(s+\sigma,x_1(s)\big)x_1'(s)\,ds+\lambda^p\int_\tau^t e(s+\sigma)x_1'(s)\,ds.\end{aligned}\tag{3.21}$$

By (3.2), (3.7), (3.16), and (3.17), we have

$$\begin{aligned}&\left|\int_\tau^t\big(\varphi_p(Ax_1)^{(n)}(s+\sigma)\big)^{(m)}x_1'(s)\,ds\right|\\&\quad\le\int_\tau^t\big|\big(\varphi_p(Ax_1)^{(n)}(s+\sigma)\big)^{(m)}\big|\big|x_1'(s)\big|\,ds\\&\quad\le|x_1'|_\infty\int_0^T\big|\big(\varphi_p(Ax_1)^{(n)}(t+\sigma)\big)^{(m)}\big|\,dt\\&\quad\le\lambda^p|x_1'|_\infty\left(\int_0^T\big|f\big(x_1(t)\big)\big|\big|x_1'(t)\big|\,dt+\int_0^T\big|g\big(t,x_1(t-\sigma)\big)\big|\,dt+\int_0^T\big|e(t)\big|\,dt\right)\\&\quad\le\lambda^p M_2\big(|f|_{M_3}M_2+2\big(|\psi|_\infty+\varepsilon\big)TM_3^{p-1}+2T^{\frac12}\big|g_\varepsilon^+\big|_2+T|e|_\infty\big).\end{aligned}$$

We have

$$\begin{aligned}&\left|\int_\tau^t f\big(x_1(s+\sigma)\big)x_1'(s+\sigma)x_1'(s)\,ds\right|\le|f|_{M_3}M_2^2T,\\&\left|\int_\tau^t g\big(s+\sigma,x_1(s)\big)x_1'(s)\,ds\right|\le|x_1'|\int_0^T\big|g(t,x(t-\sigma)\big|\,dt\le M_2^{p-1}\sqrt{T}|g_{M_3}|_2,\end{aligned}$$

where $g_{M_3}=\max_{0\le x\le M_3}|g_1(t,x)|\in L^2(0,T)$ is as in (H_3); we have

$$\left|\int_\tau^t e(t+\sigma)x_1'(t)\,dt\right|\le M_2T|e|_\infty.$$

From these inequalities we can derive from (3.21) that

$$\left|\int_{x_1(\tau)}^{x_1(t)}g_0(u)\,du\right|\le M_5',\tag{3.22}$$

for some constant M_5', which is independent on λ, x, and t. In view of the strong force condition (H_4), we know that there exists a constant $M_5 > 0$ such that

$$x_1(t) \geq M_5, \quad \forall t \in [\tau, T]. \tag{3.23}$$

The case $t \in [0, \tau]$ can be treated similarly.

From (3.14), (3.16), and (3.17) and (3.23), we let

$$\begin{aligned}\Omega = \bigl\{ & x = (x_1, x_2)^\top : E_1 \leq |x_1|_\infty \leq E_2, \bigl|x_1'\bigr|_\infty \leq E_3, |x_2|_\infty \leq E_4 \text{ and} \\ & \bigl|x_2'\bigr|_\infty \leq E_5, \forall t \in [0, T]\bigr\},\end{aligned}$$

where $0 < E_1 < \min\{M_5, D_1\}$, $E_2 > \max\{M_1, D_2\}$, $E_3 > M_2$, $E_4 > M_4$, and $E_5 > M_3$. $\Omega_2 = \{x : x \in \partial\Omega \cap \operatorname{Ker} L\}$ then $\forall x \in \partial\Omega \cap \operatorname{Ker} L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $x_1 = E_2$ or $-E_2$. But if $x_1(t) = E_2$, we know

$$0 = \int_0^T \bigl\{g(t, E_2) - e(t)\bigr\}\, dt.$$

From assumption (H_2), we have $x_1(t) \leq D_2 \leq E_2$, which yields a contradiction. Similarly if $x_1 = -E_2$. We also have $QNx \neq 0$, *i.e.*, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 2.2 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1-\mu)JQNx$, $(\mu, x) \in [0,1] \times \Omega$, then $\forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T [g(t, x_1) - e(t)]\, dt \\ -\mu x_2 - (1-\mu)|x_2|^{q-2}x_2 \end{pmatrix}.$$

We have $\int_0^T e(t)\, dt = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T g(t, x_1)\, dt \\ -\mu x_2 - (1-\mu)|x_2|^{q-2}x_2 \end{pmatrix}, \quad \forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L).$$

From (H_2), it is obvious that $x^\top H(\mu, x) < 0$, $\forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned}\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(0, x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(1, x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

So condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, *i.e.*, (1.1) has an T-periodic solution $x_1(t)$. □

Example 3.1 Consider the p-Laplacian type high-order neutral differential equation with singularity

$$\begin{aligned}&\bigl(\varphi_p\bigl(x(t)-11x(t-\tau)\bigr)'''\bigr)'''+3x^2(t)x'(t)+\frac{1}{10}(\cos 2t+4)x^3(t-\sigma)-\frac{1}{x^\kappa(t-\sigma)}\\&\quad=\sin 2t,\end{aligned} \tag{3.24}$$

where $\kappa\geq 1$ and $p=4$, σ and τ are constants, and $0\leq\sigma,\tau<T$.

It is clear that $T=\pi$, $m=n=3$, $c=11$, $g(t,x)=\frac{1}{10}(\cos 2t+4)x^3(t-\sigma)-\frac{1}{x^\kappa(t-\sigma)}$, $\psi(t)=\frac{1}{10}(\cos 2t+4)$, $|\psi|_\infty=\frac{1}{2}$, $f(x(t))=3x^2(t)$, and $|f(x(t))|\leq 3|x^2(t)|+1$; here $a=3$, $b=1$. It is obvious that (H_1)-(H_5) hold. Now we consider the assumption of the condition

$$\begin{aligned}&\frac{(a+|\psi|_\infty T)T^p}{2^p|1-|c||^{p-1}}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)}\\&\quad=\frac{(3+\frac{\pi}{2})\pi^4}{2^4\times 10^3}\times\frac{1}{2^7}\\&\quad=\frac{(3+\frac{\pi}{2})\pi^4}{2^{11}\times 10^3}<1.\end{aligned}$$

So by Theorem 3.1, we know (3.24) has at least one positive π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX, XFH, and ZBC worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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