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New existence results for some periodic and Neumann-Steklov boundary value problems with ϕ-Laplacian

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Abstract

We study the existence of solutions of the quasilinear equation

$$(\phi(u'(t)))' = f(t, u(t), u'(t)), \quad \text{a.e. } t \in [0, T],$$

with periodic or nonlinear Neumann-Steklov boundary conditions, where $\phi :]-a, a[\to \mathbb{R}$ with $0 < a < +\infty$ is an increasing homeomorphism such that $\phi(0) = 0$. Combining some sign conditions and the lower and upper solution method, we obtain the existence of solutions when there exists one lower solution or one upper solution.

MSC: 34B15

Keywords: ϕ-Laplacian; sign conditions; lower and upper solutions

1 Introduction

This work is devoted to the study of the existence of solutions of the quasilinear equation

$$(\phi(u'(t)))' = f(t, u(t), u'(t)), \quad \text{a.e. } t \in [0, T], \tag{1}$$

with periodic boundary conditions

$$u'(0) = u'(T), \qquad u(0) = u(T), \tag{2}$$

or nonlinear Neumann-Steklov boundary conditions

$$\phi(u'(0)) = g_0(u(0)), \qquad \phi(u'(T)) = g_T(u(T)), \tag{3}$$

where $\phi :]-a, a[\to \mathbb{R}$ with $0 < a < +\infty$ is an increasing homeomorphism such that $\phi(0) = 0$, $g_0, g_T : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to be an L^1-Carathéodory function.

Generally, in the lower and upper solution method, to show the existence of a solution of a problem, we need the existence of at least one lower solution and at least one upper solution. In the case of the method of sign conditions, we usually need two sign conditions to show the existence of at least one solution of a problem.

In 2007, Bereanu and Mawhin [1] proved, for continuous f, the existence of solutions of problem (1)-(2) under some sign conditions (see [1] Theorem 2) and when there exist a lower solution and an upper solution, ordered or not, of problem (1)-(2) (see [1] Theorem 4).

In 2008, Bereanu and Mawhin [2] proved, for continuous f, the existence of solutions of problem (1)-(3) under some sign conditions (see [2] Theorem 2) and when there exist a lower solution and an upper solution, ordered or not, of problem (1)-(3) (see [2] Theorem 4).

In the following results, we prove the existence of solutions of (1)-(2) and (1)-(3) when we have only one sign condition and only one lower solution or only one upper solution.

After introducing notation and preliminary results in Section 2, in Section 3, combining one sign condition and the existence of only one lower solution or only one upper solution of problem (1)-(2), we prove the existence of at least one solution of problem (1)-(2).

In Section 4, combining one sign condition and the existence of only one lower solution or only one upper solution of problem (1)-(3), we prove the existence of at least one solution of problem (1)-(3).

The results of this section enable us to obtain that, for some forced relativistic pendulum equations with friction and Neumann-Steklov boundary conditions, the existence of a lower solution or the existence of an upper solution is sufficient to obtain the existence of a solution.

2 Notation and preliminaries

We denote:

- $C = C([0, T])$, the Banach space of continuous functions on $[0, T]$;
- $\|u\|_C = \|u\|_\infty = \max\{|u(t)|; t \in [0, T]\}$, the norm of C;
- $C^1 = C^1([0, T])$, the Banach space of continuous functions on $[0, T]$ having continuous first derivative on $[0, T]$;
- $\|u\|_{C^1} = \|u\|_C + \|u'\|_C$, the norm of C^1;
- $AC = AC([0, T])$, the set of absolutely continuous functions on $[0, T]$;
- $L^1 = L^1(0, T)$, the Banach space of Lebesgue-integrable functions on $[0, T]$;
- $\|x\|_{L^1} = \int_0^T |x(t)|\, dt$, the norm of L^1;
- B_r, the open ball of C^1 with center 0 and radius r;
- d_{LS}, the Leray-Schauder degree, and d_B, the Brouwer degree;
- $u_L = \min_{[0,T]} u$ and $u_M = \max_{[0,T]} u$ for $u \in C$;
- $\mathrm{Range}(u) = \{y \in \mathbb{R}; y = u(t) \text{ with } t \in [0, T]\}$ for $u \in C$.

We introduce:

- the continuous operators $P, K : C \to C$ defined by

$$P(u) = Pu = u(0) \quad \text{and} \quad K(u) = T^{-1}\big[g_T\big(u(T)\big) - g_0\big(u(0)\big)\big];$$

- the continuous operators $Q, H : L^1 \to C$ defined by

$$Q(u) = Qu = \frac{1}{T}\int_0^T u(s)\, ds \quad \text{and} \quad (Hu)(t) = \int_0^t u(s)\, ds, \quad \forall t \in [0, T].$$

Definition 2.1 $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ is L^1-Carathéodory if:

(i) $f(\cdot,x,y):[0,T]\to\mathbb{R}$ is measurable for all $(x,y)\in\mathbb{R}^2$;

(ii) $f(t,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ is continuous for a.e. $t\in[0,T]$;

(iii) for each compact set $A\subset\mathbb{R}^2$, there is a function $\mu_A\in L^1$ such that $|f(t,x,y)|\le\mu_A(t)$ for a.e. $t\in[0,T]$ and all $(x,y)\in A$.

Definition 2.2 A solution of problem (1)-(2) (resp (1)-(3)) is a function $u\in C^1$ satisfies (1)-(2) (resp (1)-(3)) such that $\phi(u')\in AC$ and $\|u'\|_\infty<a$.

Definition 2.3 A function $\alpha\in C^1$ is a lower solution of problem (1)-(2) if $\|\alpha'\|_\infty<a$, $\phi(\alpha')\in AC$,

$$\bigl(\phi(\alpha'(t))\bigr)'\ge f\bigl(t,\alpha(t),\alpha'(t)\bigr),\quad \text{a.e. } t\in[0,T], \tag{4}$$

$$\alpha'(0)\ge\alpha'(T),\quad\text{and}\quad\alpha(0)=\alpha(T). \tag{5}$$

Definition 2.4 A function $\beta\in C^1$ is an upper solution of problem (1)-(2) if $\|\beta'\|_\infty<a$, $\phi(\beta')\in AC$,

$$\bigl(\phi(\beta'(t))\bigr)'\le f\bigl(t,\beta(t),\beta'(t)\bigr),\quad \text{a.e. } t\in[0,T], \tag{6}$$

$$\beta'(0)\le\beta'(T),\quad\text{and}\quad\beta(0)=\beta(T). \tag{7}$$

Definition 2.5 A function $\alpha\in C^1$ is a lower solution of problem (1)-(3) if $\|\alpha'\|_\infty<a$, $\phi(\alpha')\in AC$,

$$\bigl(\phi(\alpha'(t))\bigr)'\ge f\bigl(t,\alpha(t),\alpha'(t)\bigr),\quad \text{a.e. } t\in[0,T], \tag{8}$$

$$\phi\bigl(\alpha'(0)\bigr)\ge g_0\bigl(\alpha(0)\bigr),\quad\text{and}\quad\phi\bigl(\alpha'(T)\bigr)\le g_T\bigl(\alpha(T)\bigr). \tag{9}$$

Definition 2.6 A function $\beta\in C^1$ is an upper solution of problem (1)-(3) if $\|\beta'\|_\infty<a$, $\phi(\beta')\in AC$,

$$\bigl(\phi(\beta'(t))\bigr)'\le f\bigl(t,\beta(t),\beta'(t)\bigr),\quad \text{a.e. } t\in[0,T], \tag{10}$$

$$\phi\bigl(\beta'(0)\bigr)\le g_0\bigl(\beta(0)\bigr),\quad\text{and}\quad\phi\bigl(\beta'(T)\bigr)\ge g_T\bigl(\beta(T)\bigr). \tag{11}$$

Remark 2.1 It is standard to show that the Nemytskii operator associated to f, $N_f:C^1\to L^1$, defined by

$$N_f(u)=f\bigl(\cdot,u(\cdot),u'(\cdot)\bigr) \tag{12}$$

is continuous and sends bounded sets into bounded sets.

3 Existence of solutions of periodic problem

3.1 Existence of solutions under two sign conditions

Lemma 3.1 *For each $h\in C$, there exists a unique $\varrho:=Q_\phi(h)\in\operatorname{Range}(h)$ such that*

$$\int_0^T\phi^{-1}\bigl(h(t)-\varrho\bigr)\,dt=0.$$

Moreover, the function $Q_\phi:C\to\mathbb{R}$ is continuous.

Proof See [1], the proof of Lemma 1. □

Now, consider the family of boundary value problems (P_λ), $\lambda \in [0,1]$,

$$(P_\lambda)\qquad \begin{cases} (\phi(u'(t)))' = \lambda N_f(u)(t) + (1-\lambda)QN_f(u), & \text{a.e. } t \in [0,T], \\ u'(0) = u'(T), \qquad u(0) = u(T). \end{cases}$$

For each $\lambda \in [0,1]$, problem (P_λ) can be written equivalently

$$\begin{cases} (\phi(u'(t)))' = \lambda N_f(u)(t), & \text{a.e. } t \in [0,T], \\ u'(0) = u'(T), \qquad u(0) = u(T), \\ QN_f(u) = 0. \end{cases} \tag{13}$$

For each $\lambda \in [0,1]$, we associate with (P_λ) the nonlinear operator $M(\lambda,\cdot)$, where M is defined on $[0,1] \times C^1$ by

$$M(\lambda,u) = P(u) + QN_f(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ \bigl[\lambda H(I-Q)N_f\bigr](u). \tag{14}$$

Using the Arzelà-Ascoli theorem, we get that M is completely continuous.

Lemma 3.2 *Assume that there exist $R > 0$ and $\varepsilon \in \{-1,1\}$ such that*

$$u_L \geq R \quad \textit{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\bigl(t,u(t),u'(t)\bigr)\,dt\right\} > 0 \tag{15}$$

and

$$u_M \leq -R \quad \textit{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\bigl(t,u(t),u'(t)\bigr)\,dt\right\} < 0. \tag{16}$$

Then, for all sufficiently large $\rho > 0$,

$$d_{LS}\bigl[I - M(1,\cdot), B_\rho, 0\bigr] = -\varepsilon,$$

and problem (1)-(2) *has at least one solution.*

Proof Assume that there exists $(\lambda,u) \in [0,1] \times C^1$ such that $M(\lambda,u) = u$.

We have $u(0) = u(0) + [QN_f(u)]$. It follows that

$$\int_0^T f\bigl(t,u(t),u'(t)\bigr)\,dt = 0. \tag{17}$$

Since

$$u' = \bigl(M(\lambda,u)\bigr)' = \phi^{-1} \circ (I - Q_\phi) \circ \bigl[\lambda H(I-Q)N_f\bigr](u),$$

we get $\|u'\|_\infty < a$. If $u_L \geq R$ or $u_M \leq -R$, by (15) and (16) we have

$$\int_0^T f\bigl(t,u(t),u'(t)\bigr)\,dt \neq 0, \tag{18}$$

which contradicts (17); therefore, $u_L < R$ and $u_M > -R$. Since u is continuous on $[0,T]$, there exists $(t_1,t_2) \in [0,T]^2$ such that $u_L = u(t_1)$ and $u_M = u(t_2)$. We have

$$u_M - u_L = \left|\int_{t_1}^{t_2} u'(t)\,dt\right| \le \left|\int_{t_1}^{t_2} \left|u'(t)\right| dt\right| < a|t_1 - t_2| < aT. \tag{19}$$

Using (19), we have

$$u_M < u_L + aT < R + aT \quad \text{and} \quad u_L > u_M - aT > -R - aT.$$

It follows that $\|u\|_\infty < R + aT$.

Since $\|u'\|_\infty < a$ and $\|u\|_\infty < R + aT$, we have

$$\|u\|_{C^1} < R + (a+1)T. \tag{20}$$

Let M be the operator given by (14), and let $\rho > R + a(T+1)$. Using (20) and the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned} d_{LS}\big[I - M(1,\cdot), B_\rho, 0\big] &= d_{LS}\big[I - M(0,\cdot), B_\rho, 0\big] \\ &= d_{LS}\big[I - [P + QN_f], B_\rho, 0\big]. \end{aligned}$$

But the range of the mapping $u \mapsto P(u) + QN_f(u)$ is contained in the subspace of constant functions isomorphic to $\mathbb{R}$, so, using the reduction property of Leray-Schauder degree [3], we have

$$\begin{aligned} d_{LS}\big[I - [P + QN_f], B_\rho, 0\big] &= d_B\big[I - [P + QN_f]|_{\mathbb{R}},]-\rho,\rho[, 0\big] \\ &= d_B\big[-QN_f,]-\rho,\rho[, 0\big] \\ &= \frac{1}{2}\operatorname{sign}\big[-QN_f(\rho)\big] - \frac{1}{2}\operatorname{sign}\big[-QN_f(-\rho)\big] \\ &= -\varepsilon. \end{aligned}$$

By the existence property of the Leray-Schauder degree there exists $u \in B_\rho$ such that $u = M(1,u)$, which is a solution of problem (1)-(2). □

Let us decompose any $u \in C^1$ in the form $u = \overline{u} + \widetilde{u}$ ($\overline{u} = u(0)$, $\widetilde{u}(0) = 0$), and let $\widetilde{C}^1 = \{u \in C^1 : u(0) = 0\}$.

Lemma 3.3 *The set $\mathfrak{S}$ of solutions $(\overline{u},\widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1$ of problem*

$$\begin{cases} (\phi(\widetilde{u}'(t)))' = N_f(\overline{u} + \widetilde{u})(t) - QN_f(\overline{u} + \widetilde{u}), & a.e.\ t \in [0,T], \\ u'(0) = u'(T), \qquad u(0) = u(T). \end{cases} \tag{21}$$

contains a continuum subset $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1$ is contained in the ball $B_{a(T+1)}$.

Proof The proof is similar to the proof of Lemma 4 in [1]. □

Theorem 3.1 *Assume that there exist* $R > 0$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$u_L \geq R \quad \text{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\big(t, u(t), u'(t)\big)\,dt\right\} \geq 0 \tag{22}$$

and

$$u_M \leq -R \quad \text{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\big(t, u(t), u'(t)\big)\,dt\right\} \leq 0. \tag{23}$$

Then problem (1)-(2) *admits at least one solution*.

Proof The proof is similar to the proof of Theorem 2 in [1].

Let us consider the continuum $\mathcal{C}$ given by Lemma 3.3. We have $\mathcal{C} \neq \emptyset$ (see the proof of Lemma 4 in [1]). Let $(\overline{u}, \widetilde{u}) \in \mathcal{C}$. Using Lemma 3.3, it follows that

$$(R + aT, \widetilde{u}) \in \mathcal{C} \quad \text{and} \quad (-R - aT, \widetilde{u}) \in \mathcal{C}.$$

Let $v_1 = R + aT + \widetilde{u}$ and $v_2 = -R - aT + \widetilde{u}$.

Since

$$\forall t \in [0, T], \quad -aT < \widetilde{u}(t) < aT \quad \text{and} \quad -a < \widetilde{u}'(t) < a,$$

for all $t \in [0, T]$, we have

$$v_1(t) = R + aT + \widetilde{u}(t) > R + aT - aT = R, \quad -a < v_1'(t) < a,$$
$$v_2(t) = -R - aT + \widetilde{u}(t) < -R - aT + aT = -R, \quad \text{and} \quad -a < v_2'(t) < a.$$

Applying (22), we have $\varepsilon\{QN_f(v_1)\} \geq 0$, and applying (23), we have $\varepsilon\{QN_f(v_2)\} \leq 0$.

We deduce, using the intermediate value theorem for a continuous functions on a connected set, that there exists $(\overline{w}, \widetilde{w}) \in \mathcal{C}$ such that

$$QN_f(\overline{w} + \widetilde{w}) = 0.$$

Therefore, $w = \overline{w} + \widetilde{w}$ is a solution of problem (1)-(2). □

3.2 Existence of solutions under one sign condition and only one lower solution or only one upper solution

For $\alpha \in C^1$, let us define two functions $\gamma_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\gamma_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma_1(t, x) = \begin{cases} \alpha(t) & \text{if } x < \alpha(t), \\ x & \text{if } x \geq \alpha(t), \end{cases} \quad \text{and} \quad \gamma_2(t, x) = \begin{cases} \alpha(t) & \text{if } x > \alpha(t), \\ x & \text{if } x \leq \alpha(t). \end{cases}$$

We introduce the following lemma (see [4], Lemma 6.3 and Corollary 6.4).

Lemma 3.4 *For* $u \in C^1$, *the following three properties are true.*

(a) *For* $i \in \{1, 2\}$, $\frac{d}{dt}\gamma_i(t, u(t))$ *exists for a.e.* $t \in [0, T]$.

(b)

$$\frac{d}{dt}\gamma_1\big(t,u(t)\big) = \begin{cases} \alpha'(t) & \textit{if } u(t) < \alpha(t), \\ u'(t) & \textit{if } u(t) \geq \alpha(t), \end{cases}$$

and

$$\frac{d}{dt}\gamma_2\big(t,u(t)\big) = \begin{cases} \alpha'(t) & \textit{if } u(t) > \alpha(t), \\ u'(t) & \textit{if } u(t) \leq \alpha(t). \end{cases}$$

(c) *For* $i \in \{1,2\}$, *if* $(u_n)_n \subset C^1$ *is such that* $u_n \to u$ *in* C^1, *then* $\gamma_i(\cdot,u_n) \to \gamma_i(\cdot,u)$ *in* C, *and for almost every* $t \in [0,T]$, $\lim_{n\to\infty} \frac{d}{dt}\gamma_i(t,u_n(t)) = \frac{d}{dt}\gamma_i(t,u(t))$.

Theorem 3.2 *Assume that*:

(i) *there exists a lower solution* α *of problem* (1)-(2);

(ii) *there exists* $R > 0$ *such that*

$$u_L \geq R \quad \textit{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \int_0^T f\big(t,u(t),u'(t)\big)\,dt > 0. \tag{24}$$

Then problem (1)-(2) *admits at least one solution*.

Proof

Step 1: The modified problem.

Consider the function $\delta : \mathbb{R} \to \mathbb{R}$ given by $\delta(x) = \max\{-a, \min\{x,a\}\}$. Consider the function $f^* : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f^*(t,u,v) = f\left(t, \gamma_1\big(t,u(t)\big), \delta\left(\frac{d}{dt}\gamma_1\big(t,u(t)\big)\right)\right) + u(t) - \gamma_1\big(t,u(t)\big), \tag{25}$$

which is an L^1-Carathéodory function. Consider the modified problem

$$\begin{cases} (\phi(u'(t)))' = f^*(t,u(t),u'(t)), & \text{a.e. } t \in [0,T], \\ u'(0) = u'(T), \qquad u(0) = u(T). \end{cases} \tag{26}$$

Step 2: Any solution of problem (26) *is a solution of problem* (1)-(2).

Let u be a solution of problem (26). We prove that $\alpha(t) \leq u(t)$ for all $t \in [0,T]$.

Let us assume on the contrary that, for some $t_0 \in [0,T]$,

$$\max_{t\in[0,T]}\big[\alpha(t) - u(t)\big] = \alpha(t_0) - u(t_0) > 0.$$

If $t_0 \in]0,T[$, then $\alpha'(t_0) = u'(t_0)$; hence, $\phi(\alpha'(t_0)) = \phi(u'(t_0))$. We can find $\omega > 0$ such that for all $t \in]t_0, t_0+\omega[$, $\alpha(t) > u(t)$. We have

$$\forall t \in]t_0, t_0+\omega[, \quad \gamma_1\big(t,u(t)\big) = \alpha(t) \quad \text{and} \quad \delta\left(\frac{d}{dt}\gamma_1\big(t,u(t)\big)\right) = \alpha'(t)$$

for a.e. $t \in]t_0, t_0+\omega[$.

It follows that, for all $t \in]t_0, t_0 + \omega[$,

$$\begin{aligned}\phi\big(\alpha'(t)\big) - \phi\big(u'(t)\big) &= \int_{t_0}^{t} \big[\big(\phi\big(\alpha'(s)\big)\big)' - f\big(s,\alpha(s),\alpha'(s)\big) + \big(\alpha(s) - u(s)\big)\big]\,ds \\ &\geq \int_{t_0}^{t} \big(\alpha(s) - u(s)\big)\,ds > 0.\end{aligned}$$

Since ϕ is an increasing homeomorphism, $\phi(\alpha'(t)) - \phi(u'(t)) > 0 \Rightarrow \alpha'(t) - u'(t) > 0$, a contradiction.

