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A doubly degenerate diffusion equation not in divergence form with gradient term

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Abstract

In this paper, we investigate positive solutions to the doubly degenerate parabolic equation not in divergence form with gradient term $u_t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q + \gamma u^r|\nabla u|^p$, subject to the null Dirichlet boundary condition. We first establish the local existence of weak solutions to the problem, and then determine in what way the gradient term affects the behavior of solutions. The conditions for global and non-global solutions are obtained with the critical exponent $r_c = \frac{pm-q}{p-1}$. Here we introduce some precise technique for the 'concavity method' to deal with the difficult non-divergence form of the model.

MSC: Primary 35K65; 35K55; 35B40

Keywords: doubly degenerate parabolic; not in divergence form; blow-up; global solution; gradient term

1 Introduction

This paper studies the doubly degenerate parabolic equation not in divergence form with gradient term

$$\begin{cases} u_t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q + \gamma u^r|\nabla u|^p, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $m \geq 1$, $p > 2$, $q \geq m$, $r \geq m-1$, $\gamma > 0$, $\lambda > 0$, and $u_0(x) \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$, $u_0(x) > 0$ in Ω.

There has been much work contributed to the degenerate parabolic equations not in divergence form. Friedman-McLeod [1] considered the following problem:

$$\begin{cases} u_t = u^p \Delta u + u^q, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

with $p = 2$ and $q = p + 1 = 3$, for which it was shown that, for sufficiently large domains, the solutions of (1.2) must blow up in finite time regardless of the size of the initial value. The more general situation with $p > 1$ and $q > 1$ was studied by Wiegner [2, 3]. We refer to [4] for more results on (1.2).

Stinner [5] investigated the non-divergence form parabolic equation with gradient term

$$\begin{cases} u_t = u^p \Delta u + u^q + \kappa u^r |\nabla u|^2, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.3}$$

with $p > 0$, $q > 1$, $r > -1$. Quite differently from the problem (1.2) without gradient term, it was found that the additional gradient term can enforce blow-up in some cases, with the critical exponent $r_c = 2p - q$.

Recently, Jin and Yin [6] studied the doubly degenerate diffusion equation

$$\begin{cases} u_t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.4}$$

where $m \geq 1$, $p > 1$, and they obtained the critical exponent $q_c = p + m - 1$, namely, the solutions are global if $q < q_c$, and there exist both global and blow-up solutions if $q > q_c$. In the critical case $q = q_c$, blow-up or not of solutions will be determined by the size of the domain.

As for the doubly degenerate diffusion equation with gradient term

$$\begin{cases} u_t = u \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \gamma |\nabla u|^p, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.5}$$

Zhou *et al.* proved the existence conditions of solutions [7–10].

A natural question is in what way the additional gradient term in (1.1) affects the behavior of solutions. It will be shown that, depending on the complicated interaction among the multi-nonlinearity parameters m, p, q, and r, the problem (1.1) admits the critical exponent $r_c = \frac{pm-q}{p-1}$, for which there is some substantial difficulty to be overcome due to the doubly degenerate diffusion of non-divergence in (1.1). In particular, to treat the critical case $r = r_c$, we will introduce an auxiliary problem $w_t = f(w)(\operatorname{div}(|\nabla w|^{p-2}\nabla w) + cw^{p-1})$ with $f(w) = h^m(w)h'(w)^{p-2}$ and $h(s)$ solving an ODE problem. We will explore a 'concavity method' where some precise technique is necessary to deal with the difficult non-divergence form with the general $f(w)$.

Throughout the paper, denote by λ_1 the first Dirichlet eigenvalue of the problem

$$\begin{cases} -\operatorname{div}(|\nabla \varphi|^{p-2}\nabla \varphi) = \lambda \varphi^{p-1}, & x \in \Omega, \\ \varphi > 0, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

with the corresponding eigenfunction $\varphi \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$, normalized by $\varphi > 0$ in Ω, $\|\varphi\|_\infty = 1$.

2 Local existence of weak solutions

We begin with the local existence of solutions to (1.1). Denote

$$\Omega_T = \Omega \times (0,T), \qquad S_T = \partial\Omega \times (0,T), \qquad \Sigma_T = S_T \cup \{\Omega \times \{0\}\},$$

$$\mathbb{E} = \{u \in L^\infty(\Omega_T); u_t \in L^1(\Omega_T); \nabla u \in L^p_{\mathrm{loc}}(\Omega_T)\},$$
$$\mathbb{E}_0 = \{u \in \mathbb{E}; u|_{\partial\Omega} = 0\}.$$

The following comparison principle will play a crucial role in the paper, the proof of which can be found in [11].

Lemma 2.1 *Let L be the parabolic differential operator defined by*

$$Lu := \partial_t u - f(x,t,u)\Delta u + g(x,t,u)$$

with continuous functions f and g, $f \geq 0$. Let $u_i \in C^0(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$, $i = 1, 2$, be such that Lu_i is well defined in Ω_T and

$$Lu_1 \leq Lu_2 \quad \text{in } \Omega_T, \qquad u_1 \leq u_2 \quad \text{on } \Sigma_T.$$

Assume f and g are Lipschitz with respect to u in a neighborhood of $u_i(\overline{\Omega}_T)$, $i = 1$ or 2, and in addition either $u_1 < u_2$ on Σ_T or $\nabla^2 u_i \in L^\infty(\overline{\Omega}_T)$. Then

$$u_1 \leq u_2 \quad \text{in } \overline{\Omega}_T.$$

Since (1.1) degenerates when $u = 0$ or $|\nabla u| = 0$, the problem does not admit classical solutions in general. Here we deal with nonnegative weak solutions, defined as follows.

Definition 2.1 A nonnegative measurable function $u \in \mathbb{E}$ is called a weak subsolution of problem (1.1)

$$\int_0^t \int_\Omega \{u_t\phi + |\nabla u|^{p-2}\nabla u \cdot \nabla(u^m\phi) - \gamma u^r|\nabla u|^p\phi\}\,dx\,d\tau \leq \lambda \int_0^t \int_\Omega u^q\phi\,dx\,d\tau \tag{2.1}$$

for all bounded test functions $0 \leq \phi \in C_0^1(\Omega_T)$. The weak supersolution is defined by the opposite inequality, and u is a weak solution of (1.1) if it is both a subsolution and a supersolution to (1.1).

To show the local solvability of (1.1), consider the following regularized problem:

$$\begin{cases} (u_{\epsilon\eta})_t = u_{\epsilon\eta}^m \operatorname{div}\{(|\nabla u_{\epsilon\eta}|^2 + \eta)^{\frac{p-2}{2}}\nabla u_{\epsilon\eta}\} + \lambda u_{\epsilon\eta}^q & \\ \qquad + \gamma u_{\epsilon\eta}^r(|\nabla u_{\epsilon\eta}|^2 + \eta)^{\frac{p-2}{2}}|\nabla u_{\epsilon\eta}|^2 & \text{in } \Omega_T, \\ u_{\epsilon\eta} = \epsilon & \text{on } S_T, \\ u_{\epsilon\eta}(x,0) = u_0(x) + \epsilon & \text{on } \overline{\Omega}, \end{cases} \tag{2.2}$$

where $\epsilon \in (0,1)$. Denote the classical solution of problem (2.2) by $u_{\epsilon\eta}$. It is easy to prove for fixed $\eta \geq 0$ that $u_{\epsilon\eta} \geq \epsilon$, and $u_{\epsilon\eta}$ is increasing in ϵ.

Lemma 2.2 *For any $\epsilon \in (0,1)$, there exists a function u_ϵ with $u_\epsilon - \epsilon \in L^\infty(\Omega_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$ for some $T > 0$, such that u_ϵ is a weak solution of the problem*

$$\begin{cases} u_t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q + \gamma u^r|\nabla u|^p, & (x,t) \in \Omega \times (0,T), \\ u = \epsilon, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) + \epsilon, & x \in \overline{\Omega}. \end{cases} \tag{2.3}$$

Proof *Step* 1. *A priori estimates for* $u_{\epsilon\eta}$.

At first it is easy to show that there exist $T_1, M_1 > 0$ such that

$$\|u_{\epsilon\eta}\|_{L^\infty(\Omega_{T_1})} \le M_1 \quad \text{for all } \eta \ge 0.$$

In fact, let U solve

$$\frac{dU}{dt} = \lambda U^q, \qquad U(0) = \|u_0 + 1\|_{L^\infty(\Omega)}$$

in $[0, T_0)$. Set $T = T_0/2$. Then

$$\|u_{\epsilon\eta}\|_{L^\infty(\Omega_T)} \le M_1 = U(T) < \infty \tag{2.4}$$

by comparison.

