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The Euler implicit/explicit scheme for the Boussinesq equations

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Abstract

In this article, we consider the stability and convergence of the first-order implicit/explicit scheme for the Boussinesq equations. The finite element spatial discretization is based on a MINI element for the velocity and pressure, which satisfies the discrete inf-sup condition, and a linear polynomial for the temperature. The temporal terms are treated by the Euler implicit/explicit scheme, which is implicit for the linear terms and explicit for the nonlinear terms. The advantage of using the implicit/explicit scheme is that a linear system with constant coefficient matrix is obtained, which can save a lot of computational cost. The main novelties of this work are the stability of numerical solutions under the conditions $k_1\Delta t \le 1$ and $k_2\Delta t \le 1$ with two positive constants k_1, k_2 and the optimal error estimates of numerical solutions in different norms. Finally, some numerical results are provided to verify the performances of the Euler implicit/explicit scheme.

MSC: 65N15; 65N30; 76D07

Keywords: Boussinesq equations; Euler implicit/explicit scheme; stability; error estimates

1 Introduction

In this paper, we consider the following Boussinesq equations in $\mathbb{R}^2$ with coupled equations governing the viscous incompressible flow and heat transfer:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = -j\theta + f & \text{in } \Omega\times(0,T], \\ \operatorname{div} u = 0 & \text{in}\Omega\times(0,T], \\ \theta_t - \lambda\nu\Delta\theta + u\cdot\nabla\theta = g & \text{in } \Omega\times(0,T], \\ u = 0, \qquad \theta = 0 & \text{on } \partial\Omega\times(0,T], \\ u(x,0) = u_0, \qquad \theta(x,0) = \theta_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where Ω is a bounded convex polygonal domain, $u = (u_1,u_2)^T$ is the fluid velocity, p is the pressure, θ is the temperature, $\nu > 0$ is the viscosity, $\lambda = Pr^{-1}, Pr$ is the Prandtl number, $j = (0,1)^T$ is the vector of gravitational acceleration, $T > 0$ is the final time, and f and g are the forcing functions.

The Boussinesq equations (1.1) are an important dissipative nonlinear model in the atmospheric dynamics (see [1]). This system not only contains the velocity and pressure but also includes the temperature filed, and therefore finding a numerical solution of problem

(1.1) becomes a difficult task. There are numerous works devoted to the development of efficient schemes for this model, for example, the standard Galerkin finite element method (FEM) [2], the projection-based stabilized mixed FEM [3], the precondition techniques [4, 5], the two-level algorithms [6–8], and the references therein. In the literature mentioned, the suitable stability condition is a key issue for these developed schemes. Generally speaking, we can adopt the fully implicit, semi-implicit, and implicit/explicit schemes to treat the nonlinear equations. The fully implicit schemes are unconditionally stable. However, we have to solve a system of nonlinear equations at each time level. Explicit schemes are much easier in computation, but they suffer from a sever restriction of time step by the stability requirement. A popular approach is based on an implicit scheme for the linear terms and a semi-implicit scheme or an explicit scheme for the nonlinear terms. The semi-implicit scheme for the nonlinear terms results in a linear system with variable coefficient matrix of time, and an explicit treatment for the nonlinear term gives a constant matrix. Many researchers have studied the stability and convergence of these schemes for the Navier-Stokes equations [9–15]. The main results are summarized by He in his recent works [16, 17].

In this paper, we consider a first-order scheme for the Boussinesq equations. In view of the advantages of the explicit scheme for the nonlinear terms, we adopt the implicit/explicit scheme for the Boussinesq equations (1.1). Under the conditions $k_1\Delta t \le 1$ and $k_2\Delta t \le 1$ with two positive constants k_1, k_2, we present some new stabilities and establish the corresponding convergence for velocity, pressure, and temperature by the Taylor expansion and other skills. This report can be considered as an extension of the existing results [10, 16, 18, 19] from the Navier-Stokes equations to the more complex Boussinesq equations. Our main results can be stated as follows:

$$\|u - u_h^n\|_0 + \|\theta - \theta_h^n\|_0 \le C(\Delta t + h^2), \tag{1.2}$$

$$\|\nabla(u - u_h^n)\|_0 + \|\nabla(\theta - \theta_h^n)\|_0 + \|p - p_h^n\|_0 \le C(\Delta t + h), \tag{1.3}$$

where $C > 0$ is a constant depending on the parameters $f_\infty, b_\infty, u_0, \theta_0, \Omega, \nu$, and λ, but independent of h and Δt, where $f_\infty = \sup_{t\ge 0}\{|f| + |f_t|\}$, $f_t = \frac{df}{dt}$, $g_\infty = \sup_{t\ge 0}\{|g| + |g_t|\}$, $g_t = \frac{dg}{dt}$. Here and thereafter, C denotes a general positive constant, which may take different values at different places. From (1.2)-(1.3) we can see that our results are optimal for both space length h and time step Δt.

The outline of this article is as follows. Some basic notation and results for problem (1.1) are recalled in Section 2. Section 3 is devoted to develop the Euler implicit/explicit scheme. Stabilities and optimal error estimates are established in Sections 4 and 5, respectively. Finally, a series of numerical results are provided to verify the efficiency and effectiveness of the Euler implicit/explicit scheme.

2 Preliminaries

In this section, we construct a variable formulation for problem (1.1) and recall some classical results, which will be frequently used in this paper. To fix the idea, we set

$$X = H_0^1(\Omega)^2, \qquad W = H_0^1(\Omega), \qquad Y = L^2(\Omega)^2, \qquad Z = L^2(\Omega),$$

$$M = L_0^2(\Omega) = \left\{\varphi \in L^2(\Omega); \int_\Omega \varphi\, dx = 0\right\}.$$

Throughout this paper, we adopt $(\cdot,\cdot)$ and $\|\cdot\|_0$ to denote the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$. The spaces $H_0^1(\Omega)$ and X are equipped with the usual scalar product and norm $\|\nabla u\|_0^2 = (\nabla u, \nabla u)$. Define the continuous bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $\overline{a}(\cdot,\cdot)$ by

$$a(u,v) = \nu(\nabla u, \nabla v), \qquad d(v,q) = (q, \operatorname{div} v), \qquad \overline{a}(\theta,\psi) = \lambda\nu(\nabla\theta,\nabla\psi)$$

for all $u,v \in X, q \in M$, and $\theta,\psi \in W$.

Next, we introduce the closed subset V of X given by

$$V = \{v \in X, d(v,q) = 0, \forall q \in M\} = \{v \in X, \nabla\cdot v = 0 \text{ in } \Omega\}$$

and denote by H the closed subset of Y (see [17, 20]) given by

$$H = \{v \in Y, \nabla\cdot v = 0, v\cdot n|_{\partial\Omega} = 0\}.$$

We denote the Stokes operator by $A = P\Delta$, where P is the L^2-orthogonal projection of Y onto H or of Z onto W. Assume that Ω is such that the domain of A is given by (see [10, 17, 21, 22])

$$D(A) = H^2(\Omega)^2 \cap X \quad \text{or} \quad E(A) = H^2(\Omega) \cap W. \tag{2.1}$$

For instance, (2.1) holds if Γ is of class C^2 or if Ω is a convex plane polygonal domain.

Moreover, we can define the trilinear forms for all $u,v,w \in X$ and $\theta,\psi \in W$ as follows:

$$\begin{aligned}
b(u,v,w) &= \bigl((u\cdot\nabla)v,w\bigr) + \frac{1}{2}\bigl((\operatorname{div} u)v,w\bigr) = \frac{1}{2}\bigl((u\cdot\nabla)v,w\bigr) - \frac{1}{2}\bigl((u\cdot\nabla)w,v\bigr),\\
\overline{b}(u,\theta,\psi) &= \bigl((u\cdot\nabla)\theta,\psi\bigr) + \frac{1}{2}\bigl((\operatorname{div} u)\theta,\psi\bigr) = \frac{1}{2}\bigl((u\cdot\nabla)\theta,\psi\bigr) - \frac{1}{2}\bigl((u\cdot\nabla)\theta,\psi\bigr).
\end{aligned}$$

With these notations, for given $f \in L^\infty(R^+;Y)$ with $u_0 \in D(A)\cap V$ and $g \in L^\infty(R^+;Z)$ with $\theta_0 \in E(A)$, the variational formulation of (1.1) reads as follows: For all $(v,q,\psi) \in X\times M\times W$, find $(u,p,\theta) \in X\times M\times W$ with

$$\begin{aligned}
&u \in L^\infty\bigl(R^+;X\bigr)\cap L^2(0,T;V), \qquad u_t \in L^2\bigl(0,T;V'\bigr), \qquad \theta_t \in L^2\bigl(0,T;W'\bigr),\\
&p \in L^2(0,T;M), \qquad \theta \in L^\infty\bigl(R^+;W\bigr)\cap L^2(0,T;Z), \quad \forall T > 0,
\end{aligned}$$

such that

$$\begin{cases}
(u_t,v) + a(u,v) - d(v,p) + b(u,u,v) = (f,v) - (j\theta,v),\\
d(u,q) = 0,\\
(\theta_t,\psi) + \overline{a}(\theta,\psi) + \overline{b}(u,\theta,\psi) = (g,\psi),\\
u(x,0) = u_0, \qquad \theta(x,0) = \theta_0.
\end{cases} \tag{2.2}$$

Assuming that $f \in L^2(0,T;X')$, $g \in L^2(0,T;W')$, and $u_0 \in V$, $\theta_0 \in W$, problem (2.2) has at least one solution (u,p,θ) satisfying $u \in L^\infty(0,T;\Omega)\cap L^2(0,T;V)$ and $\theta \in L^\infty(0,T;\Omega)\cap$

$L^2(0,T;W)$. The uniqueness and regularity of the solution (u,p,θ) can also be proved by strengthening the assumptions on the data; see [23] for details.

We recall the following discrete Gronwall lemma, which can be found in [17, 24].

Lemma 2.1 *Let C_0 and a_k, b_k, c_k, d_k for integer $k \geq 0$ be nonnegative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \Delta t \sum_{k=0}^{n-1} d_k a_k + \Delta t \sum_{k=0}^{n-1} c_k + C_0 \quad \forall n \geq 1.$$

Then,

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \left(\Delta t \sum_{k=0}^{n-1} c_k + C_0 \right) \exp\left(\Delta t \sum_{k=0}^{n-1} d_k \right) \quad \forall n \geq 1.$$

Following the proofs provided in [1, 20, 22, 23], we can obtain that problem (2.2) possesses a unique solution (u,p,θ) with the following regularity properties.

Theorem 2.2 *Let $f \in L^\infty(R^+;Y), f_t \in L^2(0,T;Y), g \in L^\infty(R^+;Z), g_t \in L^2(0,T;Z)$ and $u_0 \in D(A) \cap V, \theta_0 \in E(A)$. Then the solution (u,p,θ) of problem* (2.2) *satisfies*

$$\|Au(t)\|_0 + \|\nabla u_t(t)\|_0 + \|Au_t(t)\|_0 + \|A\theta(t)\|_0 + \|\nabla\theta_t(t)\|_0 + \|A\theta_t(t)\|_0 \leq C.$$

Introduce the following Poincaré inequalities:

$$\|v\|_0 \leq C_1 \|\nabla v\|_0 \quad \forall v \in X \text{ or } W; \qquad \|\nabla v\|_0 \leq C_2 \|Av\|_0 \quad \forall v \in D(A) \text{ or } H^2(\Omega). \quad (2.3)$$

We end this section by recalling some properties of the trilinear forms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$, which can be found in [1, 6–8, 10, 17, 21, 22].

Lemma 2.3 *The trilinear forms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ satisfy*:

(1) *Under the condition* $\operatorname{div} u = 0$, *we have that*

$$b(u,v,v) = 0 \quad \forall u,v \in X; \qquad \overline{b}(u,\theta,\theta) = 0 \quad \forall u \in X, \theta \in W.$$

(2) *We have the following estimates for trilinear terms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$*:

$$\begin{aligned}
&|b(u,v,w)| \leq C_3 \|u\|_0^{1/2} \|A^{1/2}u\|_0^{1/2} \|A^{1/2}v\|_0^{1/2} \|Av\|_0^{1/2} \|w\|_0 \quad \forall u \in V, v \in D(A), w \in X, \\
&|b(u,v,w)| \leq C_4 \|u\|_0^{1/2} \|Av\|_0^{1/2} \|v\|_1 \|w\|_0 \quad \forall u \in V, v \in D(A), w \in X, \\
&|\overline{b}(u,\theta,\psi)| \leq C_5 \|u\|_0^{1/2} \|A^{1/2}u\|_0^{1/2} \|A^{1/2}\theta\|_0^{1/2} \|A\theta\|_0^{1/2} \|\psi\|_0 \quad \forall u \in V, \theta \in E(A), \psi \in W, \\
&|\overline{b}(u,\theta,\psi)| \leq C_6 \|u\|_0^{1/2} \|Au\|_0^{1/2} \|A^{1/2}\theta\|_0 \|\psi\|_0 \quad \forall u \in D(A), \theta, \psi \in W.
\end{aligned}$$

3 The Euler implicit/explicit scheme for the Boussinesq equations

Let $\mathcal{T}_h$ be a family of finite element partitions of Ω into triangles satisfying the usual compatibility conditions [21] with $h = \max h_K$, where h_K is the diameter of an element $K \in \mathcal{T}_h$. We assume that $\mathcal{T}_h$ is shape regular, that is, there exists a constant $\sigma > 0$ such that $h_K < \sigma \rho_K$

for all $K \in \mathcal{T}_h$, where h_K and ρ_h denote the diameter of K and the diameter of the largest ball that can be inscribed into K, respectively.

The finite element subspaces of interest in this paper are defined by the so-called MINI element with the continuous piecewise finite element subspace for the approximation of velocity and pressure and the linear polynomial for temperature, respectively:

$$X_h = \{v_h \in X_h : v_h|_K \in P_1(K)^2 \oplus \text{span}\{\lambda_1\lambda_2\lambda_3\}\ \forall K \in \mathcal{T}_h\},$$
$$M_h = \{q \in M : q|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\},$$
$$W_h = \{\psi \in W : \psi|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\}.$$

Note that λ_1, λ_2, and λ_3 are the barycentric coordinates of the reference element.