If $t_0 \in \{0, T\}$, then $\alpha'(0) - u'(0) = 0 = \alpha'(T) - u'(T)$. We can find $\omega > 0$ such that for all $t \in]0, \omega[$, $\alpha(t) > u(t)$. We have

$$\forall t \in]0,\omega[, \quad \gamma\big(t,u(t)\big) = \alpha(t) \quad \text{and} \quad \delta\bigg(\frac{d}{dt}\gamma_1\big(t,u(t)\big)\bigg) = \alpha'(t) \quad \text{for a.e. } t \in]0,\omega[.$$

It follows that, for all $t \in]0, \omega[$,

$$\begin{aligned}\phi\big(\alpha'(t)\big) - \phi\big(u'(t)\big) &= \int_{0}^{t} \big[\big(\phi\big(\alpha'(s)\big)\big)' - f\big(s,\alpha(s),\alpha'(s)\big) + \big(\alpha(s) - u(s)\big)\big]\,ds \\ &\geq \int_{0}^{t} \big(\alpha(s) - u(s)\big)\,ds > 0.\end{aligned}$$

Since ϕ is an increasing homeomorphism, $\phi(\alpha'(t)) - \phi(u'(t)) > 0 \Rightarrow \alpha'(t) - u'(t) > 0$, a contradiction.

In consequence, we have that $\alpha(t) \leq u(t)$ for all $t \in [0, T]$. Therefore, u is a solution of problem (1)-(2).

Step 3: Existence of solutions of problem (26).

Let $\Delta = [\alpha_L, \alpha_M] \times [-a, a]$. Since f is an L^1-Carathéodory function, there exists $\varphi \in L^1$ such that, for a.e. $t \in [0, T]$ and all $(x, y) \in \Delta$, $|f(t, x; y)| \leq \varphi(t)$. Let $R_1 = \max\{|\alpha_L|, |\alpha_L - \frac{1}{T}\|\varphi\|_{L^1}|, R + aT\}$. By (24), if $u \in C^1$ is such that $\|u'\|_\infty < a$ and $u_L \geq R_1$, then

$$\int_0^T f^*\big(t,u(t),u'(t)\big)\,dt > 0. \tag{27}$$

Moreover, if $u \in C^1$ is such that $\|u'\|_\infty < a$ and $u_M \leq -R_1$, then

$$u(t) \leq \alpha_L \quad \text{and} \quad u(t) \leq \alpha_L - \frac{1}{T}\|\varphi\|_{L^1} \quad \forall t \in [0,T].$$

It follows that

$$\begin{aligned}\int_0^T f^*\big(t,u(t),u'(t)\big)\,dt &= \int_0^T \big(f\big(t,\alpha(t),\alpha'(t)\big) + u(t) - \alpha(t)\big)\,dt \\ &\leq \int_0^T \bigg(f\big(t,\alpha(t),\alpha'(t)\big) + \alpha_L - \frac{1}{T}\|\varphi\|_{L^1} - \alpha(t)\bigg)\,dt \\ &\leq \int_0^T \bigg(f\big(t,\alpha(t),\alpha'(t)\big) - \frac{1}{T}\|\varphi\|_{L^1} + \big(\alpha_L - \alpha(t)\big)\bigg)\,dt \\ &\leq 0.\end{aligned} \tag{28}$$

Using (27), (28), and Theorem 3.1, we deduce that problem (26) has at least one solution, which is also a solution of problem (1)-(2) by step 2. □

Theorem 3.3 *Assume that*:

(i) *there exists an upper solution β of the problem* (1)-(2);

(ii) *there exists $R > 0$ such that*

$$u_M \leq -R \quad \text{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \int_0^T f\big(t, u(t), u'(t)\big)\,dt < 0. \tag{29}$$

Then problem (1)-(2) *admits at least one solution*.

Proof The proof is similar to the proof of Theorem 3.2. □

4 Existence of solutions of Neumann-Steklov problem

4.1 Existence of solutions under two sign conditions

Consider the family of boundary value problems (P_λ), $\lambda \in [0,1]$:

$$(P_\lambda) \quad \begin{cases} (\phi(u'(t)))' = \lambda N_f(u)(t) + (1-\lambda)[QN_f(u) - K(u)], & \text{a.e. } t \in [0,T], \\ \phi(u'(0)) = \lambda g_0(u(0)), \\ \phi(u'(T)) = \lambda g_T(u(T)). \end{cases}$$

For each $\lambda \in [0,1]$, problem (P_λ) can be written equivalently

$$\begin{cases} (\phi(u'(t)))' = \lambda N_f(u)(t), \quad \text{a.e. } t \in [0,T], \\ \phi(u'(0)) = \lambda g_0(u(0)), \qquad \phi(u'(T)) = \lambda g_T(u(T)), \\ QN_f(u) - K(u) = 0. \end{cases} \tag{30}$$

For each $\lambda \in [0,1]$, we associate with (P_λ) the nonlinear operator $M(\lambda, \cdot)$, where M is defined on $[0,1] \times C^1$ by

$$M(\lambda, u) = P(u) + QN_f(u) - K(u) + H \circ \phi^{-1} \circ \big[\lambda H(I-Q)N_f(u) + \lambda G(u)\big] \tag{31}$$

with

$$G(u)(t) = \left(1 - \frac{t}{T}\right) g_0\big(u(0)\big) + \frac{t}{T} g_T\big(u(T)\big), \quad \forall t \in [0,T]. \tag{32}$$

Using the Arzelà-Ascoli theorem, we get that M is completely continuous.

Lemma 4.1 *Assume that there exist $R > 0$ and $\varepsilon \in \{-1, 1\}$ such that*

$$u_L \geq R \quad \text{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon \left\{ \int_0^T f\big(t, u(t), u'(t)\big)\,dt - TK(u) \right\} > 0 \tag{33}$$

and

$$u_M \leq -R \quad \text{and} \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon \left\{ \int_0^T f\big(t, u(t), u'(t)\big)\,dt - TK(u) \right\} < 0. \tag{34}$$

Then, for all sufficiently large $\rho > 0$,

$$d_{LS}\big[I - M(1,\cdot), B_\rho, 0\big] = -\varepsilon,$$

and problem (1)-(3) *has at least one solution.*

Proof Assume that there exists $(\lambda, u) \in [0,1] \times C^1$ such that $M(\lambda, u) = u$.
We have

$$u(0) = u(0) + \big[QN_f(u) - T^{-1}\big(g_T\big(u(T)\big) - g_0\big(u(0)\big)\big)\big].$$

It follows that

$$\int_0^T f\big(t, u(t), u'(t)\big)\,dt - TK(u) = 0. \tag{35}$$

Since

$$u' = \big(M(\lambda, u)\big)' = \phi^{-1} \circ \big[\lambda H(I - Q)N_f(u) + \lambda G(u)\big],$$

we have $\|u'\|_\infty < a$. If $u_L \geq R$ or $u_M \leq -R$, then by (33) and (34) we have

$$\int_0^T f\big(t, u(t), u'(t)\big)\,dt - TK(u) \neq 0, \tag{36}$$

which contradicts (35); therefore, $u_L < R$ and $u_M > -R$. Since u is continuous on $[0, T]$, there exists $(t_1, t_2) \in [0, T]^2$ such that $u_L = u(t_1)$ and $u_M = u(t_2)$. We have

$$u_M - u_L = \left|\int_{t_1}^{t_2} u'(t)\,dt\right| \leq \left|\int_{t_1}^{t_2} \big|u'(t)\big|\,dt\right| < a|t_1 - t_2| < aT. \tag{37}$$

Using (37), we have

$$u_M < u_L + aT < R + aT \quad \text{and} \quad u_L > u_M - aT > -R - aT.$$

It follows that $\|u\|_\infty < R + aT$. Since $\|u'\|_\infty < a$ and $\|u\|_\infty < R + aT$, we have

$$\|u\|_{C^1} < R + (a + 1)T. \tag{38}$$

Let M be the operator given by (31) and let $\rho > R + a(T + 1)$. Using (38) and the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned} d_{LS}\big[I - M(1,\cdot), B_\rho, 0\big] &= d_{LS}\big[I - M(0,\cdot), B_\rho, 0\big] \\ &= d_{LS}\big[I - [P + QN_f - K], B_\rho, 0\big]. \end{aligned}$$

But the range of the mapping $u \mapsto P(u) + QN_f(u) - K(u)$ is contained in the subspace of constant functions isomorphic to $\mathbb{R}$, so, using the reduction property of Leray-Schauder degree [3], it follows that

$$\begin{aligned} d_{LS}\big[I-[P+QN_f-K],B_\rho,0\big] &= d_B\big[I-[P+QN_f-K]|_{\mathbb{R}},\,]-\rho,\rho[,0\big] \\ &= d_B\big[-QN_f+K,\,]-\rho,\rho[,0\big] \\ &= \frac{1}{2}\operatorname{sign}\big[-QN_f(\rho)+K(\rho)\big] \\ &\quad -\frac{1}{2}\operatorname{sign}\big[-QN_f(-\rho)+K(-\rho)\big] \\ &= -\varepsilon. \end{aligned}$$

Then, by the existence property of the Leray-Schauder degree there exists $u \in B_\rho$ such that $u = M(1,u)$, which is a solution of problem (1)-(3). □

Let us decompose any $u \in C^1$ in the form $u = \overline{u} + \widetilde{u}$ ($\overline{u} = u(0)$, $\widetilde{u}(0) = 0$), and let $\widetilde{C}^1 = \{u \in C^1 : u(0) = 0\}$.

Lemma 4.2 *The set $\mathfrak{S}$ of solutions $(\overline{u},\widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1$ of problem*

$$\begin{cases} (\phi(\widetilde{u}'(t)))' = N_f(\overline{u}+\widetilde{u})(t) - QN_f(\overline{u}+\widetilde{u}) \\ \qquad\qquad + T^{-1}[g_T(\overline{u}+\widetilde{u}(T)) - g_0(\overline{u})], \quad a.e.\ t \in [0,T], \\ \phi(\widetilde{u}'(0)) = g_0(\overline{u}), \qquad \phi(\widetilde{u}'(T)) = g_T(\overline{u}+\widetilde{u}(T)), \end{cases} \tag{39}$$

contains a continuum subset $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1$ is contained in the ball $B_{a(T+1)}$.

Proof The proof is similar to the proof of Lemma 4 in [2]. □

Theorem 4.1 *Assume that there exist $R > 0$ and $\varepsilon \in \{-1,1\}$ such that*

$$u_L \geq R \quad and \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\big(t,u(t),u'(t)\big)\,dt - TK(u)\right\} \geq 0 \tag{40}$$

and

$$u_M \leq -R \quad and \quad \|u'\|_\infty < a \quad \Rightarrow \quad \varepsilon\left\{\int_0^T f\big(t,u(t),u'(t)\big)\,dt - TK(u)\right\} \leq 0, \tag{41}$$

Then problem (1)-(3) *admits at least one solution.*

Proof The proof is similar to that of Theorem 2 in [2] and that of Theorem 3.1. □

4.2 Existence of solutions under one sign condition and only one lower solution or only one upper solution

Theorem 4.2 *Assume that*:

(i) *there exists a lower solution α of problem* (1)-(3);

(ii) *there exists* $R > 0$ *such that*

$$u_L \geq R \quad \text{and} \quad \|u'\|_\infty < a \\ \Rightarrow \quad \int_0^T f(t, u(t), u'(t))\,dt - g_T(u(T)) + g_0(u(0)) > 0. \tag{42}$$

Then problem (1)-(3) *admits at least one solution*.

Proof

Step 1: The modified problem.

Consider the functions $f^* : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g_0^* : \mathbb{R} \longrightarrow \mathbb{R}$, and $g_T^* : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f^*(t,u,v) = f\left(t, \gamma_1(t,u(t)), \delta\left(\frac{d}{dt}\gamma_1(t,u(t))\right)\right) + u(t) - \gamma_1(t,u(t)), \tag{43}$$

$$g_0^*(u) = \begin{cases} g_0(u) & \text{if } \alpha(0) \leq u, \\ g_0(\alpha(0)) + \arctan(u - \alpha(0)) & \text{if } u < \alpha(0), \end{cases} \tag{44}$$

and

$$g_T^*(u) = \begin{cases} g_T(u) & \text{if } \alpha(T) \leq u, \\ g_T(\alpha(T)) - \arctan(u - \alpha(T)) & \text{if } u < \alpha(T). \end{cases} \tag{45}$$

The function f^* is an L^1-Carathéodory function, and g_0^* and g_T^* are continuous. Consider the modified problem

$$\begin{cases} (\phi(u'(t)))' = f^*(t,u(t),u'(t)), & \text{a.e. } t \in [0,T], \\ \phi(u'(0)) = g_0^*(u(0)), \\ \phi(u'(T)) = g_T^*(u(T)). \end{cases} \tag{46}$$

Step 2: Any solution of problem (46) *is a solution of problem* (1)-(3).

Let u be a solution of problem (46). We prove that $\alpha(t) \leq u(t)$ for all $t \in [0,T]$.

Let us assume on the contrary that, for some $t_0 \in [0,T]$,

$$\max_{t\in[0,T]} [\alpha(t) - u(t)] = \alpha(t_0) - u(t_0) > 0.$$

If $t_0 \in]0,T[$, then $\alpha'(t_0) = u'(t_0)$; hence, $\phi(\alpha'(t_0)) = \phi(u'(t_0))$. We can find $\omega > 0$ such that for all $t \in]t_0, t_0+\omega[$, $\alpha(t) > u(t)$. We have

$$\forall t \in]t_0, t_0+\omega[, \quad \gamma_1(t,u(t)) = \alpha(t) \quad \text{and} \quad \delta\left(\frac{d}{dt}\gamma_1(t,u(t))\right) = \alpha'(t)$$
$$\text{for a.e. } t \in]t_0, t_0+\omega[.$$

It follows that, for all $t \in]t_0, t_0+\omega[$,

$$\begin{aligned} \phi(\alpha'(t)) - \phi(u'(t)) &= \int_{t_0}^t \left[(\phi(\alpha'(s)))' - f(s,\alpha(s),\alpha'(s)) + (\alpha(s) - u(s))\right] ds \\ &\geq \int_{t_0}^t (\alpha(s) - u(s))\,ds > 0. \end{aligned}$$

Since ϕ is an increasing homeomorphism, $\phi(\alpha'(t)) - \phi(u'(t)) > 0 \Rightarrow \alpha'(t) - u'(t) > 0$, a contradiction.

If $t_0 = 0$, then

$$\phi\bigl(\alpha'(0)\bigr) \le \phi\bigl(u'(0)\bigr) = g_0\bigl(\alpha(0)\bigr) + \arctan\bigl(u(0) - \alpha(0)\bigr) < g_0\bigl(\alpha(0)\bigr),$$

a contradiction with the definition of a lower solution.

If $t_0 = T$, then

$$\phi\bigl(\alpha'(T)\bigr) \ge \phi\bigl(u'(T)\bigr) = g_T\bigl(\alpha(T)\bigr) - \arctan\bigl(u(T) - \alpha(T)\bigr) > g_T\bigl(\alpha(T)\bigr),$$

a contradiction with the definition of a lower solution.

In consequence, we have that $\alpha(t) \le u(t)$ for all $t \in [0, T]$. Therefore, u is a solution of problem (1)-(3).

Step 3: Existence of solutions of problem (1)-(3).

Let $\Delta = [\alpha_L, \alpha_M] \times [-a, a]$. Since f is an L^1-Carathéodory function, there exists $\varphi \in L^1$ such that, for a.e. $t \in [0, T]$ and all $(x, y) \in \Delta$, $|f(t, x; y)| \le \varphi(t)$.