Choose s satisfying

$$s > C_1 := \begin{cases} \gamma M_1^{r-m+1}, & r > m-1, \\ \gamma - m, & r = m-1. \end{cases}$$

Multiply (2.2) by $u_{\epsilon\eta}^s$, and integrate over Ω_T,

$$\begin{aligned} &\int_{\Omega_T} (u_{\epsilon\eta})_t u_{\epsilon\eta}^s \,dx\,dt + (m+s)\int_{\Omega_T} u_{\epsilon\eta}^{m+s-1}\big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt \\ &\quad \le \lambda \int_{\Omega_T} (u_{\epsilon\eta})^{q+s} \,dx\,dt + \gamma \int_{\Omega_T} u_{\epsilon\eta}^{r+s}\big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt. \end{aligned}$$

We have

$$\int_{\Omega_T} u_{\epsilon\eta}^{m+s-1}\big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt \le C_2 = C_2(M_1, s, \gamma), \tag{2.5}$$

and hence

$$\int_{\Omega_T} \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt \le C_3 = C_3(M_1, s, \gamma, \epsilon), \tag{2.6}$$

due to $u_{\epsilon\eta} \ge \epsilon$. Consequently,

$$\int_{\Omega_T} |\nabla u_{\epsilon\eta}|^p \,dx\,dt \le \int_{\Omega_T} \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt \le C_3. \tag{2.7}$$

Integrate (2.2) over Ω_T. Noticing $\frac{\partial u_{\epsilon\eta}}{\partial n}|_{\partial\Omega} \le 0$, by (2.4) and (2.7), we have

$$\begin{aligned} \int_{\Omega_T} (u_{\epsilon\eta})_t \,dx\,dt &\le -m \int_{\Omega_T} u_{\epsilon\eta}^{m-1}\big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt + \lambda \int_{\Omega_T} u_{\epsilon\eta}^q \,dx\,dt \\ &\quad + \gamma \int_{\Omega_T} u_{\epsilon\eta}^r \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \,dx\,dt \\ &\le C_4 = C_4(C_3, M_1). \end{aligned} \tag{2.8}$$

Step 2. *Uniform integrability.*

For any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4C_5}$ with C_5 to be defined. Then for any measurable set $E \subset \Omega$, $\operatorname{meas}(E) < \delta$, there exists $\widetilde{E}$ such that $E \subset \widetilde{E} \subset \Omega$ with $\operatorname{meas}(\widetilde{E}) < 2\delta$. Take $\rho(x) \in C_0^1(\Omega)$ satisfying $\rho(x) = 1$ for $x \in E$, $\rho(x) = 0$ for $x \in \Omega \setminus \widetilde{E}$, and $0 \leq \rho(x) \leq 1$, $\nabla\rho(x) \leq C\rho^{\alpha}(x)$ in Ω with $\frac{p-1}{p} < \alpha < 1$. Refer to [12] for such ρ. Multiply (2.2) by $u_{\epsilon\eta}^{s}\rho$ and integrate over $\widetilde{E} \times [0, T]$ to get

$$\begin{aligned} &\int_0^T \int_{\widetilde{E}} (u_{\epsilon\eta})_t u_{\epsilon\eta}^s \rho\,dx\,dt \\ &\qquad + \int_0^T \int_{\widetilde{E}} \bigl[(m+s)u_{\epsilon\eta}^{m+s-1} - \gamma u_{\epsilon\eta}^{r+s}\bigr]\bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \rho\,dx\,dt \\ &\quad \leq \lambda \int_0^T \int_{\widetilde{E}} u_{\epsilon\eta}^{q+s} \rho\,dx\,dt - \int_0^T \int_{\widetilde{E}} u_{\epsilon\eta}^{m+s}\bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} \nabla u_{\epsilon\eta} \nabla\rho\,dx\,dt \\ &\quad \leq 2\lambda M_1^{q+s} T\delta + CM_1^{m+s} \int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}| \rho^{\alpha}\,dx\,dt. \end{aligned} \tag{2.9}$$

By Young's inequality,

$$\begin{aligned} &\int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}| \rho^{\alpha}\,dx\,dt \\ &\quad \leq \int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-1}{2}} \rho^{\alpha}\,dx\,dt \\ &\quad \leq \sigma \int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)\rho\,dx\,dt + c(\sigma) \int_0^T \int_{\widetilde{E}} \rho^{(\alpha-1)p+1}\,dx\,dt \\ &\quad \leq \sigma \int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \rho\,dx\,dt + \sigma\eta C_3 + 2c(\sigma)\delta. \end{aligned} \tag{2.10}$$

Choose $\sigma < \frac{\varepsilon}{2C_3\eta}$ satisfying $\sigma CM_1^{m+s} < \epsilon^{m+s-1}(m+s-\gamma M_1^{r-m+1})$. We have by (2.9) and (2.10)

$$\int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2 \rho\,dx\,dt \leq 2C_5\delta + \sigma\eta C_3 < \varepsilon, \tag{2.11}$$

where

$$C_5 = \frac{\frac{M_1^{s+1}}{s+1} + \lambda M_1^{q+s} T + CM_1^{m+s} c(\sigma)}{\epsilon^{m+s-1}(m+s-\gamma M_1^{r-m+1}) - \sigma CM_1^{m+s}} > 0. \tag{2.12}$$

It follows from (2.11) that

$$\int_0^T \int_E |\nabla u_{\epsilon\eta}|^p\,dx\,dt \leq \int_0^T \int_{\widetilde{E}} \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}|^2\,dx\,dt < \varepsilon. \tag{2.13}$$

By using a similarly procedure with (2.13), we can obtain

$$\int_0^T \int_E \bigl|(u_{\epsilon\eta})_t\bigr|\,dx\,dt < \varepsilon. \tag{2.14}$$

Furthermore, for any fixed $\zeta > 0$, again by (2.13), we have

$$\int_0^T \int_E u_{\epsilon\eta}^{\zeta} |\nabla u_{\epsilon\eta}|^p \, dx\, dt \le M_1^{\zeta} \int_0^T \int_E |\nabla u_{\epsilon\eta}|^p \, dx\, dt \le M_1^{\zeta} \varepsilon. \tag{2.15}$$

Step 3. *Convergence of* $u_{\epsilon\eta}$.

By Dunford-Pettis theorem, we know from the inequalities (2.4), (2.7), (2.8), (2.13), (2.14), and (2.15) that for any $\epsilon \in (0,1)$, there exist a subsequence of $u_{\epsilon\eta}$ (denoted still by $u_{\epsilon\eta}$) and a function u_ϵ with $u_\epsilon - \epsilon \in L^\infty(\Omega_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$, such that as $\eta \to 0$,

$$u_{\epsilon\eta} \to u_\epsilon \quad \text{a.e. in } \Omega_T, \tag{2.16}$$

$$\nabla u_{\epsilon\eta} \rightharpoonup \nabla u_\epsilon \quad \text{in } L^p(\Omega_T), \tag{2.17}$$

$$\frac{\partial u_{\epsilon\eta}}{\partial t} \rightharpoonup \frac{\partial u_\epsilon}{\partial t} \quad \text{in } L^1(\Omega_T), \tag{2.18}$$

$$|\nabla u_{\epsilon\eta}|^p \rightharpoonup |\nabla u_\epsilon|^p \quad \text{in } L^1(\Omega_T), \tag{2.19}$$

$$u_{\epsilon\eta}^{\zeta} |\nabla u_{\epsilon\eta}|^p \rightharpoonup u_\epsilon^{\zeta} |\nabla u_\epsilon|^p \quad \text{in } L^1(\Omega_T), \tag{2.20}$$

where $\rightharpoonup$ denotes weak convergence, and

$$\epsilon \le u_\epsilon \le M_1 \quad \text{a.e. in } \Omega_T. \tag{2.21}$$

For any $\phi(x,t) \in C_0^1(\Omega_T)$, $u_{\epsilon\eta}$ satisfies

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial u_{\epsilon\eta}}{\partial t} \phi \, dx\, dt + \int_0^T \int_\Omega \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{(p-2)/2} \nabla u_{\epsilon\eta} \cdot \nabla\big(u_{\epsilon\eta}^m \phi\big) \, dx\, dt \\ &\quad = \lambda \int_0^T \int_\Omega u_{\epsilon\eta}^q \phi \, dx\, dt + \gamma \int_0^T \int_\Omega u_{\epsilon\eta}^r \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{(p-2)/2} |\nabla u_{\epsilon\eta}|^2 \phi \, dx\, dt. \end{aligned} \tag{2.22}$$