We define the subspace V_h of X_h by

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0\ \forall q_h \in M_h\}.$$

Let $P_h : Y \to V_h$ or $Z \to W_h$ denote the L^2-orthogonal projection defined by

$$(P_h\omega, \chi_h) = (\omega, \chi_h) \quad \forall \omega \in Y \text{ or } Z, \chi_h \in V_h \text{ or } W_h.$$

We introduce a discrete analogue $A_h = -P_h\Delta_h$ of the Stokes operator A through the condition $(A_h\phi_h, \psi_h) = (\nabla\phi_h, \nabla\psi_h)$ for all $\phi_h, \psi_h \in X_h \text{or} W_h$. The restriction of A_h to V_h is invertible with the inverse A_h^{-1}. Since A_h^{-1} is self-adjoint and positive definite, we may define the "discrete" Sobolev norms on V_h of any order $r \in R$ by setting

$$\|\omega_h\|_r = \|A_h^{r/2}\omega_h\|_0, \quad \omega_h \in V_h.$$

These norms will be assumed to have various properties similar to their continuous counterparts, implicitly imposing conditions on the structure of the spaces X_h, M_h, and W_h. In particular,

$$\|\omega_h\|_1 = \|\nabla\omega_h\|_0, \qquad \|\omega_h\|_2 = \|A_h\omega_h\|_0 \quad \forall \omega_h \in V_h \text{ or } W_h.$$

The discrete Laplace operator A_h is first introduced in [20] to analyze and obtain the optimal estimates for the transient Navier-Stokes equations. Furthermore, from [1, 22] we know that there exists a positive constant $\beta > 0$ independent of h such that, for all $v_h \in X_h$ and $q_h \in M_h$,

$$b(v_h, q_h) \geq \beta\|v_h\|_1\|q_h\|_0. \tag{3.1}$$

The finite element discretization applied to problem (2.2) leads to spatial discrete equations as follows: Find $(u_h, p_h, \theta_h) \in X_h \times M_h \times W_h$ such that, for all $(v_h, q_h, \psi_h) \in X_h \times M_h \times W_h$,

$$\begin{cases} (u_{ht}, v_h) + a(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) = (f, v_h) - (j\theta_h, v_h), \\ b(u_h, q_h) = 0, \\ (\theta_{ht}, \psi) + \overline{a}(\theta_h, \psi_h) + \overline{b}(u_h, \theta_h, \psi) = (g, \psi_h). \end{cases} \tag{3.2}$$

For the stability and convergence of problem (3.2), we have the following results.

Theorem 3.1 (see [7]) *Let $f \in L^\infty(R^+;Y), f_t \in L^2(0,T;Y), g \in L^\infty(R^+;Z), g_t \in L^2(0,T;Z)$, $u_0 \in D(A) \cap V, \theta_0 \in E(A)$ and assume that $u_h(0) = u_0, \theta_h(0) = \theta_0$. By Theorem* 2.2 *problem* (3.2) *admits a unique solution $(u_h, p_h, \theta_h) \in X_h \times M_h \times W_h$ satisfying*

$$\|u_h\|_0^2 + \nu \int_0^t \|\nabla u_h\|_0^2\,ds \le \|u_0\|_0^2 + \frac{2C_1^2}{\nu}\int_0^t \|f\|_0^2\,ds + \frac{2C_1^3}{\lambda\nu^2}\left(\theta_0 + \frac{C_4^2}{\lambda\nu}\int_0^t \|g\|_0^2\,ds\right),$$

$$\|\nabla u_h\|_0^2 + \nu \int_0^t \|A_h u_h\|_0^2\,ds \le \|\nabla u_0\|_0^2 + \frac{\nu^2 C_4^2}{\lambda}\|\theta_0\|_0^2 + \frac{\nu C_4^2}{\lambda^2}\int_0^t \|g\|_0^2\,ds + \frac{1}{\nu}\int_0^t \|f\|_0^2\,ds,$$

$$\|\theta_h\|_0^2 + \lambda\nu \int_0^t \|\nabla\theta_h\|_0^2\,ds \le \|\theta_0\|_0^2 + \frac{C_1^2}{\lambda\nu}\int_0^t \|g\|_0^2\,ds,$$

$$\|\nabla\theta_h\|_0^2 + \lambda\nu \int_0^t \|A_h\theta_h\|_0^2\,ds \le \|\nabla\theta_0\|_0^2 + \frac{1}{\lambda\nu}\int_0^t \|g\|_0^2\,ds,$$

$$\|u_{ht}\|_0^2 + \nu \int_0^t \|\nabla u_h\|_0^2\,ds + \|\theta_{ht}\|_0^2 + \lambda\nu \int_0^t \|\nabla\theta_h\|_0^2\,ds \le C,$$

$$\frac{\nu}{2}\|A_h u_h\|_0^2 + \int_0^t \|\nabla u_{ht}\|_0^2\,ds + \frac{\lambda\nu}{2}\|A_h\theta_h\|_0^2 + \int_0^t \|\nabla\theta_{ht}\|_0^2\,ds \le C,$$

$$\|u_{htt}\|_0^2 + \nu \int_0^t \|\nabla u_{ht}\|_0^2\,ds + \|\theta_{htt}\|_0^2 + \lambda\nu \int_0^t \|\nabla\theta_{ht}\|_0^2\,ds \le C,$$

$$\|\nabla u_{ht}\|_0^2 + \frac{\nu}{2}\int_0^t \|A_h u_{ht}\|_0^2\,ds + \|\nabla\theta_{ht}\|_0^2 + \frac{\lambda\nu}{2}\int_0^t \|A_h\theta_{ht}\|_0^2\,ds \le C,$$

$$\frac{\nu}{2}\|A_h u_{ht}\|_0^2 + \int_0^t \|\nabla u_{htt}\|_0^2\,ds + \frac{\lambda\nu}{2}\|A_h\theta_{ht}\|_0^2 + \int_0^t \|\nabla\theta_{htt}\|_0^2\,ds \le C,$$

$$\big\|\nabla(u - u_h)\big\|_0 + \|p - p_h\|_0 + \big\|\nabla(\theta - \theta_h)\big\|_0 \le Ch, \qquad \|u - u_h\|_0 + \|\theta - \theta_h\|_0 \le Ch^2.$$

Let $\Delta t > 0$ be the time-step, and let $t_n = n\Delta t$ $(0 \le n \le N = [\frac{T}{\Delta t}])$, u_h^n, p_h^n, and θ_h^n denote the numerical solutions of u_h, p_h, and θ_h at t_n, respectively. We consider the Euler implicit/explicit scheme for the Boussinesq equations (1.1). As we have pointed out in Section 1, the advantage of adopting the Euler implicit/explicit scheme is that a linear system with constant coefficient matrix is obtained, and then a lot of computational cost can be saved.

The Euler implicit/explicit scheme for problem (3.2) reads as follows:

$$\begin{cases} (\frac{u_h^n - u_h^{n-1}}{\Delta t}, v) + \nu(\nabla u_h^n, \nabla v) + b(u_h^{n-1}, u_h^{n-1}, v) - (v, \nabla p_h^n) = (f(t_n), v) - (j\theta_h^n, v), \\ (\nabla \cdot u_h^n, q) = 0, \\ (\frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}, \psi) + \lambda\nu(\nabla\theta_h^n, \nabla\psi) + \overline{b}(u_h^{n-1}, \theta_h^{n-1}, \psi) = (g(t_n), \psi), \end{cases} \tag{3.3}$$

with $1 \le n \le N$. From (3.3) we can see that the discrete system (3.3) is a linear system; for the existence and uniqueness of u_h^n, p_h^n, and θ_h^n, we refer to [25].

For $n = 0$, equations (3.3) can be rewritten as

$$\begin{cases} (u_h^0, v) + \Delta t\nu(\nabla u_h^0, \nabla v) - \Delta t(v, \nabla p_h^0) = \Delta t(f(t_0), v) - \Delta t(j\theta_h^0, v), \\ (\nabla \cdot u_h^0, q) = 0, \\ (\theta_h^0, \psi) + \Delta t\lambda\nu(\nabla\theta_h^0, \nabla\psi) = \Delta t(g(t_0), \psi). \end{cases} \tag{3.4}$$

In order to simplify the expressions, we set $d_t\omega_h^n = \frac{\omega_h^n-\omega_h^{n-1}}{\Delta t}$, where ω is u or θ. Choosing $\psi = \theta_h^0, v_h = u_h^0$, and $q_h = p_h^0$ in (3.4), we have the following estimates:

$$\|u_h^0\|_0^2 \leq \Delta t\big(\|f\|_0 + \Delta t\|g\|_0\big), \qquad \|\theta_h^0\|_0 \leq \Delta t\|g\|_0, \tag{3.5}$$

$$\|\nabla u_h^0\|_0 \leq \frac{C_1}{\nu}\|f\|_0 + \frac{C_1^3}{\lambda\nu^2}\|g\|_0, \qquad \|\nabla\theta_h^0\|_0 \leq \frac{C_1}{\lambda\nu}\|g\|_0. \tag{3.6}$$

From (3.5)-(3.5) we can see that scheme (3.4) is stable.

4 Stability of the numerical solutions

In this section, we establish the stability of the numerical solutions u_h^n, p_h^n, and θ_h^n in the Euler implicit/explicit scheme (3.3) for the Boussinesq equations. The mathematical induction has been used to obtain the desired results; this technique has also been used to other problems, for example, for the Dirichlet problems with (p,q)-Laplacian [26] and the mixed initial-boundary value problems [27].

Theorem 4.1 *Under the conditions of Theorem* 3.1 *and the stability conditions* $k_1\Delta t \leq 1$ *and* $k_2\Delta t \leq 1$, *the solutions* u_h^n, p_h^n, *and* θ_h^n *are bounded for any integer* $0 \leq n \leq [\frac{T}{\Delta t}]$:

$$\|u_h^n\|_0^2 + \nu\Delta t\sum_{i=1}^n\|\nabla u_h^i\|_0^2 \leq \gamma_0^2, \qquad \|\theta_h^n\|_0^2 + \lambda\nu\Delta t\sum_{i=1}^n\|\nabla\theta_h^i\|_0^2 \leq \gamma_1^2, \tag{4.1}$$

$$\|\nabla u_h^n\|_0^2 + \nu\Delta t\sum_{i=1}^n\|A_h u_h^i\|_0^2 \leq k_{01}, \qquad \|\nabla\theta_h^n\|_0^2 + \lambda\nu\Delta t\sum_{i=1}^n\|A_h\theta_h^i\|_0^2 \leq k_{02}, \tag{4.2}$$

$$\|d_t u_h^n\|_0^2 + \nu\Delta t\sum_{i=1}^n\|\nabla d_t u_h^i\|_0^2 \leq k_{03}, \qquad \|d_t\theta_h^n\|_0^2 + \nu\Delta t\sum_{i=1}^n\|\nabla d_t\theta_h^i\|_0^2 \leq k_{04}, \tag{4.3}$$

$$\|A_h u_h^n\|_0^2 \leq k_{05} = 12\nu^{-2}\big(k_{03} + f_\infty^2\big) + \gamma_1 + 48\nu^{-4}C_2^4k_{01}^4\gamma_0^2, \tag{4.4}$$

$$\|A_h\theta_h^n\|_0^2 \leq k_{06} = 12(\lambda\nu)^{-2}\big(k_{04} + b_\infty^2\big) + 48(\lambda\nu)^{-4}C_3^4k_{01}^4\gamma_1^2, \tag{4.5}$$

where

$$\gamma_0^2 = 2\|u_h^0\|_0^2 + 2(1 + 4\Delta t)\|\theta_h^0\|_0^2 + 20T(T + \Delta t)g_\infty^2 + 4(1 + \Delta t)f_\infty^2,$$

$$\gamma_1^2 = 2\|\theta_h^0\|_0^2 + 4T(T + \Delta t)g_\infty^2,$$

$$k_{01} = \exp\left(\frac{8^3}{\nu^2}C_0^4\gamma_0^4\right)\Bigg(2\|\nabla u_h^0\|_0^2 + \frac{\nu}{4}\|Au_h^0\|_0^2 + 40\nu^{-1}\Bigg(T\sup_{0\leq t\leq T}\|f(t)\|_0^2 + \Delta t\sum_{i=0}^n\|\theta_h^i\|_0^2\Bigg)\Bigg),$$

$$k_{02} = \exp\left(\frac{8^3}{\lambda^2\nu^2}C_3^4\gamma_0^2\gamma_1^2\right)\left(2\|\nabla\theta_h^0\|_0^2 + \frac{\lambda\nu}{4}\|A\theta_h^n\|_0^2\Delta t + 24T(\lambda\nu)^{-1}\sup_{0\leq t\leq T}\|g(t)\|_0^2\right),$$

$$
\begin{aligned}
k_{03} = {} & \exp\biggl(\exp\bigl(16\nu^{-2}C_2^2k_{01}\bigr)\exp\biggl(\frac{8C_3^2k_{02}}{\lambda\nu}\biggr)\frac{16}{\lambda^2\nu^3}C_0^2C_3^2k_{02}T\biggr)\cdot\exp\bigl(16\nu^{-2}C_2^2k_{01}\bigr) \\
& \times\biggl(\bigl\|d_tu_h^0\bigr\|_0^2+4\nu^{-1}C_0^2T\sup_{0\le t\le T}\|f_t\|_0^2+\exp\biggl(\frac{8C_3^2k_{02}}{\lambda\nu}\biggr) \\
& \times\biggl(\bigl\|d_t\theta_h^0\bigr\|_0^2+4\frac{C_0^2T}{\lambda\nu}\sup_{0\le t\le T}\bigl\|g(t)\bigr\|_0^2\biggr)\biggr), \\
k_{04} = {} & \exp\biggl(\frac{8C_3^2k_{02}}{\lambda\nu}\biggr)\biggl(\bigl\|d_t\theta_h^0\bigr\|_0^2+\frac{4C_0^2T}{\lambda\nu}\sup_{0\le t\le T}\bigl\|g(t)\bigr\|_0^2\biggr) \\
& \times\biggl(1+\frac{4C_1^2C_3^2k_{02}}{\lambda\nu^2}\exp\biggl(\frac{8C_3^2k_{02}}{\lambda\nu}\biggr)\biggr), \\
k_1 = {} & 2\nu^{-1}C_4^2k_{05}, \qquad k_2=2(\lambda\nu)^{-1}C_6^2k_{05}.
\end{aligned}
$$

Proof We prove this theorem by induction. From (3.5)-(3.6) we know that (4.1)-(4.5) hold for $n=0$. Assume that (4.1)-(4.5) hold for $n=0,\dots,J$ with $0\le J<N=[\frac{T}{\Delta t}]$. We need to prove (4.1)-(4.5) for $n=J+1$.

First, taking $v_h=2\Delta tu_h^n, q_h=2\Delta tp_h^n$, and $\psi_h=2\Delta t\theta_h^n$ in (3.3), we obtain

$$
\begin{aligned}
& \bigl(u_h^n-u_h^{n-1},2u_h^n\bigr)+2\nu\Delta t\bigl\|\nabla u_h^n\bigr\|_0^2+2\Delta tb\bigl(u_h^{n-1},u_h^{n-1},u_h^n\bigr) \\
& \quad =2\Delta t\bigl(f(t_n),u_h^n\bigr)-2\Delta t\bigl(j\theta_h^n,u_h^n\bigr)
\end{aligned}
\tag{4.6}
$$

and

$$
\bigl(\theta_h^n-\theta_h^{n-1},2\theta_h^n\bigr)+2\lambda\nu\Delta t\bigl\|\nabla\theta_h^n\bigr\|_0^2+2\Delta t\overline{b}\bigl(u_h^{n-1},\theta_h^{n-1},\theta_h^n\bigr)=2\Delta t\bigl(g(t_n),\theta_h^n\bigr).
\tag{4.7}
$$

By using of the identities

$$
(a-b,2a)=|a|^2-|b|^2+|a-b|^2 \quad\text{and}\quad 2(a,b)=|a|^2+|b|^2-|a-b|^2,
\tag{4.8}
$$

equations (4.6)-(4.7) can be transformed into

$$
\begin{aligned}
& \bigl\|u_h^n\bigr\|_0^2-\bigl\|u_h^{n-1}\bigr\|_0^2+\bigl\|u_h^n-u_h^{n-1}\bigr\|_0^2+2\Delta t\nu\bigl\|\nabla u_h^n\bigr\|_0^2+2\Delta tb\bigl(u_h^{n-1},u_h^{n-1},u_h^n\bigr) \\
& \quad =2\Delta t\bigl(f(t_n),u_h^n\bigr)-2\Delta t\bigl(j\theta_h^n,u_h^n\bigr)
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
& \bigl\|\theta_h^n\bigr\|_0^2-\bigl\|\theta_h^{n-1}\bigr\|_0^2+\bigl\|\theta_h^n-\theta_h^{n-1}\bigr\|_0^2+2\Delta t\lambda\nu\bigl\|\nabla\theta_h^n\bigr\|_0^2+2\Delta tb\bigl(u_h^{n-1},\theta_h^{n-1},\theta_h^n\bigr) \\
& \quad =2\Delta t\bigl(g(t_n),\theta_h^n\bigr).
\end{aligned}
\tag{4.10}
$$

For the right-hand side terms of (4.9)-(4.10), we have

$$
\begin{aligned}
\bigl|2\Delta t\bigl(f,u_h^n\bigr)\bigr| & =\bigl|2\Delta t\bigl(f,u_h^{n-1}\bigr)+2\Delta t\bigl(f,u_h^n-u_h^{n-1}\bigr)\bigr| \\
& \le 2\Delta t\|f\|_0\bigl\|u_h^{n-1}\bigr\|_0+\frac{1}{4}\bigl\|u_h^n-u_h^{n-1}\bigr\|_0^2+4\Delta t^2\|f\|_0^2,
\end{aligned}
$$

$$
\begin{aligned}
\left|2\Delta t\left(j\theta_h^n, u_h^n\right)\right| &= \left|2\Delta t\left(j\theta_h^n, u_h^{n-1}\right) + 2\Delta t\left(j\theta_h^n, u_h^n - u_h^{n-1}\right)\right| \\
&\leq 2\Delta t\left\|\theta_h^n\right\|_0\left\|u_h^{n-1}\right\|_0 + \frac{1}{4}\left\|u_h^n - u_h^{n-1}\right\|_0^2 + 4\Delta t^2\left\|\theta_h^n\right\|_0^2, \\
\left|2\Delta t\left(g, \theta_h^n\right)\right| &= \left|2\Delta t\left(g, \theta_h^{n-1}\right) + 2\Delta t\left(g, \theta_h^n - \theta_h^{n-1}\right)\right| \\
&\leq 2\Delta t\|g\|_0\left\|\theta_h^{n-1}\right\|_0 + \frac{1}{2}\left\|\theta_h^n - \theta_h^{n-1}\right\|_0^2 + 2\Delta t^2\|g\|_0^2.
\end{aligned}
$$

For the trilinear terms, thanks to Lemma 2.3, we deduce that

$$
\begin{aligned}
2\left|b\left(u_h^{n-1}, u_h^{n-1}, u_h^n\right)\right| &= 2\left|b\left(u_h^{n-1}, u_h^n, u_h^n - u_h^{n-1}\right)\right| \\
&\leq 2C_4\left\|A_h u_h^{n-1}\right\|_0\left\|\nabla u_h^n\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 \\
&\leq \nu\left\|\nabla u_h^n\right\|_0^2 + \nu^{-1}C_4^2\left\|A_h u_h^{n-1}\right\|_0^2\left\|u_h^{n+1} - u_h^n\right\|_0^2, \\
2\left|\overline{b}\left(u_h^{n-1}, \theta_h^{n-1}, \theta_h^n\right)\right| &= 2\left|b\left(u_h^{n-1}, \theta_h^n, \theta_h^n - \theta_h^{n-1}\right)\right| \\
&\leq 2C_6\left\|A_h u_h^{n-1}\right\|_0\left\|\nabla\theta_h^n\right\|_0\left\|\theta_h^n - \theta_h^{n-1}\right\|_0 \\
&\leq \lambda\nu\left\|\nabla\theta_h^n\right\|_0^2 + (\lambda\nu)^{-1}C_6^2\left\|A_h u_h^{n-1}\right\|_0^2\left\|\theta_h^n - \theta_h^{n-1}\right\|_0^2.
\end{aligned}
$$

Combining these estimates with (4.9)-(4.10), we arrive at

$$
\begin{aligned}
&\left\|u_h^n\right\|_0^2 - \left\|u_h^{n-1}\right\|_0^2 + \nu\Delta t\left\|\nabla u_h^n\right\|_0^2 \\
&\quad\leq \left(\nu^{-1}C_4^2\left\|A_h u_h^{n-1}\right\|_0^2\Delta t - \frac{1}{2}\right)\left\|u_h^n - u_h^{n-1}\right\|_0^2 \\
&\qquad + 2\Delta t\left\|u_h^n\right\|_0\left(\|f\|_0 + \left\|\theta_h^n\right\|_0\right) + 4\Delta t^2\left(\|f\|_0^2 + \left\|\theta_h^n\right\|_0^2\right), \qquad (4.11) \\
&\left\|\theta_h^n\right\|_0^2 - \left\|\theta_h^{n-1}\right\|_0^2 + \lambda\nu\Delta t\left\|\nabla\theta_h^n\right\|_0^2 \\
&\quad\leq \left((\lambda\nu)^{-1}C_6^2\left\|A_h u_h^{n-1}\right\|_0^2\Delta t - \frac{1}{2}\right)\left\|\theta_h^n - \theta_h^{n-1}\right\|_0^2 \\
&\qquad + 2\Delta t\left\|\theta_h^n\right\|_0\|g\|_0 + 2\Delta t^2\|g\|_0^2, \qquad (4.12)
\end{aligned}
$$

for all $1 \leq n \leq N$. Under the stability conditions $k_1\Delta t \leq 1, k_2\Delta t \leq 1$ and the induction assumption on $n = 0, 1, \ldots, J$, we have

$$
\nu^{-1}C_4^2\left\|A_h u_h^{n-1}\right\|_0^2\Delta t - \frac{1}{2} \leq \nu^{-1}C_4^2 k_{05}\Delta t - \frac{1}{2} \leq \frac{1}{2}\Delta t k_1 - \frac{1}{2} \leq 0, \qquad (4.13)
$$

$$
(\lambda\nu)^{-1}C_6^2\left\|A_h u_h^{n-1}\right\|_0^2\Delta t - \frac{1}{2} \leq (\lambda\nu)^{-1}C_6^2 k_{05}\Delta t - \frac{1}{2} \leq \frac{1}{2}\Delta t k_2 - \frac{1}{2} \leq 0. \qquad (4.14)
$$

Summing (4.11)-(4.12) for n from 1 to $J+1$ and using (4.13)-(4.14), we obtain

$$
\left\|\theta_h^{J+1}\right\|_0^2 + \Delta t\lambda\nu\sum_{n=1}^{J+1}\left\|\nabla\theta_h^n\right\|_0^2 \leq \left\|\theta_h^0\right\|_0^2 + 2\gamma_1 T g_\infty + 2T\Delta t g_\infty \leq \gamma_1^2
$$

and

$$
\left\|u_h^{J+1}\right\|_0^2 + \Delta t\nu\sum_{n=1}^{J+1}\left\|\nabla u_h^n\right\|_0^2 \leq \left\|u_h^0\right\|_0^2 + 4\gamma_0 T f_\infty + 4T\Delta t f_\infty + 2T\gamma_0\gamma_1 + 4T\Delta t\gamma_1^2 \leq \gamma_0^2,
$$

which is (4.1) with $n = J+1$.

Next, taking $v_h = (\frac{1}{\nu} d_t u_h^n + A_h u_h^n)\Delta t \in V_h$ and $\psi_h = (\frac{1}{\lambda\nu} d_t \theta_h^n + A_h \theta_h^n)\Delta t$ in (3.3), we get

$$\begin{aligned}
&\nu^{-1}\Delta t\|d_t u_h^n\|_0^2 + \nu\Delta t\|A_h u_h^n\|_0^2 + \|\nabla u_h^n\|_0^2 - \|\nabla u_h^{n-1}\|_0^2 + \|\nabla(u_h^n - u_h^{n-1})\|_0^2 \\
&\quad + b\left(u_h^{n-1}, u_h^{n-1}, \frac{1}{\nu} d_t u_h^n + A u_h^n\right)\Delta t \\
&\quad = \left(f(t_n), \nu^{-1} d_t u_h^n + A u_h^n\right)\Delta t - \left(j\theta_h^n, \nu^{-1} d_t u_h^n + A u_h^n\right) \qquad (4.15)
\end{aligned}$$

and

$$\begin{aligned}
&(\lambda\nu)^{-1}\Delta t\|d_t\theta_h^n\|_0^2 + \lambda\nu\Delta t\|A_h\theta_h^n\|_0^2 + \|\nabla\theta_h^n\|_0^2 - \|\nabla\theta_h^{n-1}\|_0^2 + \|\nabla(\theta_h^n - \theta_h^{n-1})\|_0^2 \\
&\quad + \overline{b}\left(u_h^{n-1}, \theta_h^{n-1}, (\lambda\nu)^{-1} d_t\theta_h^n + A\theta_h^n\right)\Delta t \\
&\quad = \left(g(t_n), (\lambda\nu)^{-1} d_t\theta_h^n + A\theta_h^n\right)\Delta t. \qquad (4.16)
\end{aligned}$$

For the right-hand side terms and the trilinear terms, by using Lemma 2.3 we obtain

$$\begin{aligned}
&\left|b\left(u_h^{n-1}, u_h^{n-1}, \nu^{-1} d_t u_h^n + A_h u_h^n\right)\right| \\
&\quad \le 2C_3\|A_h^{1/2}u_h^{n-1}\|_0\|u_h^{n-1}\|_0^{1/2}\|A_h u_h^{n-1}\|_0^{1/2}\left(\nu^{-1}\|d_t u_h^n\|_0 + \|A_h u_h^n\|_0\right) \\
&\quad \le \frac{1}{4\nu}\|d_t u_h^n\|_0^2 + \frac{\nu}{4}\|A_h u_h^n\|_0^2 + \frac{8}{\nu}C_3^2\|A_h^{1/2}u_h^{n-1}\|_0^2\|u_h^{n-1}\|_0\|A_h u_h^{n-1}\|_0 \\
&\quad \le \frac{1}{4\nu}\|d_t u_h^n\|_0^2 + \frac{\nu}{4}\|A_h u_h^n\|_0^2 + \frac{\nu}{8}\|A_h u_h^{n-1}\|_0^2 + \frac{1}{2}\left(\frac{8}{\nu}\right)^3 C_3^4\|A_h^{1/2}u_h^{n-1}\|_0^4\|u_h^{n-1}\|_0^2, \\
&\left|\overline{b}\left(u_h^{n-1}, \theta_h^{n-1}, (\lambda\nu)^{-1} d_t\theta_h^n + A_h\theta_h^n\right)\right| \\
&\quad \le 2C_5\|A_h^{1/2}u_h^{n-1}\|_0^{1/2}\|u_h^{n-1}\|_0^{1/2}\|A_h^{1/2}\theta_h^{n-1}\|_0^{1/2} \\
&\qquad \times \|A_h\theta_h^{n-1}\|_0^{1/2}\left((\lambda\nu)^{-1}\|d_t\theta_h^n\|_0 + \|A_h\theta_h^n\|_0\right) \\
&\quad \le \frac{1}{4\lambda\nu}\|d_t\theta_h^n\|_0^2 + \frac{\lambda\nu}{4}\|A_h\theta_h^n\|_0^2 + \frac{\lambda\nu}{8}\|A_h\theta_h^{n-1}\|_0^2 \\
&\qquad + \frac{1}{2}\left(\frac{8}{\lambda\nu}\right)^3 C_5^4\|A_h^{1/2}u_h^{n-1}\|_0^2\|\nabla\theta_h^{n-1}\|_0^2\|u_h^{n-1}\|_0^2, \\
&\left|\left(f(t_n), \nu^{-1} d_t u_h^n + A_h u_h^n\right)\right| \le \frac{1}{4\nu}\|d_t u_h^n\|_0^2 + \frac{\nu}{16}\|A_h u_h^n\|_0^2 + \frac{20}{\nu}\|f(t_n)\|_0^2, \\
&\left|\left(j\theta_h^n, \nu^{-1} d_t u_h^n + A_h u_h^n\right)\right| \le \frac{1}{4\nu}\|d_t u_h^n\|_0^2 + \frac{\nu}{16}\|A_h u_h^n\|_0^2 + \frac{20}{\nu}\|\theta_h^n\|_0^2, \\
&\left|\left(g(t_n), (\lambda\nu)^{-1} d_t\theta_h^n + A_h\theta_h^n\right)\right| \le \frac{1}{4\lambda\nu}\|d_t\theta_h^n\|_0^2 + \frac{\lambda\nu}{8}\|A_h\theta_h^n\|_0^2 + \frac{12}{\lambda\nu}\|g(t_n)\|_0^2.
\end{aligned}$$

Combining these inequalities with (4.15)-(4.16), we find

$$\begin{aligned}
&(2\nu)^{-1}\Delta t\|d_t u_h^n\|_0^2 + 2\|\nabla u_h^n\|_0^2 - 2\|\nabla u_h^{n-1}\|_0^2 + \nu\Delta t\left(\frac{5}{4}\|A_h u_h^n\|_0^2 - \frac{1}{4}\|A_h u_h^{n-1}\|_0^2\right) \\
&\quad \le \left(\frac{8}{\nu}\right)^3 C_3^4\|\nabla u_h^{n-1}\|_0^2\|u_h^{n-1}\|_0^2\|\nabla u_h^{n-1}\|_0^2\Delta t + \frac{40}{\nu}\|f(t_n)\|_0^2\Delta t + \frac{40}{\nu}\|\theta_h^n\|_0^2\Delta t \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
&(2\lambda\nu)^{-1}\Delta t\|d_t\theta_h^n\|_0^2+2\|\nabla\theta_h^n\|_0^2-2\|\nabla\theta_h^{n-1}\|_0^2\\
&\quad+\lambda\nu\Delta t\left(\frac{5}{4}\|A_h\theta_h^n\|_0^2-\frac{1}{4}\|A_h\theta_h^{n-1}\|_0^2\right)\\
&\quad\le\left(\frac{8}{\lambda\nu}\right)^3C_5^4\|\nabla u_h^{n-1}\|_0^2\|\nabla\theta_h^{n-1}\|_0^2\|u_h^{n-1}\|_0^2\Delta t+\frac{24}{\lambda\nu}\|g(t_n)\|_0^2\Delta t. \qquad (4.18)
\end{aligned}$$

Summing (4.17)-(4.18) for n from 1 to $J+1$ and using Lemma 2.3, we finish the proof of (4.2).

Moreover, for all $v\in V_h$ and $\psi\in W_h$ with $2\le n\le[\frac{T}{\Delta t}]-1$, we deduce from (3.3) that

$$\begin{aligned}
&(d_{tt}u_h^n,v)+a(d_tu_h^n,v)+b(d_tu_h^{n-1},u_h^{n-1},v)+b(u_h^{n-2},d_tu_h^{n-1},v)\\
&\quad=\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(f_t,v)\,dt-(jd_t\theta_h^n,v), \qquad (4.19)\\
&(d_{tt}u_h^1,v)+a(d_tu_h^1,v)=-(jd_t\theta_h^1,v), \qquad (4.20)\\
&(d_{tt}\theta_h^n,\psi)+\overline{a}(d_t\theta_h^n,\psi)+\overline{b}(d_tu_h^{n-1},\theta_h^{n-1},\psi)+\overline{b}(u_h^{n-2},d_t\theta_h^{n-1},\psi)\\
&\quad=\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(g_t,\psi)\,dt, \qquad (4.21)\\
&(d_{tt}\theta_h^1,\psi)+\overline{a}(d_t\theta_h^1,\psi)=0. \qquad (4.22)
\end{aligned}$$

Choosing $v=d_tu_h^1\Delta t$ and $\psi=d_t\theta_h^1\Delta t$ in (4.20) and (4.22), respectively, we obtain

$$\begin{aligned}
&\|d_t\theta_h^1\|_0^2+\|d_{tt}\theta_h^1\|_0^2\Delta t^2+\lambda\nu\|\nabla d_t\theta_h^1\|_0^2\Delta t=\|d_t\theta_h^0\|_0^2,\\
&\|d_tu_h^1\|_0^2+\|d_{tt}u_h^1\|_0^2\Delta t^2+\nu\|\nabla d_tu_h^1\|_0^2\Delta t\le\|d_tu_h^0\|_0^2+\frac{C_1}{2\lambda\nu}\|d_t\theta_h^0\|_0^2.
\end{aligned}$$

Taking $v=2d_tu_h^n\Delta t$ in (4.19) and $\psi=2d_t\theta_h^n\Delta t$ in (4.21) with $2\le n\le[\frac{T}{\Delta t}]$, we get

$$\begin{aligned}
&\|d_tu_h^n\|_0^2-\|d_tu_h^{n-1}\|_0^2+2\nu\Delta t\|\nabla d_tu_h^n\|_0^2+2b(d_tu_h^{n-1},u_h^{n-1},d_tu_h^n)\Delta t\\
&\quad+2b(u_h^{n-2},d_tu_h^{n-1},d_tu_h^n)\Delta t\le2\int_{t_{n-1}}^{t_n}(f_t,d_tu_h^n)\,dt-(jd_t\theta_h^n,d_tu_h^n)\Delta t \qquad (4.23)
\end{aligned}$$

and

$$\begin{aligned}
&\|d_t\theta_h^n\|_0^2-\|d_t\theta_h^{n-1}\|_0^2+2\lambda\nu\Delta t\|\nabla d_t\theta_h^n\|_0^2\\
&\quad+2\overline{b}(d_tu_h^{n-1},\theta_h^{n-1},d_t\theta_h^n)\Delta t+2\overline{b}(u_h^{n-2},d_t\theta_h^{n-1},d_t\theta_h^n)\Delta t\\
&\quad\le2\int_{t_{n-1}}^{t_n}(g_t,d_t\theta_h^n)\,dt. \qquad (4.24)
\end{aligned}$$

By Lemma 2.3 and the Poincaré inequality we have

$$\begin{aligned}
&2|b(d_tu_h^{n-1},u_h^{n-1},d_tu_h^n)|+|2b(u_h^{n-2},d_tu_h^{n-1},d_tu_h^n)|\\
&\quad\le2C_4(\|A_hu_h^{n-1}\|_0+\|A_hu_h^{n-2}\|_0)\|\nabla d_tu_h^n\|_0\|d_tu_h^{n-1}\|_0
\end{aligned}$$

$$\leq \frac{\nu}{4}\|\nabla d_t u_h^n\|_0^2 + 4\nu^{-1}C_4^2\big(\|Au_h^{n-1}\|_0^2 + \|Au_h^{n-2}\|_0^2\big)\|d_t u_h^{n-1}\|_0^2,$$

$$2\big|\overline{b}\big(d_t u_h^{n-1}, \theta_h^{n-1}, d_t\theta_h^n\big)\big| + \big|2\overline{b}\big(u_h^{n-2}, d_t\theta_h^{n-1}, d_t\theta_h^n\big)\big|$$
$$\leq 2C_6\|A_h\theta_h^{n-1}\|_0\|\nabla d_t\theta_h^n\|_0\|d_t u_h^{n-1}\|_0 + 2C_6\|A_h u_h^{n-2}\|_0\|\nabla d_t\theta_h^n\|_0\|d_t\theta_h^{n-1}\|_0$$
$$\leq \frac{\lambda\nu}{4}\|\nabla d_t\theta_h^n\|_0^2 + \frac{4C_6^2}{\lambda\nu}\big(\|A_h\theta_h^{n-1}\|_0^2\|d_t u_h^{n-1}\|_0^2 + \|A_h u_h^{n-2}\|_0^2\|d_t\theta_h^{n-1}\|_0^2\big),$$

$$2\left|\int_{t_{n-1}}^{t_n}\big(f_t, d_t u_h^n\big)\,dt\right| \leq \frac{\nu\Delta t}{4}\|\nabla d_t u_h^n\|_0^2 + 4\nu^{-1}C_1^2\int_{t_{n-1}}^{t_n}\|f_t\|_0^2\,dt,$$

$$2\left|\int_{t_{n-1}}^{t_n}\big(g_t, d_t\theta_h^n\big)\,dt\right| \leq \frac{\lambda\nu\Delta t}{4}\|\nabla d_t\theta_h^n\|_0^2 + 4(\lambda\nu)^{-1}C_1^2\int_{t_{n-1}}^{t_n}\|g_t\|_0^2\,dt,$$

$$\big|\big(jd_t\theta_h^n, d_t u_h^n\big)\big| \leq \frac{\nu}{4}\|\nabla d_t u_h^n\|_0^2 + \frac{4}{\nu}C_1^2\|d_t\theta_h^n\|_0^2.$$

It follows from these inequalities that (4.23) and (4.24) can be transformed into

$$\|d_t u_h^n\|_0^2 - \|d_t u_h^{n-1}\|_0^2 + \nu\Delta t\|\nabla d_t u_h^n\|_0^2$$
$$\leq 4\nu^{-1}C_1^2\left(\int_{t_{n-1}}^{t_n}\|f_t\|_0^2\,dt + \|d_t\theta_h^n\|_0^2\right)$$
$$+ 4\nu^{-1}C_4^2\big(\|A_h u_h^{n-1}\|_0^2 + \|A_h u_h^{n-2}\|_0^2\big)\|d_t u_h^{n-1}\|_0^2 \tag{4.25}$$

and

$$\|d_t\theta_h^n\|_0^2 - \|d_t\theta_h^{n-1}\|_0^2 + \lambda\nu\Delta t\|\nabla d_t\theta_h^n\|_0^2$$
$$\leq 4(\lambda\nu)^{-1}C_1^2\int_{t_{n-1}}^{t_n}\|g_t\|_0^2\,dt$$
$$+ \frac{4C_6^2}{\lambda\nu}\big(\|A_h\theta_h^{n-1}\|_0^2\|d_t u_h^{n-1}\|_0^2 + \|A_h u_h^{n-2}\|_0^2\|d_t\theta_h^{n-1}\|_0^2\big). \tag{4.26}$$

Summing (4.25)-(4.26) for n from 1 to $J+1$ and using Lemma 2.3, we finish the proof of (4.3).

Using again Lemma 2.3 and (3.3), we deduce

$$\nu\|A_h u_h^n\|_0 \leq \|d_t u_h^n\|_0 + \|f(t_n)\|_0 + 2C_3\|\nabla u_h^{n-1}\|_0\|u_h^{n-1}\|_0^{1/2}\|A_h u_h^{n-1}\|_0^{1/2}$$

and

$$\lambda\nu\|A_h\theta_h^n\|_0 \leq \|d_t\theta_h^n\|_0 + \|g(t_n)\|_0 + 2C_5\|u_h^{n-1}\|_0^{1/2}\|\nabla u_h^{n-1}\|_0^{1/2}\|\nabla\theta_h^{n-1}\|_0^{1/2}\|A_h\theta_h^{n-1}\|_0^{1/2}.$$

If $\|A_h u_h^n\|_0 \leq \|A_h u_h^{n-1}\|_0$ and $\|A_h\theta_h^n\|_0 \leq \|A_h\theta_h^{n-1}\|_0$, then by (3.5)-(3.6) we have that (4.4)-(4.5) hold. Otherwise, setting $k_* = \sup_{0\leq n\leq J+1}\|A_h u_h^n\|_0$ and $k_{**} = \sup_{0\leq n\leq J+1}\|A_h\theta_h^n\|_0$ and using (4.1)-(4.3), the obtained inequalities give

$$\|A_h u_h^{J+1}\|_0^2 \leq k_*^2 \leq \frac{12}{\nu^2}\Big(\sup_{1\leq n\leq J+1}\|d_t u_h^n\|_0^2 + f_\infty^2\Big) + 48\nu^{-4}C_3^4\sup_{1\leq n\leq J}\|\nabla u^n\|_0^4\|u^n\|_0^2 + \gamma_1,$$

$$\|A_h\theta_h^{J+1}\|_0^2 \leq k_*^2 \leq \frac{12}{(\lambda\nu)^2}\Big(\sup_{1\leq n\leq J+1}\|d_t\theta_h^n\|_0^2 + g_\infty^2\Big) + 48(\lambda\nu)^{-4}C_5^4\sup_{1\leq n\leq J+1}\|\nabla u^n\|_0^4\|\theta^n\|_0^2.$$

Combining these estimates with (4.1)-(4.3), we finish the proof of (4.4)-(4.5) with $n = J + 1$. □

5 Error estimates

This section is devoted to present the optimal error estimates of velocity, pressure, and temperature in the Euler implicit/explicit scheme (3.3). In order to simplify the descriptions, we denote

$$E_u^n = u_h(t_n) - u_h^n, \qquad E_p^n = p_h(t_n) - p_h^n, \qquad E_\theta^n = \theta_h(t_n) - \theta_h^n,$$

where $(u_h(t_n), p_h(t_n), \theta_h(t_n))$ and $(u_h^n, p_h^n, \theta_h^n)$ be the solutions of problems (3.2) and (3.3), respectively. Furthermore, we set $E_u^0 = E_\theta^0 = 0$.

Let us define the truncation errors R_u^n and R_θ^n by

$$\begin{cases} \frac{u_h(t_n)-u_h(t_{n-1})}{\Delta t} - \nu\Delta u_h(t_n) + (u_h(t_n)\cdot\nabla)u_h(t_n) + \nabla p_h(t_n) = f(t_n) - j\theta_h(t_n) + R_u^n, \\ \nabla\cdot u_h(t_n) = 0, \\ \frac{\theta_h(t_n)-\theta_h(t_{n-1})}{\Delta t} - \lambda\nu\Delta\theta_h(t_n) + (u_h(t_n)\cdot\nabla)\theta_h(t_n) = g(t_n) + R_\theta^n, \end{cases} \tag{5.1}$$

where

$$R_u^n = -\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t-t_n)u_{htt}(t)\,dt \quad \text{and} \quad R_\theta^n = -\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t-t_n)\theta_{htt}(t)\,dt.$$

By subtracting (3.3) from (5.1) we obtain the following error equations:

$$\begin{cases} \frac{E_u^n - E_u^{n-1}}{\Delta t} - \nu\Delta E_u^n + (u_h(t_n)\cdot\nabla)u_h(t_n) - (u_h^{n-1}\cdot\nabla)u_h^{n-1} + \nabla E_p^n = R_u^n - jE_\theta^n, \\ \nabla\cdot E_u^n = 0, \\ \frac{E_\theta^n - E_\theta^{n-1}}{\Delta t} - \lambda\nu\Delta E_\theta^n + (u_h(t_n)\cdot\nabla)\theta_h(t_n) - (u_h^{n-1}\cdot\nabla)\theta_h^{n-1} = R_\theta^n. \end{cases} \tag{5.2}$$

Now, we present the error estimates for E_u^n, E_p^n, and E_θ^n in different norms. In order to simplify the expressions, we denote

$$Z_u^n = u_h(t_n) - u_h(t_{n-1}) = \int_{t_{n-1}}^{t_n} u_{ht}\,dt,$$
$$Z_\theta^n = \theta_h(t_n) - \theta_h(t_{n-1}) = \int_{t_{n-1}}^{t_n} \theta_{ht}\,dt.$$

Then

$$u_h(t_n) - u_h^{n-1} = u_h(t_n) - u_h(t_{n-1}) + u_h(t_{n-1}) - u_h^{n-1} = Z_u^n + E_u^{n-1},$$
$$\theta_h(t_n) - \theta_h^{n-1} = \theta_h(t_n) - \theta_h(t_{n-1}) + \theta_h(t_{n-1}) - \theta_h^{n-1} = Z_\theta^n + E_\theta^{n-1}.$$

As a consequence, we find

$$\begin{aligned} &\big(u_h(t_n)\cdot\nabla\big)u_h(t_n) - \big(u_h^{n-1}\cdot\nabla\big)u_h^{n-1} \\ &\quad = \big(E_u^{n-1}\cdot\nabla\big)u_h^{n-1} + \big(Z_u^n\cdot\nabla\big)u_h(t_{n-1}) + \big(u_h(t_{n-1})\cdot\nabla\big)E_u^{n-1} + \big(u_h(t_n)\cdot\nabla\big)Z_u^n \end{aligned}$$

and

$$\begin{aligned}&\big(u_h(t_n)\cdot\nabla\big)\theta_h(t_n)-\big(u_h^{n-1}\cdot\nabla\big)\theta_h^{n-1}\\&\quad=\big(E_u^{n-1}\cdot\nabla\big)\theta_h^{n-1}+\big(Z_u^n\cdot\nabla\big)\theta_h(t_{n-1})+\big(u_h(t_{n-1})\cdot\nabla\big)E_\theta^{n-1}+\big(u_h(t_n)\cdot\nabla\big)Z_\theta^n.\end{aligned}$$

Lemma 5.1 *Under the assumptions of Theorems* 3.1 *and* 4.1, *we have*

$$\big\|E_u^{J+1}\big\|_0^2+\big\|E_\theta^{J+1}\big\|_0^2+\nu\Delta t\sum_{n=1}^{J+1}\big\|\nabla E_u^n\big\|_0^2+\lambda\nu\Delta t\sum_{n=1}^{J+1}\big\|\nabla E_\theta^n\big\|_0^2\le C\Delta t^2.$$

Proof Taking the inner product of (5.2) with $2\Delta tE_u^n$ and $2\Delta tE_\theta^n$ and using the fact that $\nabla\cdot E_u^n=0$, we obtain

$$\begin{cases}\|E_u^n\|_0^2-\|E_u^{n-1}\|_0^2+2\nu\Delta t\|\nabla E_u^n\|_0^2+2\Delta tb(E_u^{n-1},u_h^{n-1},E_u^n)+2\Delta tb(Z_u^n,u_h(t_{n-1}),E_u^n)\\\quad+2\Delta tb(u_h(t_{n-1}),E_u^{n-1},E_u^n)+2\Delta tb(u_h(t_n),Z_u^n,E_u^n)\\\qquad\le 2\Delta t(R_u^n,E_u^n)-2\Delta t(jE_\theta^n,E_u^n),\\\|E_\theta^n\|_0^2-\|E_\theta^{n-1}\|_0^2+2\lambda\nu\Delta t\|\nabla E_\theta^n\|_0^2+2\Delta t\overline{b}(E_u^{n-1},\theta_h^{n-1},E_\theta^n)\\\quad+2\Delta t\overline{b}(Z_u^n,\theta_h(t_{n-1}),E_\theta^n)+2\Delta t\overline{b}(u_h(t_{n-1}),E_\theta^{n-1},E_\theta^n)+2\Delta t\overline{b}(u_h(t_n),Z_\theta^n,E_\theta^n)\\\qquad\le 2\Delta t(R_\theta^n,E_\theta^n).\end{cases}\tag{5.3}$$

The right-hand side terms of (5.3) can be treated as follows:

$$\begin{aligned}\big|2\Delta t\big(R_u^n,E_u^n\big)\big|&\le\frac{6\Delta tC_1^2}{\nu}\left\|\int_{t_{n-1}}^{t_n}(t-t_{n-1})u_{htt}\,dt\right\|_0^2+\frac{\nu\Delta t}{6}\big\|\nabla E_u^n\big\|_0^2\\&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ntt}\|_0^2\,dt+\frac{\nu\Delta t}{6}\big\|\nabla E_u^n\big\|_0^2,\\\big|2\Delta t\big(R_\theta^n,E_\theta^n\big)\big|&\le\frac{5\Delta tC_1^2}{\lambda\nu}\left\|\int_{t_{n-1}}^{t_n}(t-t_{n-1})\theta_{htt}\,dt\right\|_0^2+\frac{\lambda\nu\Delta t}{5}\big\|\nabla E_\theta^n\big\|_0^2\\&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|\theta_{htt}\|_0^2\,dt+\frac{\lambda\nu\Delta t}{5}\big\|\nabla E_\theta^n\big\|_0^2,\\\big|-2\Delta t\big(jE_\theta^n,E_u^n\big)\big|&\le 2\Delta t\big\|E_\theta^n\big\|_0\big\|E_u^n\big\|_0\le\frac{6\Delta t}{\nu}\big\|E_\theta^n\big\|_0^2+\frac{\nu\Delta t}{6}\big\|E_u^n\big\|_0^2.\end{aligned}$$

For the nonlinear terms, by Lemma 2.3 we have

$$\begin{aligned}\big|2\Delta tb\big(E_u^{n-1},u_h^{n-1},E_u^n\big)\big|&\le 2C_4\Delta t\big\|E_u^{n-1}\big\|_0\big\|A_hu_h^{n-1}\big\|_0\big\|\nabla E_u^n\big\|_0\\&\le\frac{6C_4^2}{\nu}\Delta t\big\|E_u^{n-1}\big\|_0^2\big\|A_hu_h^{n-1}\big\|_0^2+\frac{\nu\Delta t}{6}\big\|\nabla E_u^n\big\|_0^2,\\\big|2\Delta tb\big(Z_u^n,u_h(t_{n-1}),E_u^n\big)\big|&\le 2C_4\Delta t\big\|Z_u^n\big\|_0\big\|A_hu_h(t_{n-1})\big\|_0\big\|\nabla E_u^n\big\|_0\\&\le\frac{6C_4^2}{\nu}\Delta t\big\|Z_u^n\big\|_0^2\big\|A_hu_h(t_{n-1})\big\|_0^2+\frac{\nu\Delta t}{6}\big\|\nabla E_u^n\big\|_0^2\\&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt+\frac{\nu\Delta t}{6}\big\|\nabla E_u^n\big\|_0^2,\end{aligned}$$

$$\begin{aligned}
\bigl|2\Delta t b\bigl(u_h(t_{n-1}),E_u^{n-1},E_u^n\bigr)\bigr| &\le 2C_4\bigl\|A_h u_h(t_{n-1})\bigr\|_0\bigl\|E_u^{n-1}\bigr\|_0\bigl\|\nabla E_u^n\bigr\|_0\\
&\le \frac{6C_4^2}{\nu}\Delta t\bigl\|E_u^{n-1}\bigr\|_0^2\bigl\|A_h u_h(t_{n-1})\bigr\|_0^2+\frac{\nu\Delta t}{6}\bigl\|\nabla E_u^n\bigr\|_0^2,\\
\bigl|2\Delta t b\bigl(u_h(t_n),Z_u^n,E_u^n\bigr)\bigr| &\le 2C_4\Delta t\bigl\|Z_u^n\bigr\|_0\bigl\|A_h u_h(t_n)\bigr\|_0\bigl\|\nabla E_u^n\bigr\|_0\\
&\le \frac{6C_4^2}{\nu}\Delta t\bigl\|Z_u^n\bigr\|_0^2\bigl\|A_h u_h(t_n)\bigr\|_0^2+\frac{\nu\Delta t}{6}\bigl\|\nabla E_u^n\bigr\|_0^2\\
&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt+\frac{\nu\Delta t}{6}\bigl\|\nabla E_u^n\bigr\|_0^2,\\
\bigl|2\Delta t\bar b\bigl(E_u^{n-1},\theta_h^{n-1},E_\theta^n\bigr)\bigr| &\le 2C_6\Delta t\bigl\|E_u^{n-1}\bigr\|_0\bigl\|A_h\theta_h^{n-1}\bigr\|_0\bigl\|\nabla E_\theta^n\bigr\|_0\\
&\le \frac{6C_6^2}{\lambda\nu}\Delta t\bigl\|E_u^{n-1}\bigr\|_0^2\bigl\|A_h\theta_h^{n-1}\bigr\|_0^2+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2,\\
\bigl|2\Delta t\bar b\bigl(Z_u^n,\theta_h(t_{n-1}),E_\theta^n\bigr)\bigr| &\le 2C_6\Delta t\bigl\|Z_u^n\bigr\|_0\bigl\|A_h\theta_h(t_{n-1})\bigr\|_0\bigl\|\nabla E_\theta^n\bigr\|_0\\
&\le \frac{6C_6^2}{\lambda\nu}\Delta t\bigl\|Z_u^n\bigr\|_0^2\bigl\|A_h\theta_h(t_{n-1})\bigr\|_0^2+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2\\
&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2,\\
\bigl|2\Delta t\bar b\bigl(u_h(t_{n-1}),E_\theta^{n-1},E_\theta^n\bigr)\bigr| &\le 2C_6\bigl\|A_h u_h(t_{n-1})\bigr\|_0\bigl\|E_\theta^{n-1}\bigr\|_0\bigl\|\nabla E_\theta^n\bigr\|_0\\
&\le \frac{6C_6^2}{\lambda\nu}\Delta t\bigl\|E_\theta^{n-1}\bigr\|_0^2\bigl\|A_h u_h(t_{n-1})\bigr\|_0^2+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2,\\
\bigl|2\Delta t\bar b\bigl(u_h(t_n),Z_\theta^n,E_\theta^n\bigr)\bigr| &\le 2C_6\Delta t\bigl\|Z_\theta^n\bigr\|_0\bigl\|A_h u_h(t_n)\bigr\|_0\bigl\|\nabla E_\theta^n\bigr\|_0\\
&\le \frac{6C_6^2}{\lambda\nu}\Delta t\bigl\|Z_\theta^n\bigr\|_0^2\bigl\|A_h u_h(t_n)\bigr\|_0^2+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2\\
&\le C\Delta t^2\int_{t_{n-1}}^{t_n}\|\theta_{ht}\|_0^2\,dt+\frac{\lambda\nu\Delta t}{6}\bigl\|\nabla E_\theta^n\bigr\|_0^2.
\end{aligned}$$

From all these inequalities and Theorems 3.1 and 4.1 we obtain

$$\begin{aligned}
&\bigl\|E_u^n\bigr\|_0^2-\bigl\|E_u^{n-1}\bigr\|_0^2+\bigl\|E_u^n-E_u^{n-1}\bigr\|_0^2+\nu\Delta t\bigl\|\nabla E_u^n\bigr\|_0^2\\
&\quad\le C\Delta t\bigl\|E_u^{n-1}\bigr\|_0^2+C\Delta t^2\int_{t_{n-1}}^{t_n}\bigl(\|u_{htt}\|_0^2+\|u_{ht}\|_0^2\bigr)\,dt+\frac{6}{\nu}\Delta t\bigl\|E_\theta^n\bigr\|_0^2
\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}
&\bigl\|E_\theta^{n+1}\bigr\|_0^2-\bigl\|E_\theta^n\bigr\|_0^2+\bigl\|E_\theta^{n+1}-E_\theta^n\bigr\|_0^2+\lambda\nu\Delta t\bigl\|\nabla E_\theta^{n+1}\bigr\|_0^2\\
&\quad\le C\Delta t\bigl\|E_u^{n-1}\bigr\|_0^2+C\Delta t^2\int_{t_{n-1}}^{t_n}\bigl(\|\theta_{htt}\|_0^2+\|\theta_{ht}\|_0^2+\|u_{ht}\|_0^2\bigr)\,dt.
\end{aligned}\tag{5.5}$$

Summing (5.4) and (5.5) from $n=1$ to $J+1$ and using Lemma 2.1, we finish the proof. □

Lemma 5.2 *Under the assumptions of Theorems* 3.1 *and* 4.1, *we have*

$$\bigl\|\nabla E_u^{J+1}\bigr\|_0^2+\bigl\|\nabla E_\theta^{J+1}\bigr\|_0^2+\nu\Delta t\sum_{n=1}^{J+1}\bigl\|A_hE_u^n\bigr\|_0^2+\lambda\nu\Delta t\sum_{n=1}^{J+1}\bigl\|A_hE_\theta^n\bigr\|_0^2\le C\Delta t^2.$$

Proof Taking the inner product of (5.2) with $-2\Delta t A_h E_u^n \in V_h$ and $-2\Delta t A_h E_\theta^n$ and using the fact that $\nabla \cdot E_u^{n+1} = 0$, we obtain

$$\begin{cases} \|\nabla E_u^n\|_0^2 - \|\nabla E_u^{n-1}\|_0^2 + 2\nu\Delta t\|A_h E_u^n\|_0^2 \\ \quad \le 2\Delta t(b(E_u^{n-1}, u_n^{n-1}, A_h E_u^n) + b(Z_u^n, u_h(t_{n-1}), A_h E_u^n) + b(u_h(t_{n-1}), E_u^{n-1}, A_h E_u^n) \\ \qquad + b(u_h(t_n), Z_u^n, A_h E_u^n) + (jE_\theta^n, A_h E_u^n) - (R_u^n, A_h E_u^n)), \\ \|\nabla E_\theta^n\|_0^2 - \|\nabla E_\theta^{n-1}\|_0^2 + 2\lambda\nu\Delta t\|A_h E_\theta^n\|_0^2 \\ \quad \le 2\Delta t(\overline{b}(E_u^{n-1}, \theta_h^{n-1}, A_h E_\theta^n) + \overline{b}(Z_u^n, \theta_h(t_{n-1}), A_h E_\theta^n) + \overline{b}(u_h(t_{n-1}), E_\theta^{n-1}, A_h E_\theta^n) \\ \qquad + \overline{b}(u_h(t_n), Z_\theta^n, A_h E_\theta^n) - (R_\theta^n, A_h E_\theta^n)). \end{cases} \tag{5.6}$$

Now, we treat the linear terms in the right-hand side of (5.6) as follows:

$$\begin{aligned} &\left|2\Delta t\left(R_u^n, A_h E_u^n\right)\right| \le 2\Delta t\left\|R_u^n\right\|_0\left\|A_h E_u^n\right\|_0 \le C\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{htt}\|_0^2\,dt + \frac{\nu\Delta t}{2}\left\|AE_u^n\right\|_0^2, \\ &\left|2\Delta t\left(R_\theta^n, A_h E_\theta^n\right)\right| \le 2\Delta t\left\|R_\theta^n\right\|_0\left\|AE_\theta^{n+1}\right\|_0 \le C\Delta t^2\int_{t_{n-1}}^{t_n}\|\theta_{htt}\|_0^2\,dt + \frac{\lambda\nu\Delta t}{2}\left\|A_h E_\theta^n\right\|_0^2, \\ &\left|-2\Delta t\left(jE_\theta^n, A_h E_u^n\right)\right| \le 2\Delta t\left\|E_\theta^n\right\|_0\left\|AE_u^n\right\|_0 \le 4\nu^{-1}\Delta t\left\|E_\theta^n\right\|_0^2 + \frac{\nu\Delta t}{4}\left\|AE_u^n\right\|_0^2. \end{aligned}$$

For the trilinear terms of (5.6), by applying Lemma 2.3 we have

$$\begin{aligned} &\left|2\Delta t b\left(E_u^{n-1}, u_h^{n-1}, A_h E_u^n\right)\right| \le 2C_4\Delta t\left\|\nabla E_u^{n-1}\right\|_0\left\|A_h u_h^{n-1}\right\|_0\left\|A_h E_u^n\right\|_0 \\ &\qquad \le \frac{6C_4^2}{\nu}\Delta t\left\|\nabla E_u^{n-1}\right\|_0^2\left\|A_h u_h^{n-1}\right\|_0^2 + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2, \\ &\left|2\Delta t b\left(Z_u^n, u_h(t_{n-1}), A_h E_u^n\right)\right| \le 2C_4\Delta t\left\|\nabla Z_u^n\right\|_0\left\|A_h u_h(t_{n-1})\right\|_0\left\|A_h E_u^n\right\|_0 \\ &\qquad \le \frac{6C_4^2}{\nu}\Delta t\left\|\nabla Z_u^n\right\|_0^2\left\|A_h u_h(t_{n-1})\right\|_0^2 + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2 \\ &\qquad \le C\Delta t^2\int_{t_{n-1}}^{t_n}\|\nabla u_{ht}\|_0^2\,dt + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2, \\ &\left|2\Delta t b\left(u_h(t_{n-1}), E_u^{n-1}, A_h E_u^n\right)\right| \le 2C_4\left\|A_h u_h(t_{n-1})\right\|_0\left\|\nabla E_u^{n-1}\right\|_0\left\|A_h E_u^n\right\|_0 \\ &\qquad \le \frac{6C_4^2}{\nu}\Delta t\left\|\nabla E_u^{n-1}\right\|_0^2\left\|A_h u_h(t_{n-1})\right\|_0^2 + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2, \\ &\left|2\Delta t b\left(u_h(t_n), Z_u^n, A_h E_u^n\right)\right| \le 2C_4\Delta t\left\|\nabla Z_u^n\right\|_0\left\|A_h u_h(t_n)\right\|_0\left\|A_h E_u^n\right\|_0 \\ &\qquad \le \frac{6C_4^2}{\nu}\Delta t\left\|\nabla Z_u^n\right\|_0^2\left\|A_h u_h(t_n)\right\|_0^2 + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2 \\ &\qquad \le C\Delta t^2\int_{t_{n-1}}^{t_n}\|\nabla u_{ht}\|_0^2\,dt + \frac{\nu\Delta t}{6}\left\|A_h E_u^n\right\|_0^2, \\ &\left|2\Delta t\overline{b}\left(E_u^{n-1}, \theta_h^{n-1}, A_h E_\theta^n\right)\right| \le 2C_6\Delta t\left\|\nabla E_u^{n-1}\right\|_0\left\|A_h\theta_h^{n-1}\right\|_0\left\|A_h E_\theta^n\right\|_0 \\ &\qquad \le \frac{6C_6^2}{\lambda\nu}\Delta t\left\|\nabla E_u^{n-1}\right\|_0^2\left\|A_h\theta_h^{n-1}\right\|_0^2 + \frac{\lambda\nu\Delta t}{6}\left\|A_h E_\theta^n\right\|_0^2, \\ &\left|2\Delta t\overline{b}\left(Z_u^n, \theta_h(t_{n-1}), A_h E_\theta^n\right)\right| \le 2C_6\Delta t\left\|\nabla Z_u^n\right\|_0\left\|A_h\theta_h(t_{n-1})\right\|_0\left\|A_h E_\theta^n\right\|_0 \\ &\qquad \le \frac{6C_6^2}{\lambda\nu}\Delta t\left\|\nabla Z_u^n\right\|_0^2\left\|A_h\theta_h(t_{n-1})\right\|_0^2 + \frac{\lambda\nu\Delta t}{6}\left\|A_h E_\theta^n\right\|_0^2 \end{aligned}$$

$$\leq C\Delta t^2 \int_{t_{n-1}}^{t_n} \|\nabla u_{ht}\|_0^2\,dt + \frac{\lambda\nu\Delta t}{6}\|A_hE_\theta^n\|_0^2,$$

$$\begin{aligned}\bigl|2\Delta t\overline{b}\bigl(u_h(t_{n-1}),E_\theta^{n-1},A_hE_\theta^n\bigr)\bigr| &\leq 2C_6\|A_hu_h(t_{n-1})\|_0\|\nabla E_\theta^{n-1}\|_0\|A_hE_\theta^n\|_0\\ &\leq \frac{6C_6^2}{\lambda\nu}\Delta t\|\nabla E_\theta^{n-1}\|_0^2\|A_hu_h(t_{n-1})\|_0^2 + \frac{\lambda\nu\Delta t}{6}\|A_hE_\theta^n\|_0^2,\\ \bigl|2\Delta t\overline{b}\bigl(u_h(t_n),Z_\theta^n,A_hE_\theta^n\bigr)\bigr| &\leq 2C_6\Delta t\|\nabla Z_\theta^n\|_0\|A_hu_h(t_n)\|_0\|A_hE_\theta^n\|_0\\ &\leq \frac{6C_6^2}{\lambda\nu}\Delta t\|\nabla Z_\theta^n\|_0^2\|A_hu_h(t_n)\|_0^2 + \frac{\lambda\nu\Delta t}{6}\|A_hE_\theta^n\|_0^2\\ &\leq C\Delta t^2\int_{t_{n-1}}^{t_n}\|\nabla\theta_{ht}\|_0^2\,dt + \frac{\lambda\nu\Delta t}{6}\|A_hE_\theta^n\|_0^2.\end{aligned}$$

Combining these inequalities with (5.6) and summing n from 1 to $J+1$, we obtain

$$\begin{aligned}\|\nabla E_\theta^{J+1}\|_0^2 + \frac{\lambda\nu}{2}\Delta t\sum_{n=1}^{J+1}\|A_hE_\theta^n\|_0^2 &\leq C\Delta t^2\int_0^T\bigl(\|\theta_{htt}\|_0^2 + \|\nabla\theta_{ht}\|_0^2 + \|\nabla u_{ht}\|_0^2\bigr)\,dt\\ &\quad + C\Delta t\sum_{n=1}^{J+1}\|\nabla E_u^{n-1}\|_0^2 + C\Delta t\sum_{n=1}^{J+1}\|\nabla E_\theta^n\|_0^2,\\ \|\nabla E_u^{J+1}\|_0^2 + \frac{\nu}{2}\Delta t\sum_{n=1}^{J+1}\|AE_u^{n+1}\|_0^2 &\leq C\Delta t^2\int_0^T\bigl(\|u_{htt}\|_0^2 + \|\nabla u_{ht}\|_0^2\bigr)\,dt\\ &\quad + C\Delta t\sum_{n=1}^{J+1}\|\nabla E_u^{n-1}\|_0^2 + \frac{4}{\nu}\Delta t\sum_{n=1}^{J+1}\|\nabla E_\theta^{n-1}\|_0^2.\end{aligned}$$

By Lemma 2.1 we complete the proof. □

Now, we present the error estimates for E_p^n, which show that p_h^n is first-order approximations to p in the $L^\infty(L^2)$ norm. In order to achieve this aim, we provide some estimates for $d_tE_u^n = \frac{E_u^n - E_u^{n-1}}{\Delta t}$ and $d_tE_\theta^n = \frac{E_\theta^n - E_\theta^{n-1}}{\Delta t}$.

Lemma 5.3 *Under the assumptions of Theorems* 3.1 *and* 4.1, *we have*

$$\|d_tE_u^{J+1}\|_0^2 + \|d_tE_\theta^{J+1}\|_0^2 + \nu\Delta t\sum_{n=1}^{J+1}\|\nabla d_tE_u^{n+1}\|_0^2 + \lambda\nu\Delta t\sum_{n=1}^{J+1}\|\nabla d_tE_\theta^{n+1}\|_0^2 \leq C\Delta t^2.$$

Proof From problem (5.2) we obtain that, for all $v \in V$ and $\psi \in W$,

$$\begin{aligned}&\bigl(d_{tt}E_u^n,v\bigr) - \nu\bigl(\Delta d_tE_u^n,v\bigr)\\ &\quad = \bigl(d_tR_u^n,v\bigr) - \bigl(jd_tE_\theta^n,v\bigr) - b\bigl(d_tZ_u^n,u_h(t_{n-1}),v\bigr) - b\bigl(Z_h^{n-1},d_tu_h(t_{n-1}),v\bigr)\\ &\qquad - b\bigl(d_tE_u^{n-1},u_h^{n-1},v\bigr) - b\bigl(E_u^{n-2},d_tu_h^{n-1},v\bigr) - b\bigl(d_tu_h(t_{n-1}),E_u^{n-1},v\bigr)\\ &\qquad - b\bigl(u_h(t_{n-2}),d_tE_u^{n-1},v\bigr) - b\bigl(d_tu(t_n),Z_u^n,v\bigr) - b\bigl(u(t_{n-1}),d_tZ_u^n,v\bigr) \qquad (5.7)\end{aligned}$$

and

$$\begin{aligned}&\big(d_{tt}E_\theta^n, v\big) - \lambda\nu\big(\Delta d_t E_\theta^n, v\big)\\&\quad = -b\big(d_t Z_u^n, \theta_h(t_{n-1}), v\big) - b\big(Z_h^{n-1}, d_t\theta_h(t_{n-1}), v\big)\\&\qquad + \big(d_t R_\theta^n, v\big) - b\big(d_t E_u^{n-1}, \theta_h^{n-1}, v\big) - b\big(E_u^{n-2}, d_t\theta_h^{n-1}, v\big) - b\big(d_t u_h(t_{n-1}), E_\theta^{n-1}, v\big)\\&\qquad - b\big(u_h(t_{n-2}), d_t E_\theta^{n-1}, v\big) - b\big(d_t u(t_n), Z_\theta^n, v\big) - b\big(u(t_{n-1}), d_t Z_\theta^n, v\big).\end{aligned}\tag{5.8}$$

Choosing $v = 2\Delta t d_t E_u^n$ and $\psi = 2\Delta t d_t E_\theta^n$ in (5.7)-(5.8), respectively, we deduce that

$$\begin{aligned}&\|d_t E_u^n\|_0^2 - \|d_t E_u^{n-1}\|_0^2 + \|d_t E_u^n - d_t E_u^{n-1}\|_0^2 + \nu\Delta t\|\nabla d_t E_u^n\|_0^2\\&\quad = 2\Delta t\big\{\big(d_t R_u^n, d_t E_u^n\big) - \big(j d_t E_\theta^n, d_t E_u^n\big) - b\big(d_t Z_u^n, u_h(t_{n-1}), d_t E_u^n\big)\\&\qquad - b\big(Z_h^{n-1}, d_t u_h(t_{n-1}), d_t E_u^n\big) - b\big(d_t E_u^{n-1}, u_h^{n-1}, d_t E_u^n\big) - b\big(E_u^{n-2}, d_t u_h^{n-1}, v\big)\\&\qquad - b\big(d_t u_h(t_{n-1}), E_u^{n-1}, d_t E_u^n\big) - b\big(u_h(t_{n-2}), d_t E_u^{n-1}, d_t E_u^n\big)\\&\qquad - b\big(d_t u(t_n), Z_u^n, d_t E_u^n\big) - b\big(u(t_{n-1}), d_t Z_u^n, d_t E_u^n\big)\big\}\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}&\|d_t E_\theta^n\|_0^2 - \|d_t E_\theta^{n-1}\|_0^2 + \|d_t E_\theta^n - d_t E_\theta^{n-1}\|_0^2 + 2\lambda\nu\Delta t\|\nabla d_t E_\theta^n\|_0^2\\&\quad = 2\Delta t\big\{\big(d_t R_\theta^n, d_t E_\theta^n\big) - b\big(d_t Z_u^n, \theta_h(t_{n-1}), d_t E_\theta^n\big) - b\big(Z_h^{n-1}, d_t\theta_h(t_{n-1}), d_t E_\theta^n\big)\\&\qquad - b\big(d_t E_u^{n-1}, \theta_h^{n-1}, d_t E_\theta^n\big) - b\big(E_u^{n-2}, d_t\theta_h^{n-1}, d_t E_\theta^n\big) - b\big(d_t u_h(t_{n-1}), E_\theta^{n-1}, d_t E_\theta^n\big)\\&\qquad - b\big(u_h(t_{n-2}), d_t E_\theta^{n-1}, d_t E_\theta^n\big) - b\big(d_t u(t_n), Z_\theta^n, d_t E_\theta^n\big) - b\big(u(t_{n-1}), d_t Z_\theta^n, d_t E_\theta^n\big)\big\}.\end{aligned}\tag{5.10}$$

Now, we estimate the right-hand side terms of (5.9)-(5.10) separately. For $(d_t R_u^n, d_t E_u^n)$ and $(d_t R_\theta^n, d_t E_\theta^n)$, using the techniques adopted by He [16], we arrive at

$$\big(d_t R_u^n, d_t E_u^n\big) = -\frac{1}{\Delta t^2}\int_{t_{n-1}}^{t_n}(t - t_{n-1})\int_{t-\Delta t}^{t}\big(u_{httt}(s), d_t E_u^n\big)\,ds\,dt$$

and

$$\big(d_t R_\theta^n, d_t E_\theta^n\big) = -\frac{1}{\Delta t^2}\int_{t_{n-1}}^{t_n}(t - t_{n-1})\int_{t-\Delta t}^{t}\big(\theta_{httt}(s), d_t E_\theta^n\big)\,ds\,dt$$

for all $2 \le n \le J$. We deduce from these equalities that

$$\begin{aligned}&\big|2\Delta t\big(d_t R_u^n, d_t E_u^n\big)\big|\\&\quad \le 2\Delta t\|d_t R_u^n\|_0\|d_t E_u^n\|_0 \le C(\nu)\Delta t\|d_t R_u^n\|_0^2 + \frac{\nu}{4}\Delta t\|\nabla d_t E_u^n\|_0^2\\&\quad \le c(\nu)\Delta t\left[\Delta t^{-3/2}\left(\int_{t_{n-1}}^{t_n}(t - t_{n-1})^2\left\|\int_{t-\Delta t}^{t}u_{httt}(s)\,ds\right\|_0^2 dt\right)^{1/2}\right]^2 + \frac{\nu}{4}\Delta t\|\nabla d_t E_u^n\|_0^2\\&\quad \le C\Delta t^2\int_{t_{n-2}}^{t_n}\|u_{httt}\|_0^2\,dt + \frac{\nu}{4}\Delta t\|\nabla d_t E_u^n\|_0^2.\end{aligned}$$

In the same way, we have

$$\big|2\Delta t\big(d_tR_\theta^n, d_tE_\theta^{n+1}\big)\big| \le C(\lambda\nu)\Delta t^2\int_{t_{n-2}}^{t_n}\|\theta_{httt}\|_0^2\,dt + \frac{\lambda\nu}{4}\Delta t\big\|\nabla d_tE_\theta^n\big\|_0^2,$$

$$\big|\big(jd_tE_\theta^n, d_tE_u^n\big)\big| \le \big\|d_tE_\theta^n\big\|_0\big\|d_tE_u^n\big\|_0 \le C(\nu)\big\|d_tE_\theta^n\big\|_0^2 + \frac{\nu}{4}\big\|\nabla d_tE_u^n\big\|_0^2.$$

For the nonlinear terms, with the help of Lemma 2.3, we find that

$$\begin{aligned}
&2\Delta t\big|b\big(d_tZ_u^n, u_h(t_{n-1}), d_tE_u^n\big)\big| \\
&\quad\le 2C_4\Delta t\big\|d_tZ_u^n\big\|_0\big\|A_hu_h(t_{n-1})\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le 2C_4\Delta t^2\big\|u_{htt}(t_{n-1}) + \mathcal{O}\big(\Delta t^2\big)\big\|_0\big\|A_hu_h(t_{n-1})\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le \frac{20C_4^2}{\nu}\Delta t^3\big\|u_{htt}(t_{n-1}) + \mathcal{O}\big(\Delta t^2\big)\big\|_0^2\big\|A_hu_h(t_{n-1})\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(Z_h^{n-1}, d_tu_h(t_{n-1}), d_tE_u^n\big)\big| \\
&\quad\le 2C_4\Delta t\big\|Z_h^{n-1}\big\|_0\big\|A_hd_tu_h(t_{n-1})\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le 2C_4\Delta t\big\|Z_h^{n-1}\big\|_0\big\|A_hu_{ht}(t_{n-2}) + \mathcal{O}(\Delta t)\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le \frac{20C_4^2}{\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt\cdot\big\|A_hu_{ht}(t_{n-2}) + \mathcal{O}(\Delta t)\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(d_tE_u^{n-1}, u_h^{n-1}, d_tE_u^n\big)\big| \le 2C_4\Delta t\big\|d_tE_u^{n-1}\big\|_0\big\|A_hu_h^{n-1}\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\qquad\le \frac{20C_4^2}{\nu}\Delta t\big\|d_tE_u^{n-1}\big\|_0^2\big\|A_hu_h^{n-1}\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(E_u^{n-2}, d_tu_h^{n-1}, d_tE_u^n\big)\big| \le 2C_4\Delta t\big\|E_u^{n-2}\big\|_0\big\|A_hd_tu_h^{n-1}\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\qquad\le 2C_4\Delta t\big\|E_u^{n-2}\big\|_0\big\|A_hu_{ht}^{n-2} + \mathcal{O}(\Delta t)\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\qquad\le \frac{20C_4^2}{\nu}\Delta t\big\|E_u^{n-2}\big\|_0^2\big\|A_hu_{ht}^{n-2} + \mathcal{O}(\Delta t)\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(d_tu_h(t_{n-1}), E_u^{n-1}, d_tE_u^n\big)\big| \\
&\quad\le \frac{20C_4^2}{\nu}\Delta t\big\|E_u^{n-1}\big\|_0^2\big\|A_hu_{ht}(t_{n-2}) + \mathcal{O}(\Delta t)\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(u_h(t_{n-2}), d_tE_u^{n-1}, d_tE_u^n\big)\big| \le \frac{20C_4^2}{\nu}\Delta t\big\|d_tE_u^{n-1}\big\|_0^2\big\|A_hu_h(t_{n-2})\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(d_tu_h(t_n), Z_u^n, d_tE_u^n\big)\big| \\
&\quad\le 2C_4\big\|A_hd_tu_h(t_n)\big\|_0\big\|Z_u^n\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le 2C_4\big\|A_hu_{ht}(t_{n-1}) + \mathcal{O}(\Delta t)\big\|_0\big\|Z_u^n\big\|_0\big\|\nabla d_tE_u^n\big\|_0 \\
&\quad\le \frac{20C_4^2}{\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt\big\|A_hu_{ht}(t_{n-1}) + \mathcal{O}(\Delta t)\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t, \\
&2\Delta t\big|b\big(u(t_{n-1}), d_tZ_u^n, d_tE_u^n\big)\big| \\
&\quad\le \frac{20C_4^2}{\nu}\Delta t^3\big\|u_{htt}(t_{n-1}) + \mathcal{O}(\Delta t)^2\big\|_0^2\big\|A_hu_h(t_{n-1})\big\|_0^2 + \frac{\nu}{20}\big\|\nabla d_tE_u^n\big\|_0^2\Delta t.
\end{aligned}$$

In the same way, we have

$$
\begin{aligned}
&2\Delta t\left|\overline{b}\left(d_tZ_u^n,\theta_h(t_{n-1}),d_tE_\theta^n\right)\right| \\
&\quad\leq \frac{20C_6^2}{\lambda\nu}\Delta t^3\left\|u_{htt}(t_{n-1})+\mathcal{O}\left(\Delta t^2\right)\right\|_0^2\left\|A_h\theta_h(t_{n-1})\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(Z_h^{n-1},d_t\theta_h(t_{n-1}),d_tE_\theta^n\right)\right| \\
&\quad\leq \frac{20C_6^2}{\lambda\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt\cdot\left\|A_h\theta_{ht}(t_{n-2})+\mathcal{O}(\Delta t)\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(d_tE_u^{n-1},\theta_h^{n-1},d_tE_\theta^n\right)\right|\leq \frac{20C_6^2}{\lambda\nu}\Delta t\left\|d_tE_u^{n-1}\right\|_0^2\left\|A_h\theta_h^{n-1}\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(E_u^{n-2},d_t\theta_h^{n-1},d_tE_\theta^n\right)\right|\leq \frac{20C_6^2}{\lambda\nu}\Delta t\left\|E_u^{n-2}\right\|_0^2\left\|A_h\theta_{ht}^{n-2}+\mathcal{O}(\Delta t)\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(d_tu_h(t_{n-1}),E_\theta^{n-1},d_tE_\theta^n\right)\right| \\
&\quad\leq \frac{20C_6^2}{\lambda\nu}\Delta t\left\|E_\theta^{n-1}\right\|_0^2\left\|A_hu_{ht}(t_{n-2})+\mathcal{O}(\Delta t)\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(u_h(t_{n-2}),d_tE_\theta^{n-1},d_tE_\theta^n\right)\right|\leq \frac{20C_6^2}{\lambda\nu}\Delta t\left\|d_tE_\theta^{n-1}\right\|_0^2\left\|A_hu_h(t_{n-2})\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(d_tu_h(t_n),Z_\theta^n,d_tE_\theta^n\right)\right| \\
&\quad\leq \frac{20C_6^2}{\lambda\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|\theta_{ht}\|_0^2\,dt\left\|A_hu_{ht}(t_{n-1})+\mathcal{O}(\Delta t)\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t, \\
&2\Delta t\left|\overline{b}\left(u_h(t_{n-1}),d_tZ_\theta^n,d_tE_\theta^n\right)\right| \\
&\quad\leq \frac{20C_6^2}{\lambda\nu}\Delta t^3\left\|\theta_{htt}(t_{n-1})+\mathcal{O}(\Delta t)^2\right\|_0^2\left\|A_hu_h(t_{n-1})\right\|_0^2+\frac{\lambda\nu}{20}\left\|\nabla d_tE_\theta^n\right\|_0^2\Delta t.
\end{aligned}
$$

Combining these inequalities with (5.9)-(5.10) and summing from $n=1$ to $J+1$, we obtain

$$
\begin{aligned}
&\left\|d_tE_u^{J+1}\right\|_0^2+\sum_{n=1}^{J+1}\left\|d_tE_u^n-d_tE_u^{n-1}\right\|_0^2+\nu\Delta t\sum_{n=1}^{J+1}\left\|\nabla d_tE_u^n\right\|_0^2 \\
&\quad\leq \frac{40TC_4^2}{\nu}\Delta t^2\left\|u_{htt}(t_{n-1})\right\|_0^2\left\|A_hu_h(t_{n-1})\right\|_0^2+\frac{20C_4^2}{\nu}\Delta t\sum_{n=1}^{J+1}\left\|E_u^{n-1}\right\|_0^2\left\|A_hu_{ht}^{n-2}\right\|_0^2 \\
&\qquad+C\Delta t^2\int_0^T\|u_{httt}\|_0^2\,dt+\frac{20C_4^2}{\nu}\Delta t\sum_{n=1}^{J+1}\left\|d_tE_u^{n-1}\right\|_0^2\left(\left\|A_hu_h^{n-1}\right\|_0^2+\|A_hu_{ht}\|_0^2\right) \\
&\qquad+\frac{20}{\nu}\Delta t^2\|A_hu_{ht}\|_0^2\int_0^T\|u_{ht}\|_0^2\,dt+C\Delta t\sum_{n=1}^{J+1}\left\|d_tE_\theta^n\right\|_0^2
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
&\left\|d_tE_\theta^{J+1}\right\|_0^2+\sum_{n=1}^{J+1}\left\|d_tE_\theta^n-d_tE_\theta^{n-1}\right\|_0^2+\nu\Delta t\sum_{n=1}^{J+1}\left\|\nabla d_tE_\theta^n\right\|_0^2 \\
&\quad\leq \frac{20C_6^2T}{\lambda\nu}\Delta t^2\left(\|\theta_{htt}\|_0^2\|A_hu_h\|_0^2+\|u_{htt}\|_0^2\|A_h\theta_h\|_0\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{20C_6^2}{\lambda\nu}\Delta t^2\|A_h\theta_{ht}\|_0^2\int_0^T\|u_{ht}\|_0^2\,dt\\
&+\frac{20C_6^2}{\lambda\nu}\Delta t\sum_{n=1}^{J+1}\bigl(\|d_tE_u^{n-1}\|_0^2\|A_h\theta_h^{n-1}\|_0^2+\|d_tE_\theta^{n-1}\|_0^2\|A_hu_h\|_0^2+\|E_\theta^{n-1}\|_0^2\|A_hu_{ht}\|_0^2\\
&+\|E_u^{n-2}\|_0^2\|A_h\theta_{ht}^{n-2}\|_0^2\bigr)+\frac{20C_6^2}{\lambda\nu}\Delta t^2\|A_hu_{ht}\|_0^2\int_0^T\|\theta_{ht}\|_0^2\,dt\\
&+C\Delta t^2\int_0^T\|u_{httt}\|_0^2\,dt.
\end{aligned}
\tag{5.12}
$$

Substituting (5.12) into (5.11) and using Lemma 2.1, we obtain the desired results. □

Remark 5.1 In the estimates of trilinear terms, the bounds of $\int_0^T\|u_{httt}\|_0^2\,dt$ and $\int_0^T\|\theta_{httt}\|_0^2\,dt$ are used. We can prove them by differentiating (3.2) twice with respect to time and following the proofs provided in [7]. As for the bounds of $\|A_hu_{ht}^{n-2}\|_0$ and $\|A_h\theta_{ht}^{n-2}\|_0$, we can obtain them as we have done in Section 4. Here, we omit these proofs for simplification.

Now, we are in the position of establishing the optimal error estimate for pressure in $L^\infty(L^2)$ norm based on the results presented in Theorems 3.1 and 4.1 and Lemmas 5.1-5.3.

Theorem 5.4 *Under the assumptions of Theorems* 3.1 *and* 4.1, *we have*

$$
\|p_h(t_n)-p_h^n\|_0\le C\Delta t.
$$

Proof We rewrite the first equation of (5.2) as follows:

$$
-\nabla E_p^n=d_tE_u^n-\nu\Delta E_u^n-R_u^n+jE_\theta^n+\bigl(u_h(t_n)\cdot\nabla\bigr)u_h(t_n)-\bigl(u_h^{n-1}\cdot\nabla\bigr)u_h^{n-1}.
\tag{5.13}
$$

Taking the inner product of (5.13) with arbitrary $v\in X$ and using the Poincaré inequality, we obtain

$$
\begin{aligned}
&\bigl|\bigl(d_tE_u^n,v\bigr)\bigr|\le\|d_tE_u^n\|_0\|v\|_0\le C_1\|d_tE_u^n\|_0\|\nabla v\|_0,\\
&\bigl|\nu\bigl(\Delta E_u^n,v\bigr)\bigr|\le\nu\|\nabla E_u^n\|_0\|\nabla v\|_0,\\
&\bigl|\bigl(R_u^n,v\bigr)\bigr|\le\|R_u^n\|_0\|v\|_0\le C\Delta t\biggl(\int_{t_n}^{t_{n+1}}\|u_{htt}\|_0^2\biggr)^{1/2}\|\nabla v\|_0,\\
&\bigl|\bigl(jE_\theta^n,v\bigr)\bigr|\le\|E_\theta^n\|_0\|v\|_0\le C_1\|E_\theta^n\|_0\|\nabla v\|_0.
\end{aligned}
$$

For the nonlinear terms, applying Lemma 2.3 and Theorems 3.1 and 4.1, we arrive at

$$
\begin{aligned}
&\bigl|b\bigl(Z_u^n,u_h(t_{n-1}),v\bigr)\bigr|\le C_4\|Z_u^n\|_0\|A_hu_h(t_{n-1})\|_0\|\nabla v\|_0\le C\biggl(\Delta t\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt\biggr)^{1/2}\|\nabla v\|_0,\\
&\bigl|b\bigl(u_h(t_{n-1}),E_u^{n-1},v\bigr)\bigr|\le C_4\|E_u^{n-1}\|_0\|A_hu_h(t_{n-1})\|_0\|\nabla v\|_0\le C\|E_u^{n-1}\|_0\|\nabla v\|_0,\\
&\bigl|b\bigl(u_h(t_n),Z_u^n,v\bigr)\bigr|\le C\|Z_u^n\|_0\|A_hu_h(t_n)\|_0\|\nabla v\|_0\le C\biggl(\Delta t\int_{t_{n-1}}^{t_n}\|u_{ht}\|_0^2\,dt\biggr)^{1/2}\|\nabla v\|_0,
\end{aligned}
$$

$$\left|b\left(E_u^{n-1}, u_h^{n-1}, v\right)\right| \le C_4 \left\|E_u^{n-1}\right\|_0 \left\|A_h u_h^{n-1}\right\|_0 \|\nabla v\|_0 \le C \left\|E_u^{n-1}\right\|_0 \|\nabla v\|_0.$$

By the discrete inf-sup condition (3.1) we obtain

$$\begin{aligned}\left\|E_p^{n+1}\right\|_0 \le C\left(\beta^{-1}\right)\Bigg\{ &\left\|d_t E_u^{n+1}\right\|_0 + \nu\left\|\nabla E_u^n\right\|_0 + \left\|E_\theta^n\right\|_0 + \left(\Delta t \int_{t_{n-1}}^{t_n} \|u_{ht}\|_0^2\,dt\right)^{1/2} \\ &+ \Delta t\left(\int_{t_n}^{t_{n+1}} \|u_{htt}\|_0^2\,dt\right)^{1/2}\Bigg\}.\end{aligned}$$

With the results of Theorem 3.1 and Lemmas 5.1, 5.2, and 5.3, we complete the proof. □

6 Numerical experiments

In order to gain insights on the established convergence results in Section 5, in this section, we present some numerical tests. Our main interest is to verify and compare the performances of the Euler implicit/explicit scheme (3.3) for the Boussinesq equations. In all experiments, the Boussinesq equations are defined on the convex domain $\Omega = [0,1] \times [0,1]$. The mesh consists of triangular elements that are obtained by dividing Ω into subsquares of equal size and then drawing the diagonal in each subsquare. The model parameters ν and λ are simply set to 1. We use the MINI element that satisfies the discrete inf-sup condition to approximate the velocity u and pressure p and the linear polynomial to approximate the temperature θ. The boundary and initial conditions and right-hand side functions f and g are selected such that the exact solutions are given by

$$\begin{cases} u_1 = 10x^2(x-1)^2 y(y-1)(2y-1)\cos(t), \\ u_2 = -10x(x-1)(2x-1)y^2(y-1)^2\cos(t), \\ p = 10(2x-1)(2y-1)\cos(t), \\ \theta = 10x^2(x-1)^2 y(y-1)(2y-1)\cos(t) - 10x(x-1)(2x-1)y^2(y-1)^2\cos(t), \end{cases}$$

where the components of u are denoted by (u_1, u_2) for convenience.

First, we compare the errors and CPU time of the standard Galerkin finite element method using the backward Euler scheme and Newton iteration to treat the temporal term and nonlinear term and the Euler implicit/explicit scheme with varying time step Δt or mesh length h. From Tables 1-4 we can see that two kinds of numerical methods almost get the same accuracy, but the Euler implicit/explicit scheme takes less CPU time than the standard Galerkin FEM. In other words, the Euler implicit/explicit scheme is comparable with the standard Galerkin FEM but cheaper and more efficient.

Next, we focus on examining the orders of convergence for both standard Galerkin FEM and the Euler implicit/explicit scheme with respect to the time step Δt or the mesh size h. Following [28], we introduce the following way to examine the orders of convergence with respect to the time step Δt or the mesh size h due to the approximation errors $\mathcal{O}(\Delta t^\gamma) + \mathcal{O}(\Delta t^\mu)$. For example, assuming that

$$v_h^{\Delta t} \approx v(x,t_n) + C_1(x,t_n)h^\mu + C_2(x,t_n)\Delta t^\gamma,$$

we have

$$\rho_{v,h,j} = \frac{\|v_h^{\Delta t}(x,t_n) - v_{\frac{h}{2}}^{\Delta t}(x,t_n)\|_j}{\|v_{\frac{h}{2}}^{\Delta t}(x,t_n) - v_{\frac{h}{4}}^{\Delta t}(x,t_n)\|_j} \approx \frac{4^\mu - 2^\mu}{2^\mu - 1},$$

Table 1 Numerical results of the standard Galerkin FEM at time $T = 1.0$ with varying time step Δt but fixed mesh size $h = \frac{1}{32}$

Δt	$\frac{\|u-u^n\|_0}{\|u\|_0}$	$\frac{\|\nabla(u-u^n)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p^n\|_0}{\|p\|_0}$	$\frac{\|\theta-\theta^n\|_0}{\|\theta\|_0}$	$\frac{\|\nabla(\theta-\theta^n)\|_0}{\|\nabla\theta\|_0}$	CPU(S)
0.1	0.000171956	0.0127757	0.00415879	0.000115897	0.0104146	189.625
0.05	0.000162403	0.0127748	0.00416427	0.000107522	0.0104136	306.596
0.025	0.00015784	0.0127745	0.00416706	0.000103686	0.0104134	476.16
0.0125	0.000155623	0.0127744	0.0041685	0.000101859	0.0104133	965.143
0.00625	0.000154529	0.0127744	0.00416949	0.000100969	0.0104133	1,807.32

Table 2 Numerical results of the Euler implicit/explicit scheme at time $T = 1.0$ with varying time step Δt but fixed mesh size $h = \frac{1}{32}$

Δt	$\frac{\|u-u^n\|_0}{\|u\|_0}$	$\frac{\|\nabla(u-u^n)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p^n\|_0}{\|p\|_0}$	$\frac{\|\theta-\theta^n\|_0}{\|\theta\|_0}$	$\frac{\|\nabla(\theta-\theta^n)\|_0}{\|\nabla\theta\|_0}$	CPU(S)
0.1	0.000172356	0.0127759	0.00425416	0.000134799	0.0104262	190.068
0.05	0.000162794	0.0127749	0.00425975	0.000127682	0.0104252	291.062
0.025	0.000158237	0.0127747	0.00426272	0.000124474	0.0104249	457.629
0.0125	0.000156015	0.0127746	0.00426424	0.000122958	0.0104249	875.082
0.00625	0.000154919	0.0127745	0.00426517	0.000122223	0.0104248	1,665.94

Table 3 Numerical results of the standard Galerkin FEM at time $T = 1.0$ with varying mesh size h but fixed time step $\Delta t = 0.01$

$\frac{1}{h}$	$\frac{\|u-u^n\|_0}{\|u\|_0}$	$\frac{\|\nabla(u-u^n)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p^n\|_0}{\|p\|_0}$	$\frac{\|\theta-\theta^n\|_0}{\|\theta\|_0}$	$\frac{\|\nabla(\theta-\theta^n)\|_0}{\|\nabla\theta\|_0}$	CPU(S)
4	0.0225932	0.24839	0.698555	0.0209496	0.15429	14.898
9	0.0085461	0.116535	0.152122	0.00558951	0.078966	23.744
16	0.00247039	0.054131	0.0461534	0.00147279	0.0410255	46.129
25	0.000621059	0.0259783	0.0136212	0.000377335	0.0207576	208.163
36	0.000155184	0.0127744	0.00416879	0.000101501	0.0104133	1,063.48

Table 4 Numerical results of the Euler implicit/explicit scheme at time $T = 1.0$ with varying mesh size h but fixed time step $\Delta t = 0.01$

$\frac{1}{h}$	$\frac{\|u-u^n\|_0}{\|u\|_0}$	$\frac{\|\nabla(u-u^n)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p^n\|_0}{\|p\|_0}$	$\frac{\|\theta-\theta^n\|_0}{\|\theta\|_0}$	$\frac{\|\nabla(\theta-\theta^n)\|_0}{\|\nabla\theta\|_0}$	CPU(S)
4	0.0225932	0.24839	0.698555	0.0209496	0.15429	8.439
9	0.00854615	0.116536	0.152154	0.00558949	0.0789666	10.203
16	0.00247067	0.0541318	0.0461762	0.00147402	0.0410279	27.69
25	0.0006214	0.0259784	0.0136427	0.000383256	0.0207631	129.902
36	0.000155559	0.0127745	0.00420583	0.000122663	0.0104249	703.171

$$\rho_{v,\Delta t,j} = \frac{\|v_h^{\Delta t}(x,t_n) - v_h^{\frac{\Delta t}{2}}(x,t_n)\|_j}{\|v_h^{\frac{\Delta t}{2}}(x,t_n) - v_h^{\frac{\Delta t}{4}}(x,t_n)\|_j} \approx \frac{4^\gamma - 2^\gamma}{2^\gamma - 1}.$$

Here, v is u, p, or θ, and j is 0 or 1. Since $\rho_{v,h,j}$ and $\rho_{v,\Delta t,j}$ approach 4.0 or 2.0, the convergent order will be 2.0 or 1.0, respectively.

In Tables 5-6, we present the convergent orders with fixed spacing $h = \frac{1}{32}$ and varying time steps $\Delta t = 0.1, 0.05, 0.025, 0.0125$. From these results we can see that the Euler implicit/explicit scheme almost gets the same accuracy with the standard Galerkin finite element method and shows optimal convergent orders on Δt. In Tables 7-8, we study the convergence orders with fixed time step $\Delta t = 0.01$ with varying spacing $\frac{1}{h} = 2, 4, 8, 16$. Observe that $\rho_{u,h,0}$ and $\rho_{\theta,h,0}$ are close to 4.0 and $\rho_{u,h,1}$, $\rho_{\theta,h,1}$ approach 2.0, which suggests that the orders of convergence are $\mathcal{O}(h^2)$ for the L^2-norm of u and θ and $\mathcal{O}(h)$ for the H^1-norm of u and θ in space. For the convergence order of pressure, $\rho_{p,h,0}$ is close to 3.2,

Table 5 Convergence orders of the standard Galerkin FEM at time $T = 1.0$ with fixed mesh $h = \frac{1}{32}$

Δt	$\Vert u^{n,\Delta t} - u^{n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{u,\Delta t,0}$	$\Vert u^{n,\Delta t} - u^{n,\frac{\Delta t}{2}} \Vert_1$	$\rho_{u,\Delta t,1}$	$\Vert p^{n,\Delta t} - p^{n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{p,\Delta t,0}$
0.1	1.09819e–005	2.06348	7.97307e–005	2.06314	2.09931e–005	2.0025
0.05	5.32202e–006	2.04257	3.86453e–005	2.03837	1.04834e–005	1.64187
0.025	2.60556e–006	2.01799	1.89589e–005	1.91156	6.38507e–006	0.618958
0.0125	1.29116e–006		9.91804e–006		1.03158e–005	

Δt	$\Vert \theta^{n,\Delta t} - \theta^{n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{\theta,\Delta t,0}$	$\Vert \theta^{n,\Delta t} - \theta^{n,\frac{\Delta t}{2}} \Vert_1$	$\rho_{\theta,\Delta t,1}$
0.1	1.14663e–005	2.06739	8.10132e–005	2.06742
0.05	5.54628e–006	2.03617	3.91857e–005	2.03617
0.025	2.72388e–006	2.01728	1.92448e–005	2.01729
0.0125	1.35028e–006		9.53992e–006	

Table 6 Convergence orders of the Euler implicit/explicit scheme at time $T = 1.0$ with fixed mesh $h = \frac{1}{32}$

Δt	$\Vert u^{E,n,\Delta t} - u^{E,n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{u,\Delta t,0}$	$\Vert u^{E,n,\Delta t} - u^{E,n,\frac{\Delta t}{2}} \Vert_1$	$\rho_{u,\Delta t,1}$	$\Vert p^{E,n,\Delta t} - p^{E,n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{p,\Delta t,0}$
0.1	1.09855e–005	2.06806	7.97559e–005	2.06786	2.10207e–005	2.04139
0.05	5.31197e–006	2.03533	3.85693e–005	2.03005	1.02973e–005	1.57437
0.025	2.60989e–006	2.01716	1.89991e–005	1.91612	6.54057e–006	0.654705
0.0125	1.29384e–006		9.91544e–006		9.9901e–006	

Δt	$\Vert \theta^{E,n,\Delta t} - \theta^{E,n,\frac{\Delta t}{2}} \Vert_0$	$\rho_{\theta,\Delta t,0}$	$\Vert \theta^{E,n,\Delta t} - \theta^{E,n,\frac{\Delta t}{2}} \Vert_1$	$\rho_{\theta,\Delta t,1}$
0.1	1.14669e–005	2.06806	8.10171e–005	2.06809
0.05	5.54475e–006	2.0351	3.91749e–005	2.0351
0.025	2.72457e–006	2.01847	1.92496e–005	2.01848
0.0125	1.34982e–006		9.53665e–006	

Table 7 Convergence orders of the standard Galerkin FEM at time $T = 1.0$ with time step $\Delta t = 0.01$

$\frac{1}{h}$	$\Vert u^{n,h} - u^{n,\frac{h}{2}} \Vert_0$	$\rho_{u,h,0}$	$\Vert u^{n,h} - u^{n,\frac{h}{2}} \Vert_1$	$\rho_{u,h,1}$	$\Vert p^{n,h} - p^{n,\frac{h}{2}} \Vert_0$	$\rho_{p,h,0}$
2	0.0172184	2.65194	0.230412	2.14169	0.63202	4.86123
4	0.00649278	3.37905	0.107585	2.20926	0.130012	3.19887
8	0.00192148	3.98286	0.0486971	2.12864	0.0406432	3.29174
16	0.000482436		0.0228771		0.012347	

$\frac{1}{h}$	$\Vert \theta^{n,h} - \theta^{n,\frac{h}{2}} \Vert_0$	$\rho_{\theta,h,0}$	$\Vert \theta^{n,h} - \theta^{n,\frac{h}{2}} \Vert_1$	$\rho_{\theta,h,1}$
2	0.0166398	3.77383	0.131697	1.96256
4	0.00440926	3.80314	0.0671046	1.91516
8	0.00115937	3.92768	0.0350386	1.99231
16	0.000295181		0.0175869	

Table 8 Convergence orders of the Euler implicit/explicit scheme at time $T = 1.0$ with time step $\Delta t = 0.01$

$\frac{1}{h}$	$\Vert u^{E,n,h} - u^{E,n,\frac{h}{2}} \Vert_0$	$\rho_{u,h,0}$	$\Vert u^{E,n,h} - u^{E,n,\frac{h}{2}} \Vert_1$	$\rho_{u,h,1}$	$\Vert p^{E,n,h} - p^{E,n,\frac{h}{2}} \Vert_0$	$\rho_{p,h,0}$
2	0.0172184	2.65201	0.230411	2.14169	0.631987	4.86024
4	0.00649258	3.37906	0.107584	2.20925	0.130032	3.19873
8	0.00192142	3.98294	0.0486971	2.12866	0.0406512	3.29108
16	0.000482412		0.0228769		0.0123519	

$\frac{1}{h}$	$\Vert \theta^{E,n,h} - \theta^{E,n,\frac{h}{2}} \Vert_0$	$\rho_{\theta,h,0}$	$\Vert \theta^{E,n,h} - \theta^{E,n,\frac{h}{2}} \Vert_1$	$\rho_{\theta,h,1}$
2	0.01664	3.77398	0.131698	1.96258
4	0.00440914	3.80302	0.0671049	1.91516
8	0.00115938	3.92765	0.0350387	1.99231
16	0.000295184		0.017587	

which shows the superconvergence. From these numerical results we can conclude that the Euler implicit/explicit scheme not only has a good accuracy, but also saves a lot of computational cost.

Competing interests
The authors declare that there is no conflict of competing interests.

Authors' contributions
TZ carried out the main theorem and wrote the paper, JJJ revised and checked the paper, and SWX checked the article, TZ, JJJ, and SWX read and approved the final version.

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