Let

$$R_1 = \max\left\{|\alpha_L|, \left|\alpha_L + \frac{1}{T}\bigl(g_T\bigl(\alpha(T)\bigr) - g_0\bigl(\alpha(0)\bigr) - \pi\bigr) - \frac{1}{T}\|\varphi\|_{L^1}\right|, R + aT\right\}.$$

By (42), if $u \in C^1$ is such that $\|u'\|_\infty < a$ and $u_L \ge R_1$, then

$$\int_0^T f^*\bigl(t, u(t), u'(t)\bigr)\,dt - g_T^*\bigl(u(T)\bigr) + g_0^*\bigl(u(0)\bigr) > 0. \tag{47}$$

Moreover, if $u \in C^1$ is such that $\|u'\|_\infty < a$ and $u_M \le -R_1$, then

$$u(t) \le \alpha_L \quad \text{and} \quad u(t) \le \alpha_L + \frac{1}{T}\bigl(g_T\bigl(\alpha(T)\bigr) - g_0\bigl(\alpha(0)\bigr) - \pi\bigr) - \frac{1}{T}\|\varphi\|_{L^1}, \quad \forall t \in [0, T].$$

It follows that

$$\begin{aligned} &\int_0^T f^*\bigl(t, u(t), u'(t)\bigr)\,dt - g_T^*\bigl(u(T)\bigr) + g_0^*\bigl(u(0)\bigr) \\ &\quad = \int_0^T \bigl(f\bigl(t, \alpha(t), \alpha'(t)\bigr) + u(t) - \alpha(t)\bigr)\,dt - g_T\bigl(\alpha(T)\bigr) \\ &\qquad + \arctan\bigl(u(T) - \alpha(T)\bigr) + g_0\bigl(\alpha(0)\bigr) + \arctan\bigl(u(0) - \alpha(0)\bigr) \\ &\quad \le \int_0^T \left(f\bigl(t, \alpha(t), \alpha'(t)\bigr) + \alpha_L + \frac{1}{T}\bigl(g_T\bigl(\alpha(T)\bigr) - g_0\bigl(\alpha(0)\bigr) - \pi\bigr) - \frac{1}{T}\|\varphi\|_{L^1} - \alpha(t)\right) dt \\ &\qquad - g_T\bigl(\alpha(T)\bigr) + \arctan\bigl(u(T) - \alpha(T)\bigr) + g_0\bigl(\alpha(0)\bigr) + \arctan\bigl(u(0) - \alpha(0)\bigr) \\ &\quad < \int_0^T \left(f\bigl(t, \alpha(t), \alpha'(t)\bigr) - \frac{1}{T}\|\varphi\|_{L^1} + \bigl(\alpha_L - \alpha(t)\bigr)\right) dt \\ &\quad < 0. \end{aligned} \tag{48}$$

Using (47), (48), and Theorem 4.1, we deduce that problem (46) has at least one solution, which is also a solution of problem (1)-(3) by step 2. □

Theorem 4.3 *Assume that*:

(i) *there exists an upper solution* β *of problem* (1)-(3);

(ii) *there* $R > 0$ *such that*

$$\begin{aligned} & u_M \le -R \quad \text{and} \quad \|u'\|_\infty < a \\ & \quad \Rightarrow \quad \int_0^T f\big(t, u(t), u'(t)\big)\, dt - g_T\big(u(T)\big) + g_0\big(u(0)\big) < 0. \end{aligned} \tag{49}$$

Then problem (1)-(3) *admits at least one solution*.

Proof The proof is similar to that of Theorem 4.2. □

Corollary 4.1 *Assume that*:

(a) *there exists* $A \in \mathbb{R}$ *such that* $f(t,u,v) \ge A$ *for a.e.* $t \in [0,T]$ *and all* $(u,v) \in \mathbb{R} \times [-a,a]$;

(b) $\lim_{x\to+\infty}(g_0(x) - g_T(x)) = +\infty$;

(c) *there exists a lower solution* α *of problem* (1)-(3).

Then problem (1)-(3) *admits at least one solution*.

Proof By (a) we have

$$\int_0^T f\big(t, u(t), u'(t)\big)\, dt - g_T\big(u(T)\big) + g_0\big(u(0)\big) \ge AT + g_0(x) - g_T(x). \tag{50}$$

By (b) there exists $R > 0$ such that (42) is true. By Theorem 4.2 problem (1)-(3) admits at least one solution. □

Corollary 4.2 *Assume that*:

(a) *there exists* $A \in \mathbb{R}$ *such that* $f(t,u,v) \le A$ *for a.e.* $t \in [0,T]$ *and all* $(u,v) \in \mathbb{R} \times [-a,a]$;

(b) $\lim_{x\to-\infty}(g_0(x) - g_T(x)) = -\infty$;

(c) *there exists an upper solution* β *of problem* (1)-(3).

Then problem (1)-(3) *admits at least one solution*.

Proof The proof is similar to that of Corollary 4.1. □

Example 4.1 Consider the problem

$$\begin{cases} \Big(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\Big)' = t - 3 + \sin(u(t)) + u'(t) \quad \text{for a.e. } t \in [0,1], \\ \frac{u'(0)}{\sqrt{1-(u'(0))^2}} = (u(0))^2 - 2 \quad \text{and} \quad \frac{u'(1)}{\sqrt{1-(u'(1))^2}} = -e^{u(1)} + 2. \end{cases}$$

We can see that $|f(t,u,v)| \le 6$ for all $(t,u,v) \in [0,T] \times \mathbb{R} \times [-1,1]$, $\alpha(t) = 0$ is a lower solution, and $\lim_{x\to+\infty}(x^2 + e^x - 4) = +\infty$. Using Corollary 4.1, we deduce the existence of at least one solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors read and approved the final manuscript.

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