Furthermore,

$$\begin{aligned} &\int_0^T \int_\Omega \big(u_{\epsilon\eta}^m \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{(p-2)/2} \nabla u_{\epsilon\eta} - u_\epsilon^m |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon\big) \cdot \nabla\phi \, dx\, dt \\ &\quad = \int_0^T \int_\Omega u_{\epsilon\eta}^m \big(\big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{(p-2)/2} \nabla u_{\epsilon\eta} - |\nabla u_{\epsilon\eta}|^{p-2} \nabla u_{\epsilon\eta}\big) \cdot \nabla\phi \, dx\, dt \\ &\qquad + \int_0^T \int_\Omega u_{\epsilon\eta}^m \big(|\nabla u_{\epsilon\eta}|^{p-2} \nabla u_{\epsilon\eta} - |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon\big) \cdot \nabla\phi \, dx\, dt \\ &\qquad + \int_0^T \int_\Omega \big(u_{\epsilon\eta}^m - u_\epsilon^m\big) |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla\phi \, dx\, dt \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

Obviously, as $\eta \to 0$, we have $I_1 \to 0$ due to $p > 2$, $I_2 \to 0$ by (2.20), and $I_3 \to 0$ because of (2.16) and the dominated convergence theorem. Consequently,

$$\begin{aligned} &\int_0^T \int_\Omega \big(u_{\epsilon\eta}^m \big(|\nabla u_{\epsilon\eta}|^2 + \eta\big)^{(p-2)/2} \nabla u_{\epsilon\eta} - u_\epsilon^m |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon\big) \cdot \nabla\phi \, dx\, dt \\ &\quad \to 0, \quad \text{as } \eta \to 0. \end{aligned} \tag{2.23}$$

By a similar procedure, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \bigl(u_{\epsilon\eta}^r \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{(p-2)/2} |\nabla u_{\epsilon\eta}|^2 - u_\epsilon^r |\nabla u_\epsilon|^p\bigr)\phi \, dx \, dt \\
&\quad = \int_0^T \int_\Omega \bigl(u_{\epsilon\eta}^r \bigl(|\nabla u_{\epsilon\eta}|^2 + \eta\bigr)^{(p-2)/2} |\nabla u_{\epsilon\eta}|^2 - u_{\epsilon\eta}^r |\nabla u_{\epsilon\eta}|^p\bigr)\phi \, dx \, dt \\
&\qquad + \int_0^T \int_\Omega \bigl(u_{\epsilon\eta}^r |\nabla u_{\epsilon\eta}|^p - u_{\epsilon\eta}^r |\nabla u_\epsilon|^p\bigr)\phi \, dx \, dt \\
&\qquad + \int_0^T \int_\Omega \bigl(u_{\epsilon\eta}^r |\nabla u_\epsilon|^p - u_\epsilon^r |\nabla u_\epsilon|^p\bigr)\phi \, dx \, dt \to 0, \quad \text{as } \eta \to 0.
\end{aligned} \tag{2.24}$$

In summary of (2.16), (2.18), (2.22), (2.23), and (2.24), it is true for any $\phi \in C_0^1(\Omega_T)$ that

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_\epsilon}{\partial t}\phi \, dx \, dt + \int_0^T \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \cdot \nabla\bigl(u_\epsilon^m \phi\bigr) \, dx \, dt \\
&\quad = \lambda \int_0^T \int_\Omega u_\epsilon^q \phi \, dx \, dt + \gamma \int_0^T \int_\Omega u_\epsilon^r |\nabla u_\epsilon|^p \phi \, dx \, dt.
\end{aligned} \tag{2.25}$$

Together with the initial and boundary conditions, we conclude that u_ϵ is a weak solution of problem (2.3). □

Theorem 2.1 *Let $u_0 \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ with $p > 2$. Then the problem* (1.1) *admits a weak solution $u \in \mathbb{E}$.*

Proof From (2.21) and the comparison principle, u_ϵ is bounded and increasing in ϵ. So,

$$u_\epsilon \to u \quad \text{a.e. in } \Omega_T, \text{as } \epsilon \to 0. \tag{2.26}$$

Moreover, the estimate (2.5) is obviously true for $\eta = 0$, namely,

$$\int_{\Omega_T} u_\epsilon^{m+s-1} |\nabla u_\epsilon|^p \, dx \, dt \le C_2 = C_2(M_1, s, \gamma). \tag{2.27}$$

On the other hand, set $\underline{u} := c_1 \mathrm{e}^{-\xi t}\varphi(x)$ for $(x,t) \in \Omega \times [0,\infty)$, with φ defined by (1.6) and $\xi = c_1^{m+p-2}\lambda_1$. Then

$$\underline{u}_t - \underline{u}^m \operatorname{div}\bigl(|\nabla \underline{u}|^{p-2}\nabla \underline{u}\bigr) - \lambda \underline{u}^q - \gamma \underline{u}^r |\nabla \underline{u}|^p \le \underline{u}_t - \underline{u}^m \operatorname{div}\bigl(|\nabla \underline{u}|^{p-2}\nabla \underline{u}\bigr) \le 0. \tag{2.28}$$

For any $K \subset\subset \Omega$, let c_1 be small such that $c_1\varphi(x) < u_0(x)$ on K. By the comparison principle,

$$u_\epsilon \ge \underline{u} \quad \text{in } K \times [0,\infty).$$

Since $\varphi \in C^1(\overline{\Omega})$ (see [13]) and $\varphi > 0$ in Ω, there exists c_K such that

$$u_\epsilon \ge c_K \quad \text{in } K \times (0,\infty). \tag{2.29}$$

Set $\Omega_n = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{n}\}$. Then $\Omega_n \subset\subset \Omega$, and $\Omega_n \to \Omega$ as $n \to \infty$. Together with (2.27) and (2.29), we have

$$\int_0^T \int_{\Omega_1} |\nabla u_\epsilon|^p \, dx \, dt \leq C_5 = C_5(M_1, s, \gamma). \tag{2.30}$$

Thus, there exists a subsequence $\epsilon = \epsilon_{1k} \to 0$ such that

$$\nabla u_{\epsilon_{1k}} \rightharpoonup \nabla u \quad \text{in } L^p\big(\Omega_1 \times [0,T]\big).$$

It is easy to see the inequality (2.30) is still valid for $u_{\epsilon_{1k}}$ when Ω_1 is replaced by Ω_2. So, there exists a subsequence of ϵ_{2k} such that

$$\nabla u_{\epsilon_{2k}} \rightharpoonup \nabla u \quad \text{in } L^p\big(\Omega_2 \times [0,T]\big).$$

By induction, we obtain a subsequence ϵ_{nk} such that

$$\int_0^T \int_{\Omega_n} |\nabla u_{\epsilon_{nk}}|^p \, dx \, dt \leq C_5 = C_5(M_1, s, \gamma)$$

and

$$\nabla u_{\epsilon_{nk}} \rightharpoonup \nabla u \quad \text{in } L^p\big(\Omega_n \times [0,T]\big).$$

Similar to the above procedure, we see that the estimates (2.8)-(2.15) hold as well for $u_{\epsilon_{nk}}$ with Ω_n instead of Ω. Consequently,

$$\frac{\partial u_{\epsilon_{nk}}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^1\big(\Omega_n \times [0,T)\big), \tag{2.31}$$

$$|\nabla u_{\epsilon_{nk}}|^p \rightharpoonup |\nabla u|^p \quad \text{in } L^1\big(\Omega_n \times [0,T)\big), \tag{2.32}$$

$$u^{\zeta}_{\epsilon_{nk}} |\nabla u_{\epsilon_{nk}}|^p \rightharpoonup u^{\zeta} |\nabla u|^p \quad \text{in } L^1\big(\Omega_n \times [0,T)\big). \tag{2.33}$$

For any $\phi(x) \in C_0^1(\Omega_T)$, there exists $K \subset\subset \Omega$ such that $\phi = 0$ in $\Omega \setminus K$. For such K, there exists $n \in \mathbb{N}$ such that $K \subset\subset \Omega_n$. By (2.33) and the dominated convergence theorem, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \big(u^r_{\epsilon_{nk}} |\nabla u_\epsilon|^p - u^r |\nabla u|^p\big)\phi \, dx \, dt \\
&\quad = \int_0^T \int_{\Omega_n} \big(u^r_{\epsilon_{nk}} |\nabla u_{\epsilon_{nk}}|^p - u^r |\nabla u|^p\big)\phi \, dx \, dt \\
&\quad = \int_0^T \int_{\Omega_n} \big(u^r_{\epsilon_{nk}} |\nabla u_{\epsilon_{nk}}|^p - u^r_{\epsilon_{nk}} |\nabla u|^p\big)\phi \, dx \, dt \\
&\qquad + \int_0^T \int_{\Omega_n} \big(u^r_{\epsilon_{nk}} |\nabla u|^p - u^r |\nabla u|^p\big)\phi \, dx \, dt \to 0, \quad \text{as } \epsilon_{nk} \to 0.
\end{aligned} \tag{2.34}$$

Similarly,

$$\int_0^T \int_\Omega \big(u^{m-1}_{\epsilon_{nk}} |\nabla u_{\epsilon_{nk}}|^p - u^{m-1} |\nabla u|^p\big)\phi \, dx \, dt \to 0, \quad \text{as } \epsilon_{nk} \to 0 \tag{2.35}$$

and

$$\int_0^T \int_\Omega \left(u_{\epsilon_{nk}}^m |\nabla u_{\epsilon_{nk}}|^{p-2}\nabla u_{\epsilon_{nk}} - u^m|\nabla u|^{p-2}\nabla u\right)\cdot\nabla\phi\,dx\,dt \to 0, \quad \text{as } \epsilon_{nk}\to 0. \tag{2.36}$$

Notice that $u_{\epsilon_{nk}}$ satisfies

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u_{\epsilon_{nk}}}{\partial t}\phi\,dx\,dt + \int_0^T\int_\Omega u_{\epsilon_{nk}}^m|\nabla u_{\epsilon_{nk}}|^{p-2}\nabla u_{\epsilon_{nk}}\cdot\nabla\phi\,dx\,dt\\ &\quad= \lambda\int_0^T\int_\Omega u_{\epsilon_{nk}}^q\phi\,dx\,dt + \gamma\int_0^T\int_\Omega u_{\epsilon_{nk}}^r|\nabla u_{\epsilon_{nk}}|^p\phi\,dx\,dt\\ &\qquad - m\int_0^T\int_\Omega u_{\epsilon_{nk}}^{m-1}|\nabla u_{\epsilon_{nk}}|^p\phi\,dx\,dt.\end{aligned} \tag{2.37}$$

Letting $\epsilon_{nk}\to 0$ in (2.37), by (2.26)-(2.36), we conclude for the limit function u that

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u}{\partial t}\phi\,dx\,dt + \int_0^T\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\left(u^m\phi\right)dx\,dt\\ &\quad= \lambda\int_0^T\int_\Omega u^q\phi\,dx\,dt + \gamma\int_0^T\int_\Omega u^r|\nabla u|^p\phi\,dx\,dt.\end{aligned} \tag{2.38}$$

In addition, u satisfies the initial and boundary conditions of (1.1) (in the sense of trace). This proves that u is a weak solution of (1.1). □

We know by the proof of Theorem 2.1 that if $\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty$, then $\int_0^T\int_K|\nabla u|^p\,dx\,dt<\infty$ for any $K\subset\subset\Omega$, namely, $u\in L^\infty(\Omega_T)$ implies $\nabla u\in L^p_{\mathrm{loc}}(\Omega_T)$. Let T^* be the maximal existence time of the solution u. We get the following proposition immediately.

Proposition 2.1 *If $T^*<\infty$, then $\lim_{t\to T^*}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$.*

Remark 1 It is mentioned that the uniqueness of such weak solutions to the problem (1.1) cannot be ensured. In the rest of the paper, the solution of (1.1) always means the maximal solution of (1.1), for which the comparison principle is valid.

3 Global existence and nonexistence of solutions

We discuss the existence and nonexistence of global solutions to the problem (1.1) in this section, via a complete classification on the parameters m, p, q as follows:

(a) $q<p+m-1$, (b) $q=p+m-1$, (c) $q>p+m-1$.

Correspondingly, we have three theorems for them.

Theorem 3.1 *Suppose $q<p+m-1$.*

(i) *If $r<\frac{pm-q}{p-1}$, then all solutions are global and bounded.*
(ii) *If $r>\frac{pm-q}{p-1}$, then the solutions blow up for a large domain or large initial data, and they are global for a small domain with small initial data.*
(iii) *If $r=\frac{pm-q}{p-1}$, the solutions blow up for a large domain and are global for a small domain.*

We will prove Theorem 3.1 in five lemmas.

Lemma 3.1 *Suppose $q < p + m - 1$ with $r < \frac{pm-q}{p-1}$. Then all positive solutions of* (1.1) *are global and bounded.*

Proof Let $\psi \in C^{1+\beta}(\overline{\Omega})$ solve $-\operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi) = 1$ in Ω with $\psi|_{\partial\Omega} = 0$, $\beta \in (0,1)$ [13]. The condition $q < p + m - 1$ with $r < \frac{pm-q}{p-1}$ implies $m - r > \frac{q-m}{p-1}$. Choose $\alpha \in (0,1)$ such that $\frac{q-m}{p-1} < \alpha < m - r$. Let $w = M + M^{\alpha}\psi(x)$, with $M \geq 1$ to be determined. Then

$$\begin{aligned} &w_t - w^m \operatorname{div}\bigl(|\nabla w|^{p-2}\nabla w\bigr) - \lambda w^q - \gamma w^r|\nabla w|^p \\ &\quad = w^m\bigl[M^{\alpha(p-1)} - \lambda\bigl(M + M^{\alpha}\psi\bigr)^{q-m} - \gamma\bigl(M + M^{\alpha}\psi\bigr)^{r-m}M^{\alpha p}|\nabla\psi|^p\bigr] \\ &\quad \geq w^m\bigl[M^{\alpha(p-1)} - c\bigl(M^{q-m} + M^{\alpha p+r-m}\bigr)\bigr] \end{aligned}$$

with some $c > 0$. Due to the choice of α, we have

$$q - m < \alpha(p-1), \qquad \alpha p + r - m < \alpha(p-1).$$

Now let $M > 1$ be large enough such that $u_0(x) + \epsilon < M$ in Ω and $M^{\alpha(p-1)} - c(M^{q-m} + M^{\alpha p+r-m}) \geq 0$. Therefore, the comparison principle yields $u_\epsilon \leq w$ in $\Omega \times (0,\infty)$. Letting $\epsilon \to 0$, we obtain $u \leq w$ in $\Omega \times (0,\infty)$. □

Lemma 3.2 *Suppose $q < p + m - 1$ with $r > \frac{pm-q}{p-1}$. If Ω contains a ball with radius R large enough, then all solutions of* (1.1) *blow up in finite time.*

Proof Suppose for contradiction that for any $R > 0$ such Ω admits a global solution u to (1.1) with suitable initial data u_0. Without loss of generality, assume $\overline{B}_R(0) \subset \Omega$. We first show that for any fixed $M > 1$, there exists $t_0 > 0$ such that $u \geq M$ in $B_1 \times (t_0,\infty)$.

