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A mixed Legendre-Galerkin spectral method for the buckling problem of simply supported Kirchhoff plates

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Abstract

In this paper, we develop a mixed Legendre-Galerkin spectral method to approximate the buckling problem of simply supported Kirchhoff plates subjected to general plane stress tensor. By the spectral theory of compact operators, the rigorous error estimates for the approximate eigenvalues and eigenfunctions are provided. Finally, we present some numerical experiments which support our theoretical results.

Keywords: buckling problem; Legendre-Galerkin spectral method; simply supported Kirchhoff plates

1 Introduction

Buckling problem has attracted lots of interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. The buckling problem has been studied for years by many researchers (see [1–7] and the references therein). Furthermore, many numerical methods for the buckling problem have been studied, for example, finite element schemes [3, 8–12]. They are based on the well-known mixed methods to deal with the source problem of thin plates modeled by the biharmonic equation which was introduced by Ciarlet and Raviart [13]. The main idea is to introduce an auxiliary variable $\omega := \Delta\psi$ (with ψ being the transverse displacement of the mean surface of the plate) to write a variational formulation of the spectral problem. This mixed trick now has been widely used. Marin and Lupu solved the unknowns of the displacement and microrotation on harmonic vibrations in thermoelasticity of micropolar bodies [14]. Pop et al. proposed a novel algorithm for the condition detection in which the solution breaks down [15]. The author presented a spline collocation method for two different integral equations which were split by Fredholm-Hammerstein integral equations of the second kind over a rectangular region in a plane [16].

The main purpose of this paper is to propose a mixed Legendre-Galerkin spectral method to approximate the buckling problem of simply supported Kirchhoff plates subjected to general plane stress tensor. We introduce a compact operator to analyze the continuous problem. The basis functions are constructed by combining the Legendre polynomials which satisfy the boundary condition automatically. Finally, we prove the optimal order error estimate for the eigenfunctions and a double order for the eigenvalues.

The remainder of this paper is organized as follows. Section 2 describes simply supported Kirchhoff plates subjected to general plane stress tensor. The Legendre-Galerkin spectral method and the error estimate are proposed in Section 3. The details of implementation and the expression of a linear algebra system corresponding to the discrete variational formulation are given in Section 4. Section 5 presents the main numerical results of this work which demonstrate the efficiency and accuracy of this method. Finally, a conclusion of this paper is made in Section 6.

2 The spectral problem

Let Ω denote a bounded Lipschitz polygon domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. The eigenvalues of the buckling problem of a plate on a reference domain read as the following eigenvalue problem:

$$\begin{aligned} &\Delta^2\psi = -\lambda\nabla\cdot(\boldsymbol{\sigma}\nabla\psi), \quad \mathbf{x}\in\Omega,\\ &\psi = \Delta\psi = 0, \quad \mathbf{x}\in\partial\Omega, \end{aligned} \tag{1}$$

where the transverse displacement of the mean surface of the plate $\boldsymbol{\sigma}$ is a plane stress tensor field $\boldsymbol{\sigma}:\Omega\to\mathbb{R}^{2\times 2}$, $\boldsymbol{\sigma}\neq 0$, satisfying the equilibrium equations

$$\begin{aligned} &\boldsymbol{\sigma}^t = \boldsymbol{\sigma}, \quad x\in\Omega,\\ &\nabla\cdot\boldsymbol{\sigma} = 0, \quad x\in\Omega. \end{aligned}$$

Define

$$V = \left\{\psi\in H^2(\Omega):\psi = \frac{\partial^2\psi}{\partial n^2} = 0, \text{on } \partial\Omega\right\}. \tag{2}$$

A classical variational formulation of (1) reads as follows: find $\lambda,\psi\in\mathbb{R}\times V$ such that

$$\int_\Omega \Delta\psi\,\Delta\chi\,d\Omega = \lambda\int_\Omega(\boldsymbol{\sigma}\nabla\psi)\cdot\nabla\chi\,d\Omega, \quad \forall\chi\in V. \tag{3}$$

It is immediate to prove that the eigenvalues of the above problem are real and positive whenever $\boldsymbol{\sigma}$ is positive definite.

In this paper, we consider another variational formulation of (1) which is based on the splitting method. We introduce the auxiliary variable $\omega = \Delta\psi$, then (1) can be rewritten equivalently as follows:

$$\begin{cases} \omega = \Delta\psi, & \text{in } \Omega,\\ \Delta\omega = -\lambda\,\mathrm{div}(\boldsymbol{\sigma}\nabla\psi), & \text{in } \Omega,\\ \psi = \omega = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Therefore, by testing the system above with functions in $H_0^1(\Omega)$, we arrive at the following weak formulation:

Find $(\lambda,\omega,\psi)\in R\times H_0^1(\Omega)\times H_0^1(\Omega)$, $\psi\neq 0$, such that

$$\begin{cases} \int_\Omega\nabla\omega\cdot\nabla v\,d\Omega = -\lambda\int_\Omega(\boldsymbol{\sigma}\nabla\psi)\cdot\nabla v\,d\Omega, & \forall v\in H_0^1(\Omega),\\ \int_\Omega\nabla\psi\cdot\nabla\eta\,d\Omega + \int_\Omega\omