Güldü *Boundary Value Problems* (2016) 2016:135
DOI 10.1186/s13661-016-0639-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

On discontinuous Dirac operator with eigenparameter dependent boundary and two transmission conditions

Yalçın Güldü*

*Correspondence:
yguldu@gmail.com
Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas, 58140, Turkey

Abstract
In this paper, we consider a discontinuous Dirac operator with eigenparameter dependent both boundary and two transmission conditions. We introduce a suitable Hilbert space formulation and get some properties of eigenvalues and eigenfunctions. Then we investigate the Green's function, the resolvent operator, and some uniqueness theorems by using the Weyl function and some spectral data.

MSC: Primary 34A55; secondary 34B24; 34L05

Keywords: Dirac operator; eigenvalues; eigenfunctions; transmission conditions; Green's function; Weyl function

1 Introduction

Inverse problems of spectral analysis recover operators by their spectral data. Fundamental and vast studies about the classical Sturm-Liouville, Dirac operators, Schrödinger equation, and hyperbolic equations are well studied (see [1–9] and references therein).

Studies of eigenvalue dependence appearing not only in the differential equation but also in the boundary conditions have increased in recent years (see [10–18] and corresponding bibliography). Moreover, boundary conditions which depend linearly and nonlinearly on the spectral parameter are considered in [10, 18–23] and [24–30], respectively. Furthermore, boundary value problems with transmission conditions are also increasingly studied. These types of studies introduce qualitative changes in the exploration. Direct and inverse problems for Sturm-Liouville and Dirac operators with transmission conditions are investigated in some papers (see [7, 31–34] and the corresponding bibliography). Then differential equations with the spectral parameter and transmission conditions arise in heat, mechanics, mass transfer problems, in diffraction problems, and in various physical transfer problems (see [20, 31, 35–42] and corresponding bibliography).

More recently, some boundary value problems with eigenparameter in boundary and transmission conditions were extended to the case of two, more than two or a finite number of transmissions in [43–47] and the references therein.

The present paper deals with the discontinuous Dirac operator with eigenparameter dependent boundary and two transmission conditions. The aim of the present paper is to obtain the asymptotic formulas of the eigenvalues and eigenfunctions, to construct the Green's function and the resolvent operator, and to prove some uniqueness theorems.

Especially, some parameters of the considered problem can be determined by the Weyl function and some spectral data.

We consider a discontinuous boundary value problem L with function $\rho(x)$;

$$ly := \rho(x)By'(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b] = \Lambda, \tag{1}$$

where

$$\rho(x) = \begin{cases} \rho_1^{-1}, & a \le x < \xi_1, \\ \rho_2^{-1}, & \xi_1 < x < \xi_2, \\ \rho_3^{-1}, & \xi_2 < x \le b, \end{cases}$$

and ρ_1, ρ_2, and ρ_3 are given positive real numbers;

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}, \quad p(x), q(x), r(x) \in L_2[\Lambda, \mathbb{R}];$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$\lambda \in \mathbb{C}$ is a complex spectral parameter; we have boundary conditions at the endpoints,

$$l_1 y := \lambda\big(\alpha_1' y_1(a) - \alpha_2' y_2(a)\big) - \big(\alpha_1 y_1(a) - \alpha_2 y_2(a)\big) = 0, \tag{2}$$

$$l_2 y := \lambda\big(\gamma_1' y_1(b) - \gamma_2' y_2(b)\big) + \big(\gamma_1 y_1(b) - \gamma_2 y_2(b)\big) = 0, \tag{3}$$

with transmission conditions at the two points $x = \xi_1$, $x = \xi_2$,

$$l_3 y := y_1(\xi_1 + 0) - \alpha_3 y_1(\xi_1 - 0) = 0, \tag{4}$$

$$l_4 y := y_2(\xi_1 + 0) - (\alpha_4 + \lambda) y_1(\xi_1 - 0) - \alpha_3^{-1} y_2(\xi_1 - 0) = 0, \tag{5}$$

$$l_5 y := y_1(\xi_2 + 0) - \alpha_5 y_1(\xi_2 - 0) = 0, \tag{6}$$

$$l_6 y := y_2(\xi_2 + 0) - (\alpha_6 + \lambda) y_1(\xi_2 - 0) - \alpha_5^{-1} y_2(\xi_2 - 0) = 0, \tag{7}$$

where α_i, and α_j', γ_j' ($i = \overline{1,6}$, $j = 1, 2$) are real numbers; $\alpha_3 > 0$, $\alpha_5 > 0$, and

$$d_1 = \begin{vmatrix} \alpha_1 & \alpha_1' \\ \alpha_2 & \alpha_2' \end{vmatrix} > 0, \qquad d_2 = \begin{vmatrix} \gamma_1 & \gamma_1' \\ \gamma_2 & \gamma_2' \end{vmatrix} > 0.$$

2 Operator formulation and properties of spectrum

In this section, we present the inner product in the Hilbert space $H := L_2(\Lambda) \oplus L_2(\Lambda) \oplus \mathbb{C}^4$ and the operator T defined on H such that (1)-(7) can be regarded as the eigenvalue problem of operator T. We define an inner product in H by

$$\begin{aligned} \langle F, G \rangle := \rho^{-1}(x) \int_a^b \big(f_1(x)\overline{g}_1(x) + f_2(x)\overline{g}_2(x)\big)\,dx + \alpha_3 f_1(\xi_1 - 0)\overline{g}_1(\xi_1 - 0) \\ + \alpha_5 f_1(\xi_2 - 0)\overline{g}_1(\xi_2 - 0) + \frac{1}{d_1} r\overline{r_1} + \frac{1}{d_2} s\overline{s_1} \end{aligned} \tag{8}$$

for

$$F = \begin{pmatrix} f(x) \\ r \\ s \\ f_1(\xi_1 - 0) \\ f_1(\xi_2 - 0) \end{pmatrix} \in H, \qquad G = \begin{pmatrix} g(x) \\ r_1 \\ s_1 \\ g_1(\xi_1 - 0) \\ g_1(\xi_2 - 0) \end{pmatrix} \in H, \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \qquad r = \alpha_1' f_1(a) - \alpha_2' f_2(a), \qquad s = \gamma_1' f_1(b) - \gamma_2' f_2(b),$$

$$r_1 = \alpha_1' g_1(a) - \alpha_2' g_2(a), \qquad s_1 = \gamma_1' g_1(b) - \gamma_2' g_2(b).$$

Consider the operator T defined in the domain

$$D(T) = \{F \in H : f(x) \in AC([a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]), lf \in L_2(\Lambda) \oplus L_2(\Lambda), l_3 f = l_5 f = 0\}$$

such that

$$\begin{aligned} TF := \bigl(&lf, \alpha_1 f_1(a) - \alpha_2 f_2(a), -(\gamma_1 f_1(b) - \gamma_2 f_2(b)), \\ &f_2(\xi_1 + 0) - \alpha_4 f_1(\xi_1 - 0) - \alpha_3^{-1} f_2(\xi_1 - 0), f_2(\xi_2 + 0) - \alpha_6 f_1(\xi_2 - 0) - \alpha_5^{-1} f_2(\xi_2 - 0)\bigr)^T \end{aligned}$$

for

$$F = \bigl(f, \alpha_1' f_1(a) - \alpha_2' f_2(a), \gamma_1' f_1(b) - \gamma_2' f_2(b), f_1(\xi_1 - 0), f_1(\xi_2 - 0)\bigr)^T \in D(T).$$

Thus, we can rewrite the considered problem (1)-(7) in the operator form as $TF = \lambda F$, *i.e.*, the problem (1)-(7) can be considered as an eigenvalue problem of the operator T.

We define the solutions

$$\varphi(x, \lambda) = \begin{cases} \varphi_1(x, \lambda), & x \in [a, \xi_1), \\ \varphi_2(x, \lambda), & x \in (\xi_1, \xi_2), \\ \varphi_3(x, \lambda), & x \in (\xi_2, b], \end{cases} \qquad \psi(x, \lambda) = \begin{cases} \psi_1(x, \lambda), & x \in [a, \xi_1), \\ \psi_2(x, \lambda), & x \in (\xi_1, \xi_2), \\ \psi_3(x, \lambda), & x \in (\xi_2, b], \end{cases}$$

$$\varphi_1(x, \lambda) = (\varphi_{11}(x, \lambda), \varphi_{12}(x, \lambda))^T,$$

$$\varphi_2(x, \lambda) = (\varphi_{21}(x, \lambda), \varphi_{22}(x, \lambda))^T,$$

$$\varphi_3(x, \lambda) = (\varphi_{31}(x, \lambda), \varphi_{32}(x, \lambda))^T,$$

and

$$\psi_1(x, \lambda) = (\psi_{11}(x, \lambda), \psi_{12}(x, \lambda))^T,$$

$$\psi_2(x, \lambda) = (\psi_{21}(x, \lambda), \psi_{22}(x, \lambda))^T,$$

$$\psi_3(x, \lambda) = (\psi_{31}(x, \lambda), \psi_{32}(x, \lambda))^T,$$

of equation (1) satisfying the initial conditions

$$\begin{aligned}
&\varphi_{11}(a,\lambda) = \lambda\alpha_2' - \alpha_2, \qquad \varphi_{12}(a,\lambda) = \lambda\alpha_1' - \alpha_1,\\
&\varphi_{21}(\xi_1,\lambda) = \alpha_3\varphi_{11}(\xi_1,\lambda), \qquad \varphi_{22}(\xi_1,\lambda) = (\alpha_4+\lambda)\varphi_{11}(\xi_1,\lambda) + \alpha_3^{-1}\varphi_{12}(\xi_1,\lambda),\\
&\varphi_{31}(\xi_2,\lambda) = \alpha_5\varphi_{21}(\xi_2,\lambda), \qquad \varphi_{32}(\xi_2,\lambda) = (\alpha_6+\lambda)\varphi_{21}(\xi_2,\lambda) + \alpha_5^{-1}\varphi_{22}(\xi_2,\lambda),
\end{aligned} \tag{9}$$

and similarly

$$\begin{aligned}
&\psi_{31}(b,\lambda) = \lambda\gamma_2' + \gamma_2, \qquad \psi_{32}(b,\lambda) = \lambda\gamma_1' + \gamma_1,\\
&\psi_{21}(\xi_2,\lambda) = \frac{\psi_{31}(\xi_2,\lambda)}{\alpha_5}, \qquad \psi_{22}(\xi_2,\lambda) = \alpha_5\psi_{32}(\xi_2,\lambda) - (\alpha_6+\lambda)\psi_{31}(\xi_2,\lambda),\\
&\psi_{11}(\xi_2,\lambda) = \frac{\psi_{21}(\xi_1,\lambda)}{\alpha_3}, \qquad \psi_{12}(\xi_2,\lambda) = \alpha_3\psi_{22}(\xi_1,\lambda) - (\alpha_4+\lambda)\psi_{21}(\xi_1,\lambda),
\end{aligned} \tag{10}$$

respectively.

These solutions are entire functions of λ for each fixed $x \in [a,b]$ and satisfy the relation $\psi(x,\lambda_n) = \kappa_n\varphi(x,\lambda_n)$ for each eigenvalue λ_n, where

$$\kappa_n = \frac{\alpha_1'\psi_{11}(a,\lambda_n) - \alpha_2'\psi_{12}(a,\lambda_n)}{d_1}.$$

Lemma 1 *T is a self-adjoint operator. Therefore, all eigenvalues and eigenfunctions of the problem* (1)-(7) *are real and the two eigenfunctions $\varphi(x,\lambda_1) = (\varphi_1(x,\lambda_1),\varphi_2(x,\lambda_1))^T$ and $\varphi(x,\lambda_2) = (\varphi_1(x,\lambda_2),\varphi_2(x,\lambda_2))^T$ corresponding to different eigenvalues λ_1 and λ_2 are orthogonal in the sense of*

$$\begin{aligned}
&\rho^{-1}(x)\int_a^b \bigl[\varphi_1(x,\lambda_1)\varphi_1(x,\lambda_2) + \varphi_2(x,\lambda_1)\varphi_2(x,\lambda_2)\bigr]\,dx\\
&\quad + \alpha_3\varphi_1(\xi_1-0,\lambda_1)\varphi_1(\xi_1-0,\lambda_2) + \alpha_5\varphi_1(\xi_2-0,\lambda_1)\varphi_1(\xi_2-0,\lambda_2)\\
&\quad + \frac{1}{d_1}\bigl(\alpha_1'\varphi_{11}(a,\lambda_1) - \alpha_2'\varphi_{12}(a,\lambda_1)\bigr)\bigl(\alpha_1'\varphi_{11}(a,\lambda_2) - \alpha_2'\varphi_{12}(a,\lambda_2)\bigr)\\
&\quad + \frac{1}{d_2}\bigl(\gamma_1'\varphi_{31}(b,\lambda_1) - \gamma_2'\varphi_{32}(b,\lambda_1)\bigr)\bigl(\gamma_1'\varphi_{31}(b,\lambda_2) - \gamma_2'\varphi_{32}(b,\lambda_2)\bigr) = 0.
\end{aligned}$$

By the method of variation of parameters, integral equations in Lemmas 2, 3 can be obtained and with the help of these integral equations, we also have their asymptotic behaviors.

Lemma 2 *The following integral equations and asymptotic behaviors hold*:

$$\begin{aligned}
\varphi_{11}(x,\lambda) &= -\bigl(\lambda\alpha_1' - \alpha_1\bigr)\sin\lambda\rho_1(x-a) + \bigl(\lambda\alpha_2' - \alpha_2\bigr)\cos\lambda\rho_1(x-a)\\
&\quad + \int_a^x \bigl[p(t)\sin\lambda\rho_1(x-t) + q(t)\cos\lambda\rho_1(x-t)\bigr]\rho_1\varphi_{11}(t,\lambda)\,dt\\
&\quad + \int_a^x \bigl[q(t)\sin\lambda\rho_1(x-t) + r(t)\cos\lambda\rho_1(x-t)\bigr]\rho_1\varphi_{12}(t,\lambda)\,dt\\
&= -\bigl(\lambda\alpha_1' - \alpha_1\bigr)\sin\lambda\rho_1(x-a) + \bigl(\lambda\alpha_2' - \alpha_2\bigr)\cos\lambda\rho_1(x-a) + o\bigl(|\lambda| e^{|\operatorname{Im}\lambda|(x-a)\rho_1}\bigr),
\end{aligned}$$

$$
\begin{aligned}
\varphi_{12}(x,\lambda) &= \left(\lambda\alpha_1' - \alpha_1\right)\cos\lambda\rho_1(x-a) + \left(\lambda\alpha_2' - \alpha_2\right)\sin\lambda\rho_1(x-a) \\
&\quad + \int_a^x \left[-p(t)\cos\lambda\rho_1(x-t) + q(t)\sin\lambda\rho_1(x-t)\right]\rho_1\varphi_{11}(t,\lambda)\,dt \\
&\quad + \int_a^x \left[-q(t)\cos\lambda\rho_1(x-t) + r(t)\sin\lambda\rho_1(x-t)\right]\rho_1\varphi_{12}(t,\lambda)\,dt \\
&= \left(\lambda\alpha_1' - \alpha_1\right)\cos\lambda\rho_1(x-a) + \left(\lambda\alpha_2' - \alpha_2\right)\sin\lambda\rho_1(x-a) + o\left(|\lambda|e^{|\operatorname{Im}\lambda|(x-a)\rho_1}\right), \\
\varphi_{21}(x,\lambda) &= \alpha_3\varphi_{11}(\xi_1,\lambda)\cos\lambda\rho_2(x-\xi_1) \\
&\quad - \left((\alpha_4+\lambda)\varphi_{11}(\xi_1,\lambda) + \frac{1}{\alpha_3}\varphi_{12}(\xi_1,\lambda)\right)\sin\lambda\rho_2(x-\xi_1) \\
&\quad + \int_{\xi_1}^x \left[p(t)\sin\lambda\rho_2(x-t) + q(t)\cos\lambda\rho_2(x-t)\right]\rho_2\varphi_{21}(t,\lambda)\,dt \\
&\quad + \int_{\xi_1}^x \left[q(t)\sin\lambda\rho_2(x-t) + r(t)\cos\lambda\rho_2(x-t)\right]\rho_2\varphi_{22}(t,\lambda)\,dt \\
&= (\alpha_4+\lambda)\left[\left(\lambda\alpha_1' - \alpha_1\right)\sin\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(x-\xi_1)\right. \\
&\quad \left. - \left(\lambda\alpha_2' - \alpha_2\right)\cos\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(x-\xi_1)\right] + o\left(|\lambda|^2 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(x-\xi_1)\rho_2)}\right), \\
\varphi_{22}(x,\lambda) &= \alpha_3\varphi_{11}(\xi_1,\lambda)\sin\lambda\rho_2(x-\xi_1) \\
&\quad + \left((\alpha_4+\lambda)\varphi_{11}(\xi_1,\lambda) + \frac{1}{\alpha_3}\varphi_{12}(\xi_1,\lambda)\right)\cos\lambda\rho_2(x-\xi_1) \\
&\quad + \int_{\xi_1}^x \left[-p(t)\cos\lambda\rho_2(x-t) + q(t)\sin\lambda\rho_2(x-t)\right]\rho_2\varphi_{21}(t,\lambda)\,dt \\
&\quad + \int_{\xi_1}^x \left[-q(t)\cos\lambda\rho_2(x-t) + r(t)\sin\lambda\rho_2(x-t)\right]\rho_2\varphi_{22}(t,\lambda)\,dt \\
&= -(\alpha_4+\lambda)\left[\left(\lambda\alpha_1' - \alpha_1\right)\sin\lambda\rho_1(\xi_1-a)\cos\lambda\rho_2(x-\xi_1)\right. \\
&\quad \left. - \left(\lambda\alpha_2' - \alpha_2\right)\cos\lambda\rho_1(\xi_1-a)\cos\lambda\rho_2(x-\xi_1)\right] + o\left(|\lambda|^2 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(x-\xi_1)\rho_2)}\right), \\
\varphi_{31}(x,\lambda) &= \alpha_5\varphi_{21}(\xi_2,\lambda)\cos\lambda\rho_3(x-\xi_2) \\
&\quad - \left(\frac{1}{\alpha_5}\varphi_{22}(\xi_2,\lambda) + (\alpha_6+\lambda)\varphi_{21}(\xi_2,\lambda)\right)\sin\lambda\rho_3(x-\xi_2) \\
&\quad + \int_{\xi_2}^x \left[p(t)\sin\lambda\rho_3(x-t) + q(t)\cos\lambda\rho_3(x-t)\right]\rho_3\varphi_{31}(t,\lambda)\,dt \\
&\quad + \int_{\xi_2}^x \left[q(t)\sin\lambda\rho_3(x-t) + r(t)\cos\lambda\rho_3(x-t)\right]\rho_3\varphi_{32}(t,\lambda)\,dt \\
&= (\alpha_4+\lambda)(\alpha_6+\lambda)\left[-\left(\lambda\alpha_1' - \alpha_1\right)\sin\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(\xi_2-\xi_1)\right. \\
&\quad \left. + \left(\lambda\alpha_2' - \alpha_2\right)\cos\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(\xi_2-\xi_1)\right]\sin\lambda\rho_3(x-\xi_2) \\
&\quad + o\left(|\lambda|^3 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(x-\xi_2)\rho_3)}\right), \\
\varphi_{32}(x,\lambda) &= \alpha_5\varphi_{21}(\xi_2,\lambda)\sin\lambda\rho_3(x-\xi_2) \\
&\quad + \left(\frac{1}{\alpha_5}\varphi_{22}(\xi_2,\lambda) + (\alpha_6+\lambda)\varphi_{21}(\xi_2,\lambda)\right)\cos\lambda\rho_3(x-\xi_2)
\end{aligned}
$$

$$+\int_{\xi_2}^{x}\bigl[-p(t)\cos\lambda\rho_3(x-t)+q(t)\sin\lambda\rho_3(x-t)\bigr]\rho_3\varphi_{31}(t,\lambda)\,dt$$
$$+\int_{\xi_2}^{x}\bigl[-q(t)\cos\lambda\rho_3(x-t)+r(t)\sin\lambda\rho_3(x-t)\bigr]\rho_3\varphi_{32}(t,\lambda)\,dt$$
$$=-(\alpha_4+\lambda)(\alpha_6+\lambda)\bigl[-\bigl(\lambda\alpha_1'-\alpha_1\bigr)\sin\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(\xi_2-\xi_1)$$
$$+\bigl(\lambda\alpha_2'-\alpha_2\bigr)\cos\lambda\rho_1(\xi_1-a)\sin\lambda\rho_2(\xi_2-\xi_1)\bigr]\cos\lambda\rho_3(x-\xi_2)$$
$$+o\bigl(|\lambda|^3e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(x-\xi_2)\rho_3)}\bigr).$$