Let $R' > R$ be such that $\overline{B}_{R'}(0) \subset \Omega$. Set $z := c\phi_{R'}$, with $c \in (0, \lambda_{R'}^{-\frac{1}{p+m-1-q}})$ small to be determined, $\lambda_{R'}$ and $\phi_{R'}$ the first eigenvalue and eigenfunction in the domain $B_{R'}$, normalized by $\phi_{R'} > 0$ in $B_{R'}$, $\|\phi_{R'}\|_\infty = 1$. Then

$$z_t - z^m \operatorname{div}\bigl(|\nabla z|^{p-2}\nabla z\bigr) - \lambda z^q - \gamma z^r|\nabla z|^p \leq \lambda_{R'}c^{p+m-1}\phi_{R'}^{p+m-1} - c^q\phi_{R'}^q \leq 0$$

in $B_{R'}(0) \times (0,\infty)$. Since $u_0 > 0$ in Ω, $u_0 \in C(\overline{\Omega})$, choose c small enough such that $u_0 + \epsilon > c\phi_{R'}$ in $B_{R'}$. By the comparison principle, $u \geq z = c\phi_{R'}$ in $B_{R'}$. Furthermore, there exists $c_1 > 0$ small such that $u \geq c_1$ in $B_R(0) \times (0,\infty)$, due to $\phi_{R'} > 0$ in $B_{R'}$ and $\phi_{R'} \in C(\overline{B}_{R'})$ with $R < R'$. Define

$$v := y(t)\bigl(\phi_R(x) + \delta\bigr), \quad (x,t) \in B_R(0) \times (0,\infty),$$

where $y(t)$ is a nondecreasing positive function on $[0,\infty)$ to be determined with suitable $\delta > 0$. A direct computation yields

$$\begin{aligned} &v_t - v^m \operatorname{div}\bigl(|\nabla v|^{p-2}\nabla v\bigr) - \lambda v^q - \gamma v^r|\nabla v|^p \\ &\quad \leq y'\bigl(\phi_R(x) + \delta\bigr) + \lambda_R y^{p+m-1}(\phi_R + \delta)^m\phi_R^{p-1} - y^q(\phi_R + \delta)^q \\ &\quad =: L_1 + L_2 - L_3 \quad \text{in } B_R(0) \times (0,\infty), \end{aligned}$$

where

$$\frac{L_1}{L_3} = \frac{y'}{y^q}(\phi_R + \delta)^{1-q} \le \frac{y'}{y^q}\delta^{1-q}$$

and

$$\frac{L_2}{L_3} = \lambda_R y^{p+m-1-q}(\phi_R + \delta)^{m-q}\phi_R^{p-1} \le \lambda_R(2y)^{p+m-1-q}.$$

Let $y_R = \frac{1}{2(2\lambda_R)^{\frac{1}{p+m-1-q}}}$, and choose y_0 and δ such that $y_0 \le \frac{c_1}{2}$, $\delta \le \min\{\frac{c_1}{y_R}, 1\}$, and $y' \le \frac{\delta^{q-1}}{2}y^q$, $y(0) = y_0$, and $y(t) \nearrow y_R$ as $t \to \infty$. We have $v \le u_\epsilon$ on $\{t = 0\}$ and $\partial B_R(0)$. Moreover, $\frac{L_1}{L_3} \le \frac{1}{2}$, $\frac{L_2}{L_3} \le \frac{1}{2}$. By comparison, $u_\epsilon \ge v$, and hence $u \ge v$ in $B_R(0) \times (0, \infty)$.

For any fixed $M > 1$, choose $R > 1$ large such that $\frac{1}{2}y_R\phi_R \ge M$ in B_1, and hence $y(t_0) \ge \frac{1}{2}y_R\phi_R$ for some $t_0 > 0$. Consequently,

$$u \ge v \ge \frac{1}{2}y_R\phi_R \ge M \quad \text{in } B_1 \times (t_0, \infty). \tag{3.1}$$

For $n \in \mathbb{N}$ with $n > \max\{p, \frac{pq}{r(p-1)-pm+q}\}$, choose $0 \le \theta \in C_0^1(\overline{B}_1)$ such that $\int_{B_1}\theta^n\,dx = 1$. Multiply (1.1) by θ^n and integrate over B_1,

$$\begin{aligned}\frac{d}{dt}\int_{B_1} u\theta^n\,dx &= -m\int_{B_1} u^{m-1}\theta^n|\nabla u|^p\,dx - n\int_{B_1} u^m\theta^{n-1}|\nabla u|^{p-1}\nabla u\cdot\nabla\theta\,dx \\ &\quad + \lambda\int_{B_1} u^q\theta^n\,dx + \gamma\int_{B_1} u^r\theta^n|\nabla u|^p\,dx \\ &=: -I_1 - I_2 + I_3 + I_4.\end{aligned} \tag{3.2}$$

Since $r > \frac{pm-q}{p-1} > \frac{pm-p-m+1}{p-1} = m-1$, by (3.1), we have

$$I_1 \le mM^{-(r+1-m)}\int_{B_1} u^r\theta^n|\nabla u|^p\,dx \le \frac{1}{2}I_4, \quad t > t_0, \tag{3.3}$$

provided $M \ge (\frac{2m}{\gamma})^{\frac{1}{r+1-m}}$. By Young's inequality,

$$\begin{aligned}I_2 &\le n\frac{\gamma}{2n}\int_{B_1} u^r\theta^n|\nabla u|^p\,dx + n\frac{2n}{\gamma}\int_{B_1} u^{pm-r(p-1)}\theta^{n-p}|\nabla\theta|^p\,dx \\ &\le \frac{1}{2}I_4 + \frac{2n^2}{\gamma}\int_{B_1} u^{pm-r(p-1)}\theta^{n-p}|\nabla\theta|^p\,dx.\end{aligned} \tag{3.4}$$

If $r \ge \frac{pm}{p-1}$, by (3.1), (3.4), with $M > 1$, we have

$$\begin{aligned}I_2 &\le \frac{1}{2}I_4 + \frac{2n^2}{\gamma}M^{pm-r(p-1)}\int_{B_1}\theta^{n-p}|\nabla\theta|^p\,dx \\ &\le \frac{1}{2}I_4 + \frac{2n^2}{\gamma}\int_{B_1}\theta^{n-p}|\nabla\theta|^p\,dx, \quad t > t_0.\end{aligned} \tag{3.5}$$

In the case $r < \frac{pm}{p-1}$, due to $r > \frac{pm-q}{p-1}$ implying $\beta := \frac{q}{pm-r(p-1)} > 1$, again by Young's inequality,

$$\int_{B_1} u^{pm-r(p-1)}\theta^{n-p}|\nabla\theta|^p\,dx \\ \leq \frac{\lambda\gamma}{4n^2}\int_{B_1} u^q\theta^n\,dx + \frac{1}{\beta'}\left(\frac{4n^2}{\lambda\gamma\beta}\right)^{\frac{\beta'}{\beta}}\int_{B_1}\theta^{n-p\beta'}|\nabla\theta|^{p\beta'}\,dx \tag{3.6}$$

with $\beta' = \frac{\beta}{\beta-1} = \frac{q}{r(p-1)-pm+q}$.

It follows from (3.4)-(3.6) that

$$I_2 \leq \frac{1}{2}I_3 + \frac{1}{2}I_4 + \frac{\lambda(r(p-1)-pm+q)}{2(pm-r(p-1))}\cdot\left(\frac{4n^2(pm-r(p-1))}{q\gamma\lambda}\right)^{\frac{q}{r(p-1)-pm+q}} \\ \cdot\int_{B_1}\theta^{n-\frac{pq}{r(p-1)-pm+q}}|\nabla\theta|^{\frac{pq}{r(p-1)-pm+q}}\,dx. \tag{3.7}$$

Combine (3.2), (3.3), and (3.7) to get

$$\frac{d}{dt}\int_{B_1} u\theta^n\,dx \geq \frac{\lambda}{2}\int_{B_1} u^q\theta^n\,dx - C \geq \frac{\lambda}{2}\left(\int_{B_1} u\theta^n\,dx\right)^q - C \tag{3.8}$$

with

$$C = \begin{cases} \frac{\lambda(r(p-1)-pm+q)}{2(pm-r(p-1))}\cdot\left(\frac{4n^2(pm-r(p-1))}{q\gamma\lambda}\right)^{\frac{q}{r(p-1)-pm+q}} \\ \quad\cdot\int_{B_1}\theta^{n-\frac{pq}{r(p-1)-pm+q}}|\nabla\theta|^{\frac{pq}{r(p-1)-pm+q}}\,dx, & r < \frac{pm}{p-1}, \\ \frac{2n^2}{\gamma}\int_{B_1}\theta^{n-p}|\nabla\theta|^p\,dx, & r \geq \frac{pm}{p-1}. \end{cases}$$

Choose $M \geq \max\{(\frac{2m}{\gamma})^{\frac{1}{r+1-m}}, (\frac{2C}{\lambda})^{\frac{1}{q}}\}$. By (3.1), we have

$$\frac{\lambda}{2}\left(\int_{B_1} u(\cdot,t_0)\theta^n\,dx\right)^q \geq \frac{\lambda}{2}\left(M\int_{B_1}\theta^n\,dx\right)^q = \frac{\lambda}{2}M^q > C. \tag{3.9}$$

We conclude from (3.8) and (3.9) that $\int_{B_1} u\theta^n\,dx$ must blow up in finite time. □

Lemma 3.3 *Suppose $q \leq p+m-1$ with $r > \frac{pm-q}{p-1}$. Let $u_0 = b\omega$ with $\omega \in W_0^{1,p}(\Omega)\cap C(\overline{\Omega})$ and $\omega > 0$ in Ω. Then there is $b_0 > 0$ such that the solution u of* (1.1) *blows up in finite time provided $b \geq b_0$.*

Proof Suppose for contradiction that u is global with $u_0 \geq b_0\omega$ for any $b_0 > 0$. Pick n and M defined in the proof of Lemma 3.2. Let Ω contain a ball with radius $2R$, without loss generality, $B_{2R}(0) \subset \Omega$. It suffices to show there exists $t_0 > 0$ such that $u \geq M$ in $B_{R/2} \times (t_0,\infty)$.

For $\sigma > \max\{\frac{p}{p-1}, \frac{p(r+1-m)}{(p-1)r+q-pm}\}$, set $w(x) := \delta e^{-z}$ for $x \in B_R(0)$ with $z := \frac{|x|^\sigma}{R^2-|x|^2}$ and $\delta > 0$ to be determined. For $x \in B_R(0)$, we have

$$w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) + \lambda w^q + \gamma w^r|\nabla w|^p \\ = \delta^{p+m-1}e^{-(p+m-1)z}\frac{|x|^{\sigma p-\sigma-p}(\sigma R^2-(\sigma-2)|x|^2)^{p-2}}{(R^2-|x|^2)^{2p}}\big\{(p-1)|x|^\sigma\big(\sigma R^2-(\sigma-2)|x|^2\big)^2$$

$$
\begin{aligned}
&- n\big(R^2-|x|^2\big)^2\big(\sigma R^2-(\sigma-2)|x|^2\big) - 4|x|^2\big(R^2-|x|^2\big)\big(\sigma R^2-(\sigma-2)|x|^2\big)\\
&- \sigma(\sigma p-\sigma-p)\big(R^2-|x|^2\big)^3 - (2p-4)|x|^2\big(R^2-|x|^2\big)^2\big\}\\
&+ \lambda\delta^q e^{-qz} + \gamma\delta^{r+p}e^{-(r+p)z}\frac{|x|^{p(\sigma-1)}(\sigma R^2-(\sigma-2)|x|^2)^p}{(R^2-|x|^2)^{2p}}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
f(s) := {}&(p-1)s^\sigma\big(\sigma R^2-(\sigma-2)s^2\big)^2 - n\big(R^2-s^2\big)^2\big(\sigma R^2-(\sigma-2)s^2\big)\\
&- 4s^2\big(R^2-s^2\big)\big(\sigma R^2-(\sigma-2)s^2\big)2\\
&- \sigma(\sigma p-\sigma-p)\big(R^2-s^2\big)^3 - (2p-4)s^2\big(R^2-s^2\big)^2, \quad s\in[0,R].
\end{aligned}
$$

Then there exists $K>0$ such that $|f(s)|\le K$ for $s\in[0,R]$. Due to $f(R)=(p-1)2^pR^{\sigma+p}>0$, there exists $c_2\in(\frac{1}{2},1)$ such that $f(s)\ge 0$ in $[c_2R,R]$, and hence

$$
w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) + \lambda w^q + \gamma w^r|\nabla w|^p \ge 0, \quad |x|\in[c_2R,R].
$$

For the above σ with $r>\frac{pm-q}{p-1}$, set

$$
l\in\left(\frac{p+m-1-q}{\sigma p-\sigma-p}, \frac{r+1-m}{\sigma}\right).
$$

Then there exists $\delta_0\ge 1$ such that $c_1=\delta^{-l}\in(0,\frac{1}{2})$ for all $\delta\ge\delta_0$. Hence, there exists $\delta_1\ge\delta_0$ such that

$$
\begin{aligned}
&w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) + \lambda w^q + \gamma w^r|\nabla w|^p\\
&\ge -\delta^{p+m-1}\frac{c_1^{\sigma p-\sigma-p}R^{\sigma p-\sigma-5p}K}{(1-c_1^2)^{2p}} + \lambda\delta^q e^{-\frac{qc_1^\sigma R^{\sigma-2}}{1-c_1^2}}\\
&\ge -\delta^{p+m-1-l(\sigma p-\sigma-p)}\frac{R^{\sigma p-\sigma-5p}K}{(\frac{3}{4})^{2p}} + \lambda\delta^q e^{-\frac{q}{3}(\frac{R}{2})^{\sigma-2}}\\
&\ge 0, \quad |x|\in[0,c_1R], \delta\ge\delta_1.
\end{aligned}
$$

Furthermore, there is $\delta_2\ge\delta_1$ such that

$$
\begin{aligned}
&w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) + \lambda w^q + \gamma w^r|\nabla w|^p\\
&\ge \frac{|x|^{\sigma p-\sigma-p}(\sigma R^2-(\sigma-2)|x|^2)^{p-2}}{(R^2-|x|^2)^{2p}}\Big\{-\delta^{p+m-1}K + 4\gamma\delta^{r+p}c_1^\sigma R^{\sigma+4}e^{-(r+p)\frac{(c_2R)^\sigma}{(1-c_2^2)R^2}}\Big\}\\
&\ge \frac{|x|^{\sigma p-\sigma-p}(\sigma R^2-(\sigma-2)|x|^2)^{p-2}}{(R^2-|x|^2)^{2p}}\Big\{-\delta^{p+m-1}K + 4\gamma\delta^{r+p-l\sigma}R^{\sigma+4}e^{-(r+p)\frac{(c_2R)^\sigma}{(1-c_2^2)R^2}}\Big\}\\
&\ge 0, \quad |x|\in[c_1R,c_2R], \delta\ge\delta_2.
\end{aligned}
$$

Finally, choose $\delta\ge\delta_2$ such that $w(x)\ge M$ in $B_{R/2}$. In summary, we obtain

$$
w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) + \lambda w^q + \gamma w^r|\nabla w|^p \ge 0 \quad \text{in } B_R(0),
$$

and $w(x)\ge M$ in $B_{R/2}$.

Let $b_0 > 0$ be large such that $b_0\omega \geq w$ in $B_R(0)$. By the comparison principle, we have $u_\epsilon \geq w$ in $B_R(0) \times (0,\infty)$. Hence $u(x,t) \geq w(x) \geq M$ in $B_{R/2} \times (0,\infty)$.

The rest of the lemma can be proved by the same arguments as those for Lemma 3.2. □

Lemma 3.4 *Suppose $q < p + m - 1$ with $r \geq \frac{pm-q}{p-1}$. Then all positive solutions of* (1.1) *are bounded, if Ω is contained in a ball with radius R small.*

Proof Without loss of generality, assume $\Omega \subset \{x \in \mathbb{R}^N | R < x_1 < 2R\}$. Set $z := Kx_1^\kappa$ for $(x,t) \in \Omega \times (0,\infty)$, with $K > 0$ and $0 < \kappa < 1$ to be determined. A simple calculation yields

$$\begin{aligned} & z_t - z^m \operatorname{div}\bigl(|\nabla z|^{p-2}\nabla z\bigr) - \lambda z^q - \gamma z^r |\nabla z|^p \\ &\quad = (p-1)(1-\kappa)K^{p+m-1}\kappa^{p-1}x_1^{(p-1)(\kappa-1)-1+m\kappa} - \lambda K^q x_1^{q\kappa} - \gamma K^{r+p}\kappa^p x_1^{p(\kappa-1)+r\kappa} \\ &\quad =: L_1 - L_2 - L_3. \end{aligned}$$

Noticing $r \geq \frac{pm-q}{p-1} > m-1$, we have

$$\begin{aligned} \frac{2L_3}{L_1} &= \frac{2\gamma\kappa}{(p-1)(1-\kappa)}K^{r-m+1}x_1^{\kappa(r-m+1)} \\ &\leq \frac{2\gamma\kappa}{(p-1)(1-\kappa)}K^{r-m+1}(2R)^{\kappa(r-m+1)} \\ &\leq 1, \end{aligned}$$

provided $K = (\frac{(p-1)(1-\kappa)}{2\gamma\kappa})^{\frac{1}{r-m+1}}(2R)^{-\kappa}$.

Choose $\kappa = \min\{\frac{p-1}{4\gamma M^{r-m+1}+p-1}, \frac{p}{p+m-1-q}\}$, with $M := \|u_0\|_{L^\infty(\Omega)} + 1$. Then there exists R small such that

$$K \geq \left(\frac{2}{(p-1)(1-\kappa)\kappa^{p-1}}\right)^{\frac{1}{p+m-1-q}}(2R)^{\frac{p}{p+m-1-q}-\kappa},$$

and hence

$$\begin{aligned} \frac{2L_2}{L_1} &= \frac{2}{(p-1)(1-\kappa)\kappa^{p-1}}K^{q-(p+m-1)}x_1^{q\kappa-m\kappa-(p-1)(\kappa-1)+1} \\ &\leq \frac{2}{(p-1)(1-\kappa)\kappa^{p-1}}K^{q-(p+m-1)}(2R)^{p-\kappa(p+m-1-q)} \\ &\leq 1. \end{aligned}$$

In addition,

$$z = Kx_1^\kappa \geq \left(\frac{(p-1)(1-\kappa)}{2\gamma\kappa}\right)^{\frac{1}{r-m+1}}(2R)^{-\kappa}R^\kappa \geq M$$

on the parabolic boundary of $\Omega \times (0,\infty)$ due to the choice of κ. We conclude that z is a time-independent supersolution of (1.1). □

Lemma 3.5 *Suppose $q < p + m - 1$ with $r = \frac{pm-q}{p-1}$. Then all solutions of* (1.1) *blow up in finite time provided Ω large enough.*

Proof The lemma will be proved in three steps.

Step 1. Set $u = h(v)$, and substitute into (1.1),

$$\begin{aligned} h'(v)v_t &= \left[h^m(v)h'(v)^{p-2}\right]\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + \lambda h^q(v) \\ &\quad + \left[(p-1)h(v)^m h'(v)^{p-2}h''(v) + \gamma h^r(v)h'(v)^p\right]|\nabla v|^p. \end{aligned} \tag{3.10}$$

Here $h \in C^0([0,+\infty)) \cup C^2((0,+\infty))$ satisfies

$$h'(s) = \mathrm{e}^{-\frac{\gamma}{\beta(p-1)}h^\beta(s)}, \qquad h(0) = 0$$

with $\beta = r - m + 1 = \frac{p+m-1-q}{p-1} \in (0,1]$, and hence $(p-1)h(v)^m h''(v) + \gamma h^r(v)h'(v)^2 = 0$. Thus,

$$v_t = h^m(v)h'(v)^{p-2}\left(\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + \gamma h^{(p-1)(1-\beta)}(v)\mathrm{e}^{\frac{\lambda}{\beta}h^\beta(s)}\right).$$

Set $g(v) = h^{(p-1)(1-\beta)}(v)\mathrm{e}^{\frac{\lambda}{\beta}h^\beta(s)}$. Similarly to the proof of Theorem 2.5 in [5], we can find constants $v_0 \geq 2$ and $c_0 > 0$, only depending on β, λ, and γ, such that

$$g(v) \geq c_0(v+1)^{p-1} \quad \text{for } v \geq v_0. \tag{3.11}$$

So, v is a supersolution of

$$w_t = h^m(w)h'(w)^{p-2}\left(\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) + c_0 w^{p-1}\right) \quad \text{in } \Omega \times (t_0, T)$$

whenever $v \geq v_0$.

Step 2. Let w solve

$$\begin{cases} w_t = f(w)(\operatorname{div}(|\nabla w|^{p-2}\nabla w) + c_0 w^{p-1}), & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{3.12}$$

with $f(w) = h^m(w)h'(w)^{p-2}$. We claim that w blows up in finite time for any initial data $w_0 > 0$ provided Ω large such that $\lambda_1 < c_0$.

We prove the claim by using the so-called 'concavity' method. Define

$$\mathscr{E}(w) = \frac{1}{p}\int_\Omega |\nabla w|^p\,dx - \frac{c_0}{p}\int_\Omega w^p\,dx, \qquad \mathscr{H}(w) = \int_\Omega \Phi(w)\,dx,$$

where $\Phi(w) = \int_0^w \frac{\varrho}{f(\varrho)}\,d\varrho$ with $f(\varrho) = h^m(\varrho)h'(\varrho)^{p-2}$. Such Φ is well defined. In fact, since $h(0) = 0$, there exists $\varrho_1 > 0$ such that $h'(\varrho) = \mathrm{e}^{-\frac{\gamma}{\beta(p-1)}h^\beta(\varrho)} \geq \frac{1}{2}$ for $\varrho \in [0,\varrho_1]$. Set $h_1(\varrho) = (\alpha\ln(\varrho+1))^{\frac{1}{\beta}}$, with $0 < \alpha < \frac{\beta(p-1)}{\gamma}$. Then

$$\frac{h_1'(\varrho)}{\mathrm{e}^{-\frac{\gamma}{\beta(p-1)}h_1^\beta(\varrho)}} = \frac{\frac{1}{\beta}(\alpha\ln(\varrho+1))^{\frac{1-\beta}{\beta}}\frac{1}{\varrho+1}}{(\varrho+1)^{-\frac{\alpha\gamma}{\beta(p-1)}}} = \frac{1}{\beta}\left(\alpha\ln(\varrho+1)\right)^{\frac{1-\beta}{\beta}}(\varrho+1)^{\frac{\alpha\gamma}{\beta(p-1)}-1}.$$

Choose $\varrho_2 = \min\{\varrho_1, e^{\frac{\beta^{\frac{\beta}{1-\beta}}}{\alpha}-1}\}$. We have $(\alpha\ln(\varrho+1))^{\frac{1-\beta}{\beta}} \le \beta$ for $0 \le \varrho \le \varrho_2$. Since $0 < \alpha < \frac{\beta(p-1)}{\gamma}$ implies $(\varrho+1)^{\frac{\alpha\gamma}{\beta(p-1)}-1} \le 1$ for $0 \le \varrho \le \varrho_2$, we have

$$\frac{h_1'(\varrho)}{e^{-\frac{\gamma}{\beta(p-1)}h_1^{\beta}(\varrho)}} \le 1 \quad \text{for } \varrho \in [0, \varrho_2].$$

Hence, by the comparison principle with $h(0) = h_1(0) = 0$, we obtain

$$h(\varrho) \ge h_1(\varrho) \quad \text{for } \varrho \in [0, \varrho_2].$$

Consequently,

$$\int_0^{\varrho_2} \frac{\varrho}{f(\varrho)}\,d\varrho \le \int_0^{\varrho_2} \frac{\varrho}{2h_1^m(\varrho)}\,d\varrho \le \int_0^{\varrho_2} \frac{\varrho}{2\alpha^{\frac{m}{\beta}}\ln^{\frac{m}{\beta}}(\varrho+1)}\,d\varrho < \infty,$$

and so, $\Phi(w) = \int_0^w \frac{\varrho}{f(\varrho)}\,d\varrho$ is well defined.

A simple calculation yields

$$\mathscr{H}'(w) = \int_\Omega \frac{w}{f(w)} w_t\,dx = -p\mathscr{E}(w)$$

and

$$\mathscr{H}''(w) = -p\mathscr{E}'(w) = p\int_\Omega \frac{w_t^2}{f(w)}\,dx.$$

Noticing f is nondecreasing, we have by Hölder's inequality

$$\begin{aligned}
\bigl(\mathscr{H}'(w)\bigr)^2 &= \left(\int_\Omega \frac{w}{\sqrt{f(w)}}\frac{w_t}{\sqrt{f(w)}}\,dx\right)^2 \le \left(\int_\Omega \frac{w^2}{f(w)}\,dx\right)\left(\int_\Omega \frac{w_t^2}{f(w)}\,dx\right) \\
&\le \frac{2}{p}\left(\int_\Omega\int_0^w \frac{\varrho}{f(\varrho)}\,d\varrho\,dx\right)\left(p\int_\Omega \frac{w_t^2}{f(w)}\,dx\right) \\
&= \frac{2}{p}\mathscr{H}(w)\mathscr{H}''(w),
\end{aligned}$$

which implies

$$\frac{d^2}{dt^2}\mathscr{H}^{1-\frac{p}{2}}(t) \le 0.$$

It follows that there exists $T < \infty$ such that if

$$\mathscr{E}(w_0) = \frac{1}{p}\int_\Omega |\nabla w_0|^p\,dx - \frac{c_0}{p}\int_\Omega w_0^p\,dx < 0,$$

then

$$\limsup_{t\to T} \mathscr{H}(t) = \infty,$$

which implies

$$\limsup_{t\to T}\left\|\Phi\big(w(\cdot,t)\big)\right\|_{L^\infty(\Omega)}=\infty.$$

Thus, we have

$$\limsup_{t\to T}\left\|w(\cdot,t)\right\|_{L^\infty(\Omega)}=\infty. \tag{3.13}$$

Let $\varphi>0$ in Ω be the first eigenfunction of (1.6). We have, for any $k>0$,

$$\mathscr{E}(k\psi)=\frac{1}{p}\int_\Omega\left|\nabla(k\psi)\right|^p\,dx-\frac{c_0}{p}\int_\Omega(k\psi)^p\,dx=\frac{\lambda_1-c_0}{p}\int_\Omega(k\psi)^p\,dx<0,$$

provided Ω large such that the first eigenvalue $\lambda_1<c_0$. Choose k small enough such that $w_0\geq k\psi$. Then w blows up in finite time by comparison.