Lemma 3 *The following integral equations and asymptotic behaviors hold*:

$$\psi_{31}(x,\lambda)=\bigl(\lambda\gamma_2'+\gamma_2\bigr)\cos\lambda\rho_3(x-b)-\bigl(\lambda\gamma_1'+\gamma_1\bigr)\sin\lambda\rho_3(x-b)$$
$$-\int_x^b\bigl[p(t)\sin\lambda\rho_3(x-t)+q(t)\cos\lambda\rho_3(x-t)\bigr]\rho_3\psi_{31}(t,\lambda)\,dt$$
$$-\int_x^b\bigl[q(t)\sin\lambda\rho_3(x-t)+r(t)\cos\lambda\rho_3(x-t)\bigr]\rho_3\psi_{32}(t,\lambda)\,dt$$
$$=\bigl(\lambda\gamma_2'+\gamma_2\bigr)\cos\lambda\rho_3(x-b)-\bigl(\lambda\gamma_1'+\gamma_1\bigr)\sin\lambda\rho_3(x-b)+o\bigl(|\lambda|e^{|\operatorname{Im}\lambda|(b-x)\rho_3}\bigr),$$
$$\psi_{32}(x,\lambda)=\bigl(\lambda\gamma_2'+\gamma_2\bigr)\sin\lambda\rho_3(x-b)+\bigl(\lambda\gamma_1'+\gamma_1\bigr)\cos\lambda\rho_3(x-b)$$
$$+\int_x^b\bigl[p(t)\cos\lambda\rho_3(x-t)-q(t)\sin\lambda\rho_3(x-t)\bigr]\rho_3\psi_{31}(t,\lambda)\,dt$$
$$+\int_x^b\bigl[q(t)\cos\lambda\rho_3(x-t)-r(t)\sin\lambda\rho_3(x-t)\bigr]\rho_3\psi_{32}(t,\lambda)\,dt$$
$$=\bigl(\lambda\gamma_2'+\gamma_2\bigr)\sin\lambda\rho_3(x-b)+\bigl(\lambda\gamma_1'+\gamma_1\bigr)\cos\lambda\rho_3(x-b)+o\bigl(|\lambda|e^{|\operatorname{Im}\lambda|(b-x)\rho_3}\bigr),$$
$$\psi_{21}(x,\lambda)=\bigl[(\alpha_6+\lambda)\psi_{31}(\xi_2,\lambda)-\alpha_5\psi_{32}(\xi_2,\lambda)\bigr]\sin\lambda\rho_2(x-\xi_2)$$
$$+\frac{1}{\alpha_5}\psi_{31}(\xi_2,\lambda)\cos\lambda\rho_2(x-\xi_2)$$
$$-\int_x^{\xi_2}\bigl[p(t)\sin\lambda\rho_2(x-t)+q(t)\cos\lambda\rho_2(x-t)\bigr]\rho_2\psi_{21}(t,\lambda)\,dt$$
$$-\int_x^{\xi_2}\bigl[q(t)\sin\lambda\rho_2(x-t)+r(t)\cos\lambda\rho_2(x-t)\bigr]\rho_2\psi_{22}(t,\lambda)\,dt$$
$$=(\alpha_6+\lambda)\bigl[\bigl(\lambda\gamma_2'+\gamma_2\bigr)\cos\lambda\rho_3(\xi_2-b)\sin\lambda\rho_2(x-\xi_2)$$
$$-\bigl(\lambda\gamma_1'+\gamma_1\bigr)\sin\lambda\rho_3(\xi_2-b)\sin\lambda\rho_2(x-\xi_2)\bigr]+o\bigl(|\lambda|^2e^{|\operatorname{Im}\lambda|((b-\xi_2)\rho_3+(\xi_2-x)\rho_2)}\bigr),$$
$$\psi_{22}(x,\lambda)=\bigl[-(\alpha_6+\lambda)\psi_{31}(\xi_2,\lambda)+\alpha_5\psi_{32}(\xi_2,\lambda)\bigr]\cos\lambda\rho_2(x-\xi_2)$$
$$+\frac{1}{\alpha_5}\psi_{31}(\xi_2,\lambda)\sin\lambda\rho_2(x-\xi_2)$$
$$+\int_x^{\xi_2}\bigl[p(t)\cos\lambda\rho_2(x-t)-q(t)\sin\lambda\rho_2(x-t)\bigr]\rho_2\psi_{21}(t,\lambda)\,dt$$
$$+\int_x^{\xi_2}\bigl[q(t)\cos\lambda\rho_2(x-t)-r(t)\sin\lambda\rho_2(x-t)\bigr]\rho_2\psi_{22}(t,\lambda)\,dt$$
$$=-(\alpha_6+\lambda)\bigl[\bigl(\lambda\gamma_2'+\gamma_2\bigr)\cos\lambda\rho_3(\xi_2-b)\cos\lambda\rho_2(x-\xi_2)$$

$$
\begin{aligned}
&\quad - \left(\lambda\gamma_1' + \gamma_1\right) \sin\lambda\rho_3(\xi_2 - b) \cos\lambda\rho_2(x - \xi_2)\big] + o\left(|\lambda|^2 e^{|\operatorname{Im}\lambda|((b-\xi_2)\rho_3 + (\xi_2 - x)\rho_2)}\right),\\
\psi_{11}(x,\lambda) &= \left(\alpha_3\psi_{22}(\xi_1,\lambda) - (\alpha_4+\lambda)\psi_{21}(\xi_1,\lambda)\right)\sin\lambda\rho_1(x-\xi_1)\\
&\quad - \frac{1}{\alpha_3}\psi_{21}(\xi_1,\lambda)\cos\lambda\rho_1(x-\xi_1)\\
&\quad - \int_x^{\xi_1}\big[p(t)\sin\lambda\rho_1(x-t) + q(t)\cos\lambda\rho_1(x-t)\big]\rho_2\psi_{11}(t,\lambda)\,dt\\
&\quad - \int_x^{\xi_1}\big[q(t)\sin\lambda\rho_1(x-t) + r(t)\cos\lambda\rho_1(x-t)\big]\rho_2\psi_{12}(t,\lambda)\,dt\\
&= -(\alpha_4+\lambda)(\alpha_6+\lambda)\big[\left(\lambda\gamma_2'+\gamma_2\right)\cos\lambda\rho_3(\xi_2-b)\\
&\quad - \left(\lambda\gamma_1'+\gamma_1\right)\sin\lambda\rho_3(\xi_2-b)\big]\sin\lambda\rho_2(\xi_1-\xi_2)\sin\lambda\rho_1(x-\xi_1)\\
&\quad + o\left(|\lambda|^3 e^{|\operatorname{Im}\lambda|((b-\xi_2)\rho_3+(\xi_2-\xi_1)\rho_2+(\xi_1-x)\rho_1)}\right),\\
\psi_{12}(x,\lambda) &= \left((\alpha_4+\lambda)\psi_{21}(\xi_1,\lambda) - \alpha_3\psi_{22}(\xi_1,\lambda)\right)\cos\lambda\rho_1(x-\xi_1)\\
&\quad - \frac{1}{\alpha_3}\psi_{21}(\xi_1,\lambda)\sin\lambda\rho_1(x-\xi_1)\\
&\quad + \int_x^{\xi_1}\big[p(t)\cos\lambda\rho_1(x-t) - q(t)\sin\lambda\rho_1(x-t)\big]\rho_1\psi_{11}(t,\lambda)\,dt\\
&\quad + \int_x^{\xi_1}\big[q(t)\cos\lambda\rho_1(x-t) - r(t)\sin\lambda\rho_1(x-t)\big]\rho_1\psi_{12}(t,\lambda)\,dt\\
&= (\alpha_4+\lambda)(\alpha_6+\lambda)\big[\left(\lambda\gamma_2'+\gamma_2\right)\cos\lambda\rho_3(\xi_2-b)\\
&\quad - \left(\lambda\gamma_1'+\gamma_1\right)\sin\lambda\rho_3(\xi_2-b)\big]\sin\lambda\rho_2(\xi_1-\xi_2)\cos\lambda\rho_1(x-\xi_1)\\
&\quad + o\left(|\lambda|^3 e^{|\operatorname{Im}\lambda|((b-\xi_2)\rho_3+(\xi_2-\xi_1)\rho_2+(\xi_1-x)\rho_1)}\right).
\end{aligned}
$$

Denote

$$\Delta_i(\lambda) := W(\varphi_i,\psi_i,x) := \varphi_{i1}\psi_{i2} - \varphi_{i2}\psi_{i1}, \quad x\in\Lambda_i\ (i=\overline{1,3}),$$

which are independent of $x \in \Lambda_i$ and are entire functions such that $\Lambda_1 = [a,\xi_1)$, $\Lambda_2 = (\xi_1,\xi_2)$, $\Lambda_3 = (\xi_2,b]$.

Let

$$\Delta_3(\lambda) = \Delta(\lambda) = W(\varphi,\psi,b) = \left(\lambda\gamma_1'+\gamma_1\right)\varphi_{31}(b,\lambda) - \left(\lambda\gamma_2'+\gamma_2\right)\varphi_{32}(b,\lambda) \tag{11}$$

and

$$
\begin{aligned}
\mu_n &:= \rho^{-1}(x)\int_a^b\big[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\big]\,dx\\
&\quad + \alpha_3\varphi_1^2(\xi_1-0,\lambda_n) + \alpha_5\varphi_1^2(\xi_2-0,\lambda_n) + \frac{1}{d_1}\left(\alpha_1'\varphi_{11}(a,\lambda_n) - \alpha_2'\varphi_{12}(a,\lambda_n)\right)^2\\
&\quad + \frac{1}{d_2}\left(\gamma_1'\varphi_{31}(b,\lambda_n) - \gamma_2'\varphi_{32}(b,\lambda_n)\right)^2.
\end{aligned}
\tag{12}
$$

The function $\Delta(\lambda)$ is called the characteristic function and numbers $\{\mu_n\}_{n\in\mathbb{Z}}$ are called the normalizing constants of the problem (1)-(7).

Lemma 4 *The following equality holds for each eigenvalue* λ_n:

$$\dot{\Delta}(\lambda_n) = -\kappa_n \mu_n.$$

Proof Since

$$\rho(x)\varphi_2'(x,\lambda_n) + p(x)\varphi_1(x,\lambda_n) + q(x)\varphi_2(x,\lambda_n) = \lambda_n \varphi_1(x,\lambda_n),$$
$$\rho(x)\psi_2'(x,\lambda) + p(x)\psi_1(x,\lambda) + q(x)\psi_2(x,\lambda) = \lambda\psi_1(x,\lambda),$$

and

$$-\rho(x)\varphi_1'(x,\lambda_n) + q(x)\varphi_1(x,\lambda_n) + r(x)\varphi_2(x,\lambda_n) = \lambda_n \varphi_2(x,\lambda_n),$$
$$-\rho(x)\psi_1'(x,\lambda) + q(x)\psi_1(x,\lambda) + r(x)\psi_2(x,\lambda) = \lambda\psi_2(x,\lambda),$$

we obtain

$$\begin{aligned}
&\varphi_1(x,\lambda_n)\psi_2(x,\lambda) - \varphi_2(x,\lambda_n)\psi_1(x,\lambda)\Big(\Big|_a^{\xi_1} + \Big|_{\xi_1}^{\xi_2} + \Big|_{\xi_2}^{b}\Big) \\
&\quad = (\lambda - \lambda_n)\rho_1 \int_a^{\xi_1} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\qquad + (\lambda - \lambda_n)\rho_2 \int_{\xi_1}^{\xi_2} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\qquad + (\lambda - \lambda_n)\rho_3 \int_{\xi_2}^{b} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx.
\end{aligned}$$

After adding and subtracting $\Delta(\lambda)$ on the left-hand side of the last equality and by using the conditions (2)-(7) one can obtain

$$\begin{aligned}
&\Delta(\lambda) - (\lambda - \lambda_n)\big(\alpha_2'\psi_2(a,\lambda) - \alpha_1'\psi_1(a,\lambda)\big) + (\lambda - \lambda_n)\big(\gamma_2'\varphi_2(b,\lambda_n) - \gamma_1'\varphi_1(b,\lambda_n)\big) \\
&\qquad + \alpha_3(\lambda - \lambda_n)\varphi_1(\xi_1 - 0,\lambda_n)\psi_1(\xi_1 - 0,\lambda) + \alpha_5(\lambda - \lambda_n)\varphi_1(\xi_2 - 0,\lambda_n)\psi_1(\xi_2 - 0,\lambda) \\
&\quad = (\lambda - \lambda_n)\rho_1 \int_a^{\xi_1} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\qquad + (\lambda - \lambda_n)\rho_2 \int_{\xi_1}^{\xi_2} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\qquad + (\lambda - \lambda_n)\rho_3 \int_{\delta_2}^{b} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx,
\end{aligned}$$

or

$$\begin{aligned}
\frac{\Delta(\lambda)}{\lambda - \lambda_n} &= \rho_1 \int_a^{\xi_1} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\quad + \rho_2 \int_{\xi_1}^{\xi_2} \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx \\
&\quad + \rho_3 \big[\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)\big]\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\alpha_1'\psi_1(a,\lambda)-\alpha_2'\psi_2(a,\lambda))(\alpha_1'\varphi_1(a,\lambda_n)-\alpha_2'\varphi_2(a,\lambda_n))}{d_1}\\
&+\frac{(\gamma_2'\psi_2(b,\lambda)-\gamma_1'\psi_1(b,\lambda))(\gamma_2'\varphi_2(b,\lambda_n)-\gamma_1'\varphi_1(b,\lambda_n))}{d_2}\\
&+\alpha_3\varphi_1(\xi_1-0,\lambda_n)\psi_1(\xi_1-0,\lambda)\\
&+\alpha_5\varphi_1(\xi_2-0,\lambda_n)\psi_1(\xi_2-0,\lambda).
\end{aligned}
$$

For $\lambda \to \lambda_n$, $-\dot{\Delta}(\lambda_n) = \kappa_n\mu_n$ is obtained by using the equality $\psi(x,\lambda_n) = \kappa_n\varphi(x,\lambda_n)$ and (12). □

From Lemma 4, we see that $\dot{\Delta}(\lambda_n) \neq 0$. Thus, the eigenvalues of problem L are simple.

Lemma 5 (cf. [48]) *Let $\{\alpha_i\}_{i=1}^p$ be the set of real numbers satisfying the inequalities $\alpha_0 > \cdots > \alpha_{p-1} > 0$ and $\{a_i\}_{i=1}^p$ be the set of complex numbers. If $a_p \neq 0$ then the roots of the equation $e^{\alpha_0\lambda} + a_1e^{\alpha_1\lambda} + \cdots + a_{p-1}e^{\alpha_{p-1}\lambda} + a_p = 0$ have the form*

$$\lambda_n = \frac{2\pi ni}{\alpha_0} + \Psi(n) \quad (n = 0, \pm 1, \ldots),$$

where $\Psi(n)$ is a bounded sequence.

Now, from Lemma 2 and (11), we can note that

$$\Delta(\lambda) - \Delta_0(\lambda) = o\big(|\lambda|^4 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(b-\xi_2)\rho_3)}\big),$$

where

$$
\begin{aligned}
\Delta_0(\lambda) &= \lambda^4 \sin\lambda\rho_2(\xi_2-\xi_1)\big[\gamma_1'\sin\lambda\rho_3(b-\xi_2) + \gamma_2'\cos\lambda\rho_3(b-\xi_2)\big]\\
&\quad\times\big[\alpha_2'\cos\lambda\rho_1(\xi_1-a) - \alpha_1'\sin\lambda\rho_1(\xi_1-a)\big].
\end{aligned}
$$

On the other hand, we can see non-zero roots, namely the λ_n^0 of the equation $\Delta_0(\lambda) = 0$ are real and simple.

Furthermore, it can be proved by using Lemma 5 that

$$\lambda_n^0 = \frac{n\pi}{(\xi_1-a)\rho_1 + (\xi_2-\xi_1)\rho_2 + (b-\xi_2)\rho_3} + \Psi_n, \quad \sup_n|\Psi_n| < \infty, n = 0, \mp 1, \mp 2, \ldots. \tag{13}$$

Theorem 1 *The eigenvalues $\{\lambda_n\}$ which are located on the positive side of the real axis satisfy the following asymptotic behavior*:

$$\lambda_n = \lambda_{n-4}^0 + o(1), \quad n \to \infty. \tag{14}$$

Proof Denote

$$G_n := \big\{\lambda : 0 \le \operatorname{Re}\lambda \le \lambda_n^0 - \delta, |\operatorname{Im}\lambda| \le \lambda_n^0 - \delta, n = 0,1,2,\ldots\big\} \cup \big\{\lambda : |\lambda| < \delta\big\},$$

where δ is a sufficiently small number. The relations

$$\big|\Delta_0(\lambda)\big| \ge C|\lambda|^4 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(b-\xi_2)\rho_3)}$$

and

$$\Delta(\lambda) - \Delta_0(\lambda) = o\left(|\lambda|^4 e^{|\operatorname{Im} k|((\xi_1 - a)\rho_1 + (\xi_2 - \xi_1)\rho_2 + (b - \xi_2)\rho_3)}\right)$$

are valid for $\lambda \in \partial G_n$. Then, by Rouche's theorem, we see that the number of zeros of $\Delta_0(\lambda)$ coincides with the number of zeros of $\Delta(\lambda)$ in G_n, namely $n + 4$ zeros, $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{n+3}$. In the annulus between G_n and G_{n+1}, $\Delta(\lambda)$ has accurately one zero, namely $k_n : k_n = \lambda_n^0 + \delta_n$, for $n \geq 1$. So, it follows that $\lambda_{n+4} = k_n$. Applying Rouche's theorem in $\eta_\varepsilon = \{\lambda : |\lambda - \lambda_n^0| \leq \varepsilon\}$ for sufficiently small ε and sufficiently large n, we get $\delta_n = o(1)$. Finally, we obtain the asymptotic formula $\lambda_n = \lambda_{n-4}^0 + o(1)$. □

3 Construction of Green's function

In this section, we get the resolvent of the boundary value problem (1)-(7) for λ, not an eigenvalue. Hence, we find the solution of the non-homogeneous differential equation

$$\rho(x)By'(x) + \Omega(x)y(x) = \lambda y(x) + f(x), \quad x \in \Lambda, \tag{15}$$

which satisfies the conditions (2)-(7).

We can find the general solution of homogeneous differential equation

$$\rho(x)By'(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in \Lambda,$$

in the form

$$U_1(x,\lambda) = \begin{pmatrix} c_1\varphi_{11}(x,\lambda) + c_2\chi_{11}(x,\lambda) \\ c_1\varphi_{12}(x,\lambda) + c_2\chi_{12}(x,\lambda) \end{pmatrix}, \quad [a, \xi_1),$$

$$U_2(x,\lambda) = \begin{pmatrix} c_3\varphi_{21}(x,\lambda) + c_4\chi_{21}(x,\lambda) \\ c_3\varphi_{22}(x,\lambda) + c_4\chi_{22}(x,\lambda) \end{pmatrix}, \quad (\xi_1, \xi_2),$$

$$U_3(x,\lambda) = \begin{pmatrix} c_5\varphi_{31}(x,\lambda) + c_6\chi_{31}(x,\lambda) \\ c_5\varphi_{32}(x,\lambda) + c_6\chi_{32}(x,\lambda) \end{pmatrix}, \quad (\xi_2, b],$$

where c_i, $i = \overline{1,6}$ are arbitrary constants. By using the method of variation of parameters, we shall investigate the general solution of the non-homogeneous linear differential equation (15) in the form

$$U_1(x,\lambda) = \begin{pmatrix} c_1(x,\lambda)\varphi_{11}(x,\lambda) + c_2(x,\lambda)\chi_{11}(x,\lambda) \\ c_1(x,\lambda)\varphi_{12}(x,\lambda) + c_2(x,\lambda)\chi_{12}(x,\lambda) \end{pmatrix}, \quad \text{for } x \in [a, \xi_1),$$

$$U_2(x,\lambda) = \begin{pmatrix} c_3(x,\lambda)\varphi_{21}(x,\lambda) + c_4(x,\lambda)\chi_{21}(x,\lambda) \\ c_3(x,\lambda)\varphi_{22}(x,\lambda) + c_4(x,\lambda)\chi_{22}(x,\lambda) \end{pmatrix}, \quad \text{for } x \in (\xi_1, \xi_2), \tag{16}$$

$$U_3(x,\lambda) = \begin{pmatrix} c_5(x,\lambda)\varphi_{31}(x,\lambda) + c_6(x,\lambda)\chi_{31}(x,\lambda) \\ c_5(x,\lambda)\varphi_{32}(x,\lambda) + c_6(x,\lambda)\chi_{32}(x,\lambda) \end{pmatrix}, \quad \text{for } x \in (\xi_2, b],$$

where the functions $c_i(x,\lambda)$ ($i = \overline{1\text{-}6}$) satisfy the following linear system of equations:

$$\begin{pmatrix} c_1'(x,\lambda)\varphi_{11}(x,\lambda) + c_2'(x,\lambda)\chi_{11}(x,\lambda) = f_1(x) \\ c_1'(x,\lambda)\varphi_{12}(x,\lambda) + c_2'(x,\lambda)\chi_{12}(x,\lambda) = f_2(x) \end{pmatrix}, \quad \text{for } x \in [a, \xi_1),$$

$$\begin{pmatrix} c_3'(x,\lambda)\varphi_{21}(x,\lambda) + c_4'(x,\lambda)\chi_{21}(x,\lambda) = f_1(x) \\ c_3'(x,\lambda)\varphi_{22}(x,\lambda) + c_4'(x,\lambda)\chi_{22}(x,\lambda) = f_2(x) \end{pmatrix}, \quad \text{for } x \in (\xi_1, \xi_2),$$

$$\begin{pmatrix} c_5'(x,\lambda)\varphi_{31}(x,\lambda) + c_6'(x,\lambda)\chi_{31}(x,\lambda) = f_1(x) \\ c_5'(x,\lambda)\varphi_{32}(x,\lambda) + c_6'(x,\lambda)\chi_{32}(x,\lambda) = f_2(x) \end{pmatrix}, \quad \text{for } x \in (\xi_2, b].$$

Since λ is not an eigenvalue, each of the linear system of equations has a unique solution. Thus,

$$\begin{vmatrix} \varphi_{11}(x,\lambda) & \chi_{11}(x,\lambda) \\ \varphi_{12}(x,\lambda) & \chi_{12}(x,\lambda) \end{vmatrix} \neq 0, \qquad \begin{vmatrix} \varphi_{21}(x,\lambda) & \chi_{21}(x,\lambda) \\ \varphi_{22}(x,\lambda) & \chi_{22}(x,\lambda) \end{vmatrix} \neq 0,$$

and

$$\begin{vmatrix} \varphi_{31}(x,\lambda) & \chi_{31}(x,\lambda) \\ \varphi_{32}(x,\lambda) & \chi_{32}(x,\lambda) \end{vmatrix} \neq 0.$$

It is obvious that

$$\begin{aligned}
c_1(x,\lambda) &= \frac{1}{\Delta_1(\lambda)} \int_x^{\xi_1} \bigl(\chi_{11}(t,\lambda)f_2(t) - \chi_{12}(t,\lambda)f_1(t)\bigr)\,dt + c_1, \\
c_2(x,\lambda) &= \frac{1}{\Delta_1(\lambda)} \int_a^{x} \bigl(\varphi_{11}(t,\lambda)f_2(t) - \varphi_{12}(t,\lambda)f_1(t)\bigr)\,dt + c_2, \\
c_3(x,\lambda) &= \frac{1}{\Delta_2(\lambda)} \int_x^{\xi_2} \bigl(\chi_{21}(t,\lambda)f_2(t) - \chi_{22}(t,\lambda)f_1(t)\bigr)\,dt + c_3, \\
c_4(x,\lambda) &= \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{x} \bigl(\varphi_{21}(t,\lambda)f_2(t) - \varphi_{22}(t,\lambda)f_1(t)\bigr)\,dt + c_4, \\
c_5(x,\lambda) &= \frac{1}{\Delta_3(\lambda)} \int_x^{b} \bigl(\chi_{31}(t,\lambda)f_2(t) - \chi_{32}(t,\lambda)f_1(t)\bigr)\,dt + c_5, \\
c_6(x,\lambda) &= \frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{x} \bigl(\varphi_{31}(t,\lambda)f_2(t) - \varphi_{32}(t,\lambda)f_1(t)\bigr)\,dt + c_6,
\end{aligned}$$

where c_i, $i = \overline{1,6}$ are arbitrary constants. Substituting these above expressions in (16), then we obtain the general solution of non-homogeneous linear differential equation (15) in the form

$$\begin{aligned}
\text{for } x \in [a,\xi_1), \quad U_1(x,\lambda) = {} & \frac{1}{\Delta_1(\lambda)} \int_x^{\xi_1} \bigl(\chi_{11}(t,\lambda)f_2(t) - \chi_{12}(t,\lambda)f_1(t)\bigr)\varphi_{11}(x,\lambda)\,dt \\
& + \frac{1}{\Delta_1(\lambda)} \int_x^{\xi_1} \bigl(\chi_{11}(t,\lambda)f_2(t) - \chi_{12}(t,\lambda)f_1(t)\bigr)\varphi_{12}(x,\lambda)\,dt \\
& + \frac{1}{\Delta_1(\lambda)} \int_a^{x} \bigl(\varphi_{11}(t,\lambda)f_2(t) - \varphi_{12}(t,\lambda)f_1(t)\bigr)\chi_{11}(x,\lambda)\,dt \\
& + \frac{1}{\Delta_1(\lambda)} \int_a^{x} \bigl(\varphi_{11}(t,\lambda)f_2(t) - \varphi_{12}(t,\lambda)f_1(t)\bigr)\chi_{12}(x,\lambda)\,dt \\
& + c_1\varphi_{11}(x,\lambda) + c_2\chi_{11}(x,\lambda) + c_1\varphi_{12}(x,\lambda) + c_2\chi_{12}(x,\lambda),
\end{aligned}$$

$$\begin{aligned}
\text{for } x \in (\xi_1, \xi_2), \quad U_2(x,\lambda) &= \frac{1}{\Delta_2(\lambda)} \int_x^{\xi_2} \bigl(\chi_{21}(t,\lambda) f_2(t) - \chi_{22}(t,\lambda) f_1(t)\bigr) \varphi_{21}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_2(\lambda)} \int_x^{\xi_2} \bigl(\chi_{21}(t,\lambda) f_2(t) - \chi_{22}(t,\lambda) f_1(t)\bigr) \varphi_{22}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{x} \bigl(\varphi_{21}(t,\lambda) f_2(t) - \varphi_{22}(t,\lambda) f_1(t)\bigr) \chi_{21}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{x} \bigl(\varphi_{21}(t,\lambda) f_2(t) - \varphi_{22}(t,\lambda) f_1(t)\bigr) \chi_{22}(x,\lambda)\,dt \\
&+ c_3 \varphi_{21}(x,\lambda) + c_4 \chi_{21}(x,\lambda) + c_3 \varphi_{22}(x,\lambda) + c_4 \chi_{22}(x,\lambda), \\
\text{for } x \in (\xi_2, b], \quad U_3(x,\lambda) &= \frac{1}{\Delta_3(\lambda)} \int_x^{b} \bigl(\chi_{31}(t,\lambda) f_2(t) - \chi_{32}(t,\lambda) f_1(t)\bigr) \varphi_{31}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_3(\lambda)} \int_x^{b} \bigl(\chi_{31}(t,\lambda) f_2(t) - \chi_{32}(t,\lambda) f_1(t)\bigr) \varphi_{32}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{x} \bigl(\varphi_{31}(t,\lambda) f_2(t) - \varphi_{32}(t,\lambda) f_1(t)\bigr) \chi_{31}(x,\lambda)\,dt \\
&+ \frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{x} \bigl(\varphi_{31}(t,\lambda) f_2(t) - \varphi_{32}(t,\lambda) f_1(t)\bigr) \chi_{32}(x,\lambda)\,dt \\
&+ c_5 \varphi_{31}(x,\lambda) + c_6 \chi_{31}(x,\lambda) + c_5 \varphi_{32}(x,\lambda) + c_6 \chi_{32}(x,\lambda). \qquad (17)
\end{aligned}$$

Therefore, we easily see that $l_1(U_1) = -c_2 \Delta_1(\lambda)$, $l_2(U_3) = c_5 \Delta_3(\lambda)$. Since $\Delta_1(\lambda) \neq 0$, $\Delta_2(\lambda) \neq 0$, and from the boundary conditions (2)-(3), $c_2 = 0$, and $c_5 = 0$.

On the other hand, considering these results and transmission conditions (4)-(7), the following linear equation system according to the variables c_1, c_3, c_4, and c_6 is acquired:

$$\begin{aligned}
&-c_1 \varphi_{21}(\xi_1 + 0) + c_3 \varphi_{21}(\xi_1 + 0) + c_4 \chi_{21}(\xi_1 + 0) \\
&\quad = -\frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{\xi_2} \bigl(\chi_{21}(t,\lambda) f_2(t) - \chi_{22}(t,\lambda) f_1(t)\bigr) \varphi_{21}(\xi_1 + 0,\lambda)\,dt \\
&\qquad + \frac{1}{\Delta_1(\lambda)} \int_a^{\xi_1} \bigl(\varphi_{11}(t,\lambda) f_2(t) - \varphi_{12}(t,\lambda) f_1(t)\bigr) \chi_{21}(\xi_1 + 0,\lambda)\,dt, \\
&-c_1 \varphi_{22}(\xi_1 + 0) + c_3 \varphi_{22}(\xi_1 + 0) + c_4 \chi_{22}(\xi_1 + 0) \\
&\quad = -\frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{\xi_2} \bigl(\chi_{21}(t,\lambda) f_2(t) - \chi_{22}(t,\lambda) f_1(t)\bigr) \varphi_{22}(\xi_1 + 0,\lambda)\,dt \\
&\qquad + \frac{1}{\Delta_1(\lambda)} \int_a^{\xi_1} \bigl(\varphi_{11}(t,\lambda) f_2(t) - \varphi_{12}(t,\lambda) f_1(t)\bigr) \chi_{22}(\xi_1 + 0,\lambda)\,dt, \\
&-c_3 \varphi_{31}(\xi_2 + 0) - c_4 \chi_{31}(\xi_2 + 0) + c_6 \chi_{31}(\xi_2 + 0) \\
&\quad = -\frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{b} \bigl(\chi_{31}(t,\lambda) f_2(t) - \chi_{32}(t,\lambda) f_1(t)\bigr) \varphi_{31}(\xi_2 + 0,\lambda)\,dt \\
&\qquad + \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{\xi_2} \bigl(\varphi_{21}(t,\lambda) f_2(t) - \varphi_{22}(t,\lambda) f_1(t)\bigr) \chi_{31}(\xi_2 + 0,\lambda)\,dt, \\
&-c_3 \varphi_{32}(\xi_2 + 0) - c_4 \chi_{32}(\xi_2 + 0) + c_6 \chi_{32}(\xi_2 + 0)
\end{aligned}$$

$$= -\frac{1}{\Delta_3(\lambda)}\int_{\xi_2}^{b}\big(\chi_{31}(t,\lambda)f_2(t) - \chi_{32}(t,\lambda)f_1(t)\big)\varphi_{32}(\xi_2+0,\lambda)\,dt$$
$$+ \frac{1}{\Delta_2(\lambda)}\int_{\xi_1}^{\xi_2}\big(\varphi_{21}(t,\lambda)f_2(t) - \varphi_{22}(t,\lambda)f_1(t)\big)\chi_{32}(\xi_2+0,\lambda)\,dt. \tag{18}$$

Recalling the definitions of the solutions $\varphi_{ij}(x,\lambda)$ and $\chi_{ij}(x,\lambda)$ ($i = 2,3, j = 1,2$), the following relation is gotten for the determinant of this linear equation system:

$$\begin{vmatrix} -\varphi_{21}(\xi_1+0) & \varphi_{21}(\xi_1+0) & \chi_{21}(\xi_1+0) & 0 \\ -\varphi_{22}(\xi_1+0) & \varphi_{22}(\xi_1+0) & \chi_{22}(\xi_1+0) & 0 \\ 0 & -\varphi_{31}(\xi_2+0) & -\chi_{31}(\xi_2+0) & \chi_{31}(\xi_2+0) \\ 0 & -\varphi_{32}(\xi_2+0) & -\chi_{32}(\xi_2+0) & \chi_{32}(\xi_2+0) \end{vmatrix} = -\Delta_2(\lambda)\Delta_3(\lambda).$$

Since the above determinant is different from zero, the solution of (18) is unique. When we solve system (18), we obtain the following equality for the coefficients c_1, c_3, c_4, and c_6:

$$c_1 = \frac{1}{\Delta_2(\lambda)}\int_{\xi_1}^{\xi_2}\big(\chi_{21}(t,\lambda)f_2(t) - \chi_{22}(t,\lambda)f_1(t)\big)\,dt$$
$$+ \frac{1}{\Delta_3(\lambda)}\int_{\xi_2}^{b}\big(\chi_{31}(t,\lambda)f_2(t) - \chi_{32}(t,\lambda)f_1(t)\big)\,dt,$$
$$c_3 = \frac{1}{\Delta_3(\lambda)}\int_{\xi_2}^{b}\big(\chi_{31}(t,\lambda)f_2(t) - \chi_{32}(t,\lambda)f_1(t)\big)\,dt,$$
$$c_4 = \frac{1}{\Delta_1(\lambda)}\int_{a}^{\xi_1}\big(\varphi_{11}(t,\lambda)f_2(t) - \varphi_{12}(t,\lambda)f_1(t)\big)\,dt,$$
$$c_6 = \frac{1}{\Delta_1(\lambda)}\int_{a}^{\xi_1}\big(\varphi_{11}(t,\lambda)f_2(t) - \varphi_{12}(t,\lambda)f_1(t)\big)\,dt$$
$$+ \frac{1}{\Delta_2(\lambda)}\int_{\xi_1}^{\xi_2}\big(\varphi_{21}(t,\lambda)f_2(t) - \varphi_{22}(t,\lambda)f_1(t)\big)\,dt.$$

As a result, if we substitute the coefficients c_i ($i = 1,3,4,6$) in (17), then we get the resolvent $U(x,\lambda)$ as follows:

$$U(x,\lambda) = \frac{\chi(x,\lambda)}{\Delta_i(\lambda)}\int_a^x (f_2\varphi_{i1} - f_1\varphi_{i2})\,dt + \frac{\varphi(x,\lambda)}{\Delta_i(\lambda)}\int_x^b (f_2\chi_{i1} - f_1\chi_{i2})\,dt, \quad i = \overline{1,3} \tag{19}$$

such that

$$\varphi(x,\lambda) = \begin{cases} \binom{\varphi_{11}(x,\lambda)}{\varphi_{12}(x,\lambda)}, & x \in [a,\xi_1), \\ \binom{\varphi_{21}(x,\lambda)}{\varphi_{22}(x,\lambda)}, & x \in (\xi_1,\xi_2), \\ \binom{\varphi_{31}(x,\lambda)}{\varphi_{32}(x,\lambda)}, & x \in (\xi_2,b], \end{cases}$$

$$\chi(x,\lambda) = \begin{cases} \binom{\chi_{11}(x,\lambda)}{\chi_{12}(x,\lambda)}, & x \in [a,\xi_1), \\ \binom{\chi_{21}(x,\lambda)}{\chi_{22}(x,\lambda)}, & x \in (\xi_1,\xi_2), \\ \binom{\chi_{31}(x,\lambda)}{\chi_{32}(x,\lambda)}, & x \in (\xi_2,b]. \end{cases}$$

We can easily find the Green's function from the resolvent (19) as follows:

$$G(x,t;\lambda) = \begin{cases} \frac{\chi(x,\lambda)}{\Delta_i(\lambda)} \begin{pmatrix} -\varphi_{i2}(x,\lambda) \\ \varphi_{i1}(x,\lambda) \end{pmatrix}^T, & a \le t \le x \le b, x \ne \xi_1, \xi_2, t \ne \xi_1, \xi_2, \\ \frac{\varphi(x,\lambda)}{\Delta_i(\lambda)} \begin{pmatrix} -\chi_{i2}(x,\lambda) \\ \chi_{i1}(x,\lambda) \end{pmatrix}^T, & a \le t \le x \le b, x \ne \xi_1, \xi_2, t \ne \xi_1, \xi_2. \end{cases}$$

We can rewrite equation (19) in the following form:

$$U(x,\lambda) = \int_a^b G(x,t;\lambda) f(t)\,dt \quad \text{such that } f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

Now, we define the resolvent operator

$$R(T,\lambda) := (T - \lambda I)^{-1} : H_1 \to H_1.$$

It is easy to see that the operator equation $(T - \lambda I)Y = F$, $F \in H_1$ is equivalent to the boundary value problem (15), (2)-(7) where λ is not an eigenvalue,

$$Y = \bigl(y_1(x), y_2(x), y_3, y_4, y_5, y_6\bigr)^T \quad \text{such that } y_3 = \alpha_1' y_1(a) - \alpha_2' y_2(a),$$
$$y_4 = \gamma_1' y_1(b) - \gamma_2' y_2(b), \qquad y_5 = y_1(\xi_1 - 0), \qquad y_6 = y_1(\xi_2 - 0) \quad \text{and}$$
$$F = \bigl(f_1(x), f_2(x), z_3, z_4, z_5, z_6\bigr)^T \quad \text{where } z_3 = z_4 = z_5 = z_6 = 0.$$

4 Inverse problems

In this section, we study the inverse problems for the reconstruction of boundary value problem (1)-(7) by the Weyl function and spectral data.

We consider the boundary value problem $\tilde{L}$ which has the same form as L but with different coefficients $\tilde{\Omega}(x)$, $\tilde{\alpha}_j$, $\tilde{\gamma}_j$, $\tilde{\alpha}_j'$, $\tilde{\gamma}_j'$, $j = 1, 2$, such that

$$\tilde{\Omega}(x) = \begin{pmatrix} \tilde{p}(x) & q(x) \\ q(x) & \tilde{r}(x) \end{pmatrix}.$$

If a certain symbol σ denotes an object related to L, then the symbol $\tilde{\sigma}$ denotes the corresponding object related to $\tilde{L}$.

Let $\Phi(x,\lambda)$ be a solution of equation (1) which satisfies the condition $(\lambda\alpha_2' - \alpha_2)\Phi_2(a,\lambda) - (\lambda\alpha_1' - \alpha_1)\Phi_1(a,\lambda) = 1$ and the transmissions (4)-(7).

Assume that the function $\phi(x,\lambda) = (\phi_1(x,\lambda), \phi_2(x,\lambda))^T$ is the solution of equation (1) that satisfies the conditions $\phi_1(a,\lambda) = d_1^{-1}\alpha_2'$, $\phi_2(a,\lambda) = d_1^{-1}\alpha_1'$ and the transmission conditions (4)-(7).

Since $W[\varphi,\phi] = 1$, the functions ϕ and φ are linearly independent. Therefore, the function $\psi(x,\lambda)$ can be represented by

$$\psi(x,\lambda) = d_1^{-1}\bigl(\alpha_1'\psi_1(a,\lambda) - \alpha_2'\psi_2(a,\lambda)\bigr)\varphi(x,\lambda) + \Delta(\lambda)\phi(x,\lambda)$$

or

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)} = \phi(x,\lambda) + \frac{\alpha_1'\psi_1(a,\lambda) - \alpha_2'\psi_2(a,\lambda)}{d_1\Delta(\lambda)}\varphi(x,\lambda), \tag{20}$$

which is called the Weyl solution, and

$$\frac{\alpha_1'\psi_1(a,\lambda)-\alpha_2'\psi_2(a,\lambda)}{d_1\Delta(\lambda)} = M(\lambda) = d_1^{-1}\big(\alpha_1'\Phi_1(a,\lambda)-\alpha_2'\Phi_2(a,\lambda)\big) \tag{21}$$

is called the Weyl function.

Lemma 6 *The following representation is true*:

$$M(\lambda) = \sum_{n=-\infty}^{\infty}\frac{1}{\mu_n(\lambda_n-\lambda)}.$$

Proof The Weyl function $M(\lambda)$ is a meromorphic function with respect to λ, which has simple poles at λ_n. Therefore, we calculate

$$\operatorname{Re}s_{\lambda=\lambda_n}M(\lambda) = \frac{\alpha_1'\psi_1(a,\lambda_n)-\alpha_2'\psi_2(a,\lambda_n)}{d_1\dot{\Delta}(\lambda_n)}.$$

Since

$$\kappa_n = \frac{\alpha_1'\psi_1(a,\lambda_n)-\alpha_2'\psi_2(a,\lambda_n)}{d_1}$$

and $\dot{\Delta}(\lambda_n) = -\kappa_n\mu_n$,

$$\operatorname{Re}s_{\lambda=\lambda_n}M(\lambda) = -\frac{1}{\mu_n}. \tag{22}$$

Let $\Gamma_n = \{\lambda : |\lambda| \le |\lambda_n^o| + \varepsilon\}$, where ε is a sufficiently small number. Consider the contour integral $I_n(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)}{\mu-\lambda}\,d\mu$, $\lambda \in \operatorname{int}\Gamma_n$.

Since $\Delta(\lambda) \ge C_\delta\lambda^4 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(b-\xi_2)\rho_3)}$ and $M(\lambda) = \frac{\alpha_1'\psi_1(a,\lambda)-\alpha_2'\psi_2(a,\lambda)}{d_1\Delta(\lambda)}$, $|M(\lambda)| \le \frac{C_\delta}{|\lambda|}$ for $\lambda \in F_\delta = \{\lambda : |\lambda-\lambda_n| \ge \delta, n = 0, \pm 1, \ldots\}$, where δ is a sufficiently small number. Thus, $\lim_{n\to\infty} I_n(\lambda) = 0$. Then the residue theorem yields

$$M(\lambda) = \sum_{n=-\infty}^{\infty}\frac{1}{\mu_n(\lambda_n-\lambda)}.$$

□

Theorem 2 *If* $M(\lambda) = \tilde{M}(\lambda)$, *then* $L = \tilde{L}$, *i.e.*, $\Omega(x) = \tilde{\Omega}(x)$, *a.e. and* $\alpha_j = \tilde{\alpha}_j$, $\gamma_j = \tilde{\gamma}_j$, $\alpha_j' = \tilde{\alpha}_j'$, $\gamma_j' = \tilde{\gamma}_j'$, $j = 1, 2$.

Proof We introduce a matrix $P(x,\lambda) = [P_{kj}(x,\lambda)]_{k,j=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix}\tilde{\varphi}_1 & \tilde{\Phi}_1\\ \tilde{\varphi}_2 & \tilde{\Phi}_2\end{pmatrix} = \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_2 & \Phi_2\end{pmatrix} \tag{23}$$

or

$$\begin{pmatrix}P_{11}(x,\lambda) & P_{12}(x,\lambda)\\ P_{21}(x,\lambda) & P_{22}(x,\lambda)\end{pmatrix} = \begin{pmatrix}\varphi_1\tilde{\Phi}_2-\Phi_1\tilde{\varphi}_2 & -\varphi_1\tilde{\Phi}_1+\Phi_1\tilde{\varphi}_1\\ \varphi_2\tilde{\Phi}_2-\tilde{\varphi}_2\Phi_2 & -\varphi_2\tilde{\Phi}_1+\tilde{\varphi}_1\Phi_2\end{pmatrix}, \tag{24}$$

where $\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}$ and $W(\tilde{\Phi},\tilde{\varphi}) = 1$. Thus, we find

$$\begin{aligned}
P_{11}(x,\lambda) &= \varphi_1(x,\lambda)\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(x,\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\tilde{\varphi}_2(x,\lambda), \\
P_{12}(x,\lambda) &= -\varphi_1(x,\lambda)\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} + \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\tilde{\varphi}_1(x,\lambda), \\
P_{21}(x,\lambda) &= \varphi_2(x,\lambda)\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\tilde{\varphi}_2(x,\lambda), \\
P_{22}(x,\lambda) &= -\varphi_2(x,\lambda)\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} + \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\tilde{\varphi}_1(x,\lambda).
\end{aligned} \tag{25}$$

On the other hand, from (20), we get

$$\begin{aligned}
P_{11}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{\phi}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\phi_1(x,\lambda) + \big(\tilde{M}(\lambda) - M(\lambda)\big)\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda), \\
P_{12}(x,\lambda) &= -\varphi_1(x,\lambda)\tilde{\phi}_1(x,\lambda) + \tilde{\varphi}_1(x,\lambda)\phi_1(x,\lambda) - \big(\tilde{M}(\lambda) - M(\lambda)\big)\varphi_1(x,\lambda)\tilde{\varphi}_1(x,\lambda), \\
P_{21}(x,\lambda) &= \varphi_2(x,\lambda)\tilde{\phi}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\phi_2(x,\lambda) + \big(\tilde{M}(\lambda) - M(\lambda)\big)\varphi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda), \\
P_{22}(x,\lambda) &= -\varphi_2(x,\lambda)\tilde{\phi}_1(x,\lambda) + \tilde{\varphi}_1(x,\lambda)\phi_2(x,\lambda) - \big(\tilde{M}(\lambda) - M(\lambda)\big)\tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda).
\end{aligned} \tag{26}$$

Thus, if $M(\lambda) \equiv \tilde{M}(\lambda)$ then the functions $P_{ij}(x,\lambda)$ $(i,j = 1,2)$ are entire in λ for each fixed x. Moreover, since asymptotic behaviors of $\varphi_i(x,\lambda)$, $\tilde{\varphi}_i(x,\lambda)$, $\psi_i(x,\lambda)$, $\tilde{\psi}_i(x,\lambda)$, and $|\Delta(\lambda)| \geq C_\delta|\lambda|^4 e^{|\operatorname{Im}\lambda|((\xi_1-a)\rho_1+(\xi_2-\xi_1)\rho_2+(b-\xi_2)\rho_3)}$ in $F_\delta \cap \tilde{F}_\delta$, we can easily see that the functions $P_{ij}(x,\lambda)$ are bounded with respect to λ. As a result, these functions do not depend on λ. Here, we denote $\tilde{F}_\delta = \{\lambda : |\lambda - \tilde{\lambda}_n| \geq \delta, n = 0, \pm1, \pm2, \ldots\}$ where n is sufficiently small number, $\tilde{\lambda}_n$ are eigenvalues of the problem $\tilde{L}$.

From (25), we get

$$\begin{aligned}
P_{11}(x,\lambda) - 1 &= \frac{\tilde{\psi}_2(x,\lambda)(\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda))}{\tilde{\Delta}(\lambda)} - \tilde{\varphi}_2(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right), \\
P_{12}(x,\lambda) &= \frac{\psi_1(x,\lambda)(\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda))}{\Delta(\lambda)} + \varphi_1(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right), \\
P_{21}(x,\lambda) &= \varphi_2(x,\lambda)\left(\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right) + \psi_2(x,\lambda)\left(\frac{\varphi_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)}{\Delta(\lambda)}\right), \\
P_{22}(x,\lambda) - 1 &= \frac{\psi_2(x,\lambda)(\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda))}{\Delta(\lambda)} + \varphi_2(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right).
\end{aligned}$$

It follows from the representations of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, that

$$\lim_{\lambda\to\infty} \frac{\tilde{\psi}_2(x,\lambda)(\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda))}{\tilde{\Delta}(\lambda)} = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \tilde{\varphi}_2(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right) = 0$$

for all x in Λ. Thus, $\lim_{\lambda\to\infty}(P_{11}(x,\lambda) - 1) = 0$ is valid uniformly with respect to x. So we have $P_{11}(x,\lambda) \equiv 1$ and similarly $P_{12}(x,\lambda) \equiv 0$, $P_{21}(x,\lambda) \equiv 0$, $P_{22}(x,\lambda) \equiv 1$.

From (23), we obtain $\varphi_1(x,\lambda) \equiv \tilde{\varphi}_1(x,\lambda)$, $\Phi_1 \equiv \tilde{\Phi}_1$, $\varphi_2(x,\lambda) \equiv \tilde{\varphi}_2(x,\lambda)$, and $\Phi_2 \equiv \tilde{\Phi}_2$ for all x and λ. Moreover, from $\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}$, we get $\frac{\psi_2(x,\lambda)}{\psi_1(x,\lambda)} = \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\psi}_1(x,\lambda)}$. Hence, $\Omega(x) = \tilde{\Omega}(x)$, *i.e.*,

$p(x) = \tilde{p}(x)$, $r(x) = \tilde{r}(x)$ almost everywhere. On the other hand, since

$$\begin{pmatrix} \varphi_{11}(a,\lambda) \\ \varphi_{12}(a,\lambda) \end{pmatrix} = \begin{pmatrix} \lambda\alpha_2' - \alpha_2 \\ \lambda\alpha_1' - \alpha_1 \end{pmatrix}, \qquad \begin{pmatrix} \psi_{31}(b,\lambda) \\ \psi_{32}(b,\lambda) \end{pmatrix} = \begin{pmatrix} \lambda\gamma_2' + \gamma_2 \\ \lambda\gamma_1' + \gamma_1 \end{pmatrix},$$

we easily see that $\alpha_2' = \tilde{\alpha}_2'$, $\alpha_2 = \tilde{\alpha}_2$, $\alpha_1' = \tilde{\alpha}_1'$, $\alpha_1 = \tilde{\alpha}_1$, and $\gamma_2' = \tilde{\gamma}_2'$, $\gamma_2 = \tilde{\gamma}_2$, $\gamma_1' = \tilde{\gamma}_1'$, $\gamma_1 = \tilde{\gamma}_1$. Therefore, $L \equiv \tilde{L}$. □

Theorem 3 *If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all n, then $L \equiv \tilde{L}$, i.e., $\Omega(x) = \tilde{\Omega}(x)$, a.e., $\alpha_j = \tilde{\alpha}_j$, $\gamma_i = \tilde{\gamma}_i$, $\alpha_j' = \tilde{\alpha}_j'$, $\gamma_j' = \tilde{\gamma}_j'$, $j = 1, 2$. Hence, the problem* (1)-(7) *is uniquely determined by the spectral data $\{\lambda_n, \mu_n\}$.*

Proof If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all n, then $M(\lambda) = \tilde{M}(\lambda)$ by Lemma 6. Therefore, we get $L = \tilde{L}$ by Theorem 2. □

Let us consider the boundary value problem L_1 with the condition $\alpha_1' y_1(a,\lambda) - \alpha_2' y_2(a,\lambda) = 0$ instead of the condition (2) in L. Let $\{\tau_n\}_{n\in\mathbb{Z}}$ be the eigenvalues of the problem L_1. It is clear that the τ_n are zeros of $\Delta_1(\tau) := \alpha_1'\psi_1(a,\tau) - \alpha_2'\psi_2(a,\tau)$, which is the characteristic function of L_1.

Theorem 4 *If $\lambda_n = \tilde{\lambda}_n$ and $\tau_n = \tilde{\tau}_n$ for all n, then $L(\Omega, \gamma_i, \gamma_j') = L(\tilde{\Omega}, \tilde{\gamma}_i, \tilde{\gamma}_j')$, $j = 1, 2$.*

Hence, the problem L is uniquely determined by the sequences $\{\lambda_n\}$ and $\{\tau_n\}$ except coefficients α_j and α_j'.

Proof Since the characteristic functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are entire of order 1, the functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are uniquely determined up to a multiplicative constant with their zeros by Hadamard's factorization theorem [49],

$$\Delta(\lambda) = C \prod_{n=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right),$$
$$\Delta_1(\tau) = C_1 \prod_{n=-\infty}^{\infty} \left(1 - \frac{\tau}{\tau_n}\right),$$

where C and C_1 are constants depending on $\{\lambda_n\}$ and $\{\tau_n\}$, respectively. When $\lambda_n = \tilde{\lambda}_n$ and $\tau_n = \tilde{\tau}_n$ for all n, $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$ and $\Delta_1(\tau) \equiv \tilde{\Delta}_1(\tau)$. Hence, $\alpha_1'\psi_1(a,\tau) - \alpha_2'\psi_2(a,\tau) = \tilde{\alpha}_1'\tilde{\psi}_1(a,\tau) - \tilde{\alpha}_2'\tilde{\psi}_2(a,\tau)$. As a result, we get $M(\lambda) = \tilde{M}(\lambda)$ by (21). So, the proof is completed by Theorem 2. □

Competing interests

The author declares to have no competing interests.

Acknowledgements

The author cordially thanks the anonymous referee for his (her) valuable comments and suggestions, which lead to the improvement of this paper.

Received: 18 March 2016 Accepted: 1 July 2016 Published online: 20 July 2016

References

1. Levitan, BM, Sargsyan, IS: Sturm-Liouville and Dirac Operators. Nauka, Moscow (1988) (in Russian)
2. Berezanskii, YM: Uniqueness theorem in the inverse spectral problem for the Schrödinger equation. Tr. Mosk. Mat. Obŝ. **7**, 3-51 (1958)

3. Gasymov, MG, Dzhabiev, TT: Determination of a system of Dirac differential equations using two spectra. In: Proceedings of School-Seminar on the Spectral Theory of Operators and Representations of Group Theory, pp. 46-71. Elm, Baku (1975) (in Russian)
4. Marchenko, VA: Sturm-Liouville Operators and Their Applications. Naukova Dumka, Kiev (1977) (in Russian)
5. Nizhnik, LP: Inverse Scattering Problems for Hyperbolic Equations. Naukova Dumka, Kiev (1977) (in Russian)
6. Gasymov, MG: Inverse problem of the scattering theory for Dirac system of order 2*n*. Tr. Mosk. Mat. Obŝ. **19**, 41-112 (1968) (in Russian)
7. Guseinov, IM: On the representation of Jost solutions of a system of Dirac differential equations with discontinuous coefficients. Izv. Akad. Nauk Azerb. SSR **5**, 41-45 (1999)
8. Sat, M, Panakhov, ES: Spectral problem for diffusion operator. Appl. Anal. **93**(6), 1178-1186 (2014)
9. Sat, M, Panakhov, ES: A uniqueness theorem for Bessel operator from interior spectral data. Abstr. Appl. Anal. **2013**, Article ID 713654 (2013)
10. Fulton, CT: Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. Proc. R. Soc. Edinb. **77A**, 293-308 (1977)
11. Shkalikov, AA: Boundary value problems for ordinary differential equations with a parameter in boundary conditions. Tr. Semin. Im. I.G. Petrovskogo **9**, 190-229 (1983)
12. Yakubov, S: Completeness of Root Functions of Regular Differential Operators. Longman Scientific and Technical, New York (1994)
13. Kerimov, NB, Memedov, KK: On a boundary value problem with a spectral parameter in the boundary conditions. Sib. Mat. Zh. **40**(2), 325-335 (1999); English translation: Sib. Math. J. **40**(2), 281-290 (1999)
14. Binding, PA, Browne, PJ, Watson, BA: Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II. J. Comput. Appl. Math. **148**, 147-169 (2002)
15. Mukhtarov, OS, Kadakal, M, Muhtarov, FS: On discontinuous Sturm-Liouville problem with transmission conditions. J. Math. Kyoto Univ. **444**, 779-798 (2004)
16. Tunç, E, Muhtarov, OS: Fundamental solution and eigenvalues of one boundary value problem with transmission conditions. Appl. Comput. Math. **157**, 347-355 (2004)
17. Akdoğan, Z, Demirci, M, Mukhtarov, OS: Sturm-Liouville problems with eigendependent boundary and transmissions conditions. Acta Math. Sci. **25B**(4), 731-740 (2005)
18. Akdoğan, Z, Demirci, M, Mukhtarov, OS: Discontinuous Sturm-Liouville problem with eigenparameter-dependent boundary and transmission conditions. Acta Appl. Math. **86**, 329-344 (2005)
19. Fulton, CT: Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions. Proc. R. Soc. Edinb. A **87**, 1-34 (1980)
20. Amirov, RK, Ozkan, AS, Keskin, B: Inverse problems for impulsive Sturm-Liouville operator with spectral parameter linearly contained in boundary conditions. Integral Transforms Spec. Funct. **20**, 607-618 (2009)
21. Guliyev, NJ: Inverse eigenvalue problems for Sturm-Liouville equations with spectral parameter linearly contained in one of the boundary conditions. Inverse Probl. **21**, 1315-1330 (2005)
22. Mukhtarov, OS: Discontinuous boundary value problem with spectral parameter in boundary conditions. Turk. J. Math. **18**, 183-192 (1994)
23. Tuna, H, Eryılmaz, A: Dissipative Sturm-Liouville operators with transmission conditions. Abstr. Appl. Anal. **2013**, Article ID 248740 (2013)
24. Russakovskii, EM: Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions. Funct. Anal. Appl. **9**, 358-359 (1975)
25. Binding, PA, Browne, PJ, Seddighi, K: Sturm-Liouville problems with eigenparameter dependent boundary conditions. Proc. Edinb. Math. Soc. **37**, 57-72 (1993)
26. Russakovskii, EM: Matrix boundary value problems with eigenvalue dependent boundary conditions. In: Topics in Interpolation Theory. Oper. Theory Adv. Appl., vol. 95, pp. 453-462. Birkhäuser, Basel (1997)
27. Mennicken, R, Schmid, H, Shkalikov, AA: On the eigenvalue accumulation of Sturm-Liouville problems depending nonlinearly on the spectral parameter. Math. Nachr. **189**, 157-170 (1998)
28. Binding, PA, Browne, PJ, Watson, BA: Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions. J. Lond. Math. Soc. **62**, 161-182 (2000)
29. Schmid, H, Tretter, C: Singular Dirac systems and Sturm-Liouville problems nonlinear in the spectral parameter. J. Differ. Equ. **181**(2), 511-542 (2002)
30. Binding, PA, Browne, PJ, Watson, BA: Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. J. Math. Anal. Appl. **29**, 246-261 (2004)
31. Hald, OH: Discontinuous inverse eigenvalue problems. Commun. Pure Appl. Math. **37**, 539-577 (1984)
32. Kobayashi, M: A uniqueness proof for discontinuous inverse Sturm-Liouville problems with symmetric potentials. Inverse Probl. **5**, 767-781 (1989)
33. Shepelsky, D: The inverse problem of reconstruction of the medium's conductivity in a class of discontinuous and increasing functions. In: Spectral Operator Theory and Related Topics. Adv. Sov. Math., vol. 19, pp. 209-232. Am. Math. Soc., Providence (1994)
34. Amirov, RK, Güldü, Y: Inverse problems for Dirac operator with discontinuity conditions inside an interval. Int. J. Pure Appl. Math. **37**(2), 215-226 (2007)
35. Likov, AV, Mikhailov, YA: The Theory of Heat and Mass Transfer. Qosenergaizdat, Moscow (1963) (in Russian)
36. Meschanov, VP, Feldstein, AL: Automatic Design of Directional Couplers. Sviaz, Moscow (1980)
37. Tikhonov, AN, Samarskii, AA: Equations of Mathematical Physics. Pergamon, Oxford (1990)
38. McLaughlin, J, Polyakov, P: On the uniqueness of a spherical symmetric speed of sound from transmission eigenvalues. J. Differ. Equ. **107**, 351-382 (1994)
39. Voitovich, NN, Katsenelbaum, BZ, Sivov, AN: Generalized Method of Eigen-Vibration in the Theory of Diffraction. Nauka, Moscow (1997) (in Russian)
40. Titeux, I, Yakubov, Y: Completeness of root functions for thermal conduction in a strip with peicewise continuous coefficients. Math. Models Methods Appl. Sci. **7**, 1035-1050 (1997)
41. Yurko, VA: Integral transforms connected with discontinuous boundary value problems. Integral Transforms Spec. Funct. **10**, 141-164 (2000)

42. Freiling, G, Yurko, VA: Inverse Sturm-Liouville Problems and Their Applications. Nova Science, New York (2001)
43. Kadakal, M, Mukhtarov, OS: Sturm-Liouville problems with discontinuities at two points. Comput. Math. Appl. **54**, 1367-1379 (2007)
44. Yang, Q, Wang, W: Asymptotic behavior of a differential operator with discontinuities at two points. Math. Methods Appl. Sci. **34**, 373-383 (2011)
45. Shahriari, M, Akbarfam, AJ, Teschl, G: Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions. J. Math. Anal. Appl. **395**, 19-29 (2012)
46. Güldü, Y: Inverse eigenvalue problems for a discontinuous Sturm-Liouville operator with two discontinuities. Bound. Value Probl. **2013**, 209 (2013)
47. Yang, C-F: Uniqueness theorems for differential pencils with eigenparameter boundary conditions and transmission conditions. J. Differ. Equ. **255**, 2615-2635 (2013)
48. Zhdanovich, VF: Formulae for the zeros of Dirichlet polynomials and quasi-polynomials. Dokl. Akad. Nauk SSSR **135**(8), 1046-1049 (1960)
49. Titchmarsh, EC: The Theory of Functions. Oxford University Press, London (1939)