Step 3. By the assumption, Ω contains a closed ball with radius R. Let $\overline{B}_R(0)\subset\Omega$, without loss of generality. Suppose there is initial value $u_0>0$ such that the solution u is global in time. Then we can show in a way similar to the proof of Lemma 3.2 that there is $t_0>0$ such that $u\geq M>f(v_0)$ for $(x,t)\in B_1\times(t_0,\infty)$, and hence $v\geq f^{-1}(u)\geq f^{-1}(M)\geq v_0$ in $B_1\times(t_0,\infty)$. Therefore v is a non-global supersolution of (3.12) in $B_1\times(t_0,\infty)$ by Step 1. $\square$

Theorem 3.2 *Suppose that $q>p+m-1$. Then the solutions of* (1.1) *blow up for large initial data and are global for small initial data.*

Proof Let ψ solve $-\operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi)=1$ in Ω with $\psi|_{\partial\Omega}=0$. Since $q>p+m-1$ implies $\frac{pm-q}{p-1}<m-1\leq r$, it follows that $\frac{p-1}{q-m}<\frac{1}{m-r}$ when $m>r$. Fix $\delta>0$ such that

$$\frac{p-1}{q-m}<\delta<\min\left\{\frac{1}{m-r},1\right\}\quad\text{for } m>r, \tag{3.14}$$

and set

$$w=\mu^\delta+\mu\psi(x),\quad (x,t)\in\Omega\times(0,\infty),$$

with $\mu\in(0,1)$ small to be determined. A simple calculation yields

$$w_t-w^m\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big)-w^q-\gamma w^r|\nabla w|^p=\mu^{p-1}w^m-w^q-\gamma\mu^p w^r|\nabla\psi|^p.$$

Since $(q-m)\delta-p+1>0$ by (3.14), we have

$$\frac{2w^q}{\mu^{p-1}w^m}=2\big(\mu^\delta+\mu\psi\big)^{q-m}\mu^{1-p}\leq 2\mu^{(q-m)\delta-p+1}\big(1+\|\psi\|_{L^\infty(\Omega)}\big)\leq 1,$$

provided μ is small enough. If $r<m$, by (3.14), we have

$$\frac{2\gamma\mu^p w^r|\nabla\psi|^p}{\mu^{p-1}w^m}=2\gamma\mu\big(\mu^\delta+\mu\psi\big)^{r-m}|\nabla\psi|^p\leq 2\gamma\mu^{1-(m-r)\delta}|\nabla\psi|^p\leq 1$$

for μ small. In the case $r \geq m$,

$$\frac{2\gamma\mu^p w^r |\nabla\psi|^p}{\mu^{p-1} w^m} \leq 2\gamma\mu^{1+(r-m)\delta}\big(1+\|\psi\|_{L^\infty(\Omega)}\big)^{r-m}|\nabla\psi|^p \leq 1 \tag{3.15}$$

for μ small. In one word, there exists $\mu > 0$ small enough that w is a time-independent supersolution provided $u_0 < \mu^\delta + \mu\psi$.

To deal with large initial data, consider the following problem without gradient term:

$$\begin{cases} u_t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

the solutions of which blow up in finite time for large initial data [6]. By the comparison principle, the solutions of (1.1) blow up as well for large initial data. □

Theorem 3.3 *Suppose $q = p + m - 1$.*

(i) *If $r > m - 1 = \frac{pm-q}{p-1}$, there exist both global and non-global solutions.*

(ii) *Suppose $r = m - 1 = \frac{pm-q}{p-1}$. If $\lambda_1 < \lambda(\frac{\gamma+p-1}{p-1})^{\frac{2(p-1)}{p}}$, all solutions of* (1.1) *blow up in finite time. If $\lambda_1 \geq \lambda(\frac{\gamma+p-1}{p-1})^{\frac{2(p-1)}{p}}$, the solutions are global and bounded.*

Proof (i) It follows from Lemma 3.3 immediately that the solutions of (1.1) blow up in finite time for large initial data.

Next, we will show the solutions are global for small domain and small initial data. Let Ω be small with $\lambda_1(\Omega) > \lambda$, and choose $\widetilde{\Omega}$ with $\Omega \subset\subset \widetilde{\Omega}$ such that $\tilde{\lambda}_1 := \lambda_1(\widetilde{\Omega}) > \lambda$ (see [14]). Normalize $\tilde{\varphi}$, the eigenfunction corresponding to $\tilde{\lambda}_1$, by $\tilde{\varphi} > 0$ in Ω', $\|\tilde{\varphi}\|_{L^\infty(\widetilde{\Omega})} = 1$. Then $\tilde{\varphi} \geq \rho$ in Ω for some $\rho > 0$.

Define $w = a\tilde{\varphi}^b$ with $a = (\frac{(1-b)(p-1)}{b\gamma})^{\frac{1}{r-m+1}}$, $b \in (\lambda/\tilde{\lambda}_1, 1)$. Then

$$\begin{aligned} & w_t - w^m \operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) - \lambda w^q - \gamma u^r |\nabla w|^p \\ &\quad = w^m\big[(ab)^{p-1}\tilde{\lambda}_1\tilde{\varphi}^{(b-1)(p-1)+(p-1)} - (b-1)(p-1)(ab)^{p-1}\tilde{\varphi}^{(b-1)(p-1)-1}|\nabla\tilde{\varphi}|^p \\ &\qquad - \lambda a^{p-1}\tilde{\varphi}^{b(p-1)} - \gamma a^{p+r-m}b^p\tilde{\varphi}^{(b-1)p+b(r-m)}|\nabla\tilde{\varphi}|^p\big] \\ &\quad = w^m\big[a^{p-1}\tilde{\varphi}^{b(p-1)}\big(b^{p-1}\tilde{\lambda}_1 - \lambda\big) \\ &\qquad - (ab)^{p-1}\tilde{\varphi}^{(b-1)p-b}|\nabla\tilde{\varphi}|^p\big((b-1)(p-1) + \gamma b a^{r-m+1}\tilde{\varphi}^{b(r-m+1)}\big)\big] \\ &\quad \geq 0. \end{aligned}$$

If in addition u_0 is small such that $\|u_0\|_{L^\infty(\Omega)} < a\rho^b$, then $u_\epsilon \leq w$ in $\Omega \times (0,\infty)$ by the comparison principle. Thus $u \leq w$ in $\Omega \times (0,\infty)$.

(ii) By the re-scaling $v(x,t) = u^\alpha(\frac{x}{\beta}, \frac{t}{\alpha})$ with $\alpha = \frac{\gamma}{p-1} + 1$ and $\beta = \alpha^{\frac{p-1}{p}}$, we transform (1.1) into the Dirichlet problem

$$\begin{cases} v_t = v^{m_1} \operatorname{div}(|\nabla v|^{p-2}\nabla v) + \lambda v^{p+m_1-1}, & (x,t) \in U \times (0,\alpha T), \\ v = 0, & (x,t) \in \partial U \times (0,\alpha T), \\ v(x,0) = u_0^\alpha(x), & x \in \overline{U}, \end{cases} \tag{3.16}$$

where $m_1 = \frac{\alpha+m-1-(\alpha-1)(p-1)}{\alpha}$, $U = \{\beta x | x \in \Omega\}$. We know from Theorem 3.2 of [6] that all positive solutions of (3.16) are global and bounded if $\lambda_1(U) \geq \lambda$, and so do the positive solutions of (1.1) whenever

$$\lambda_1 := \lambda_1(\Omega) = \beta^2 \lambda_1(U) \geq \lambda\beta^2 = \lambda\left(\frac{\gamma + p - 1}{p-1}\right)^{\frac{2(p-1)}{p}}.$$

Also by Theorem 3.2 of [6], similarly, if $\lambda_1 < \lambda(\frac{\gamma+p-1}{p-1})^{\frac{2(p-1)}{p}}$, all solutions of (1.1) blow up in finite time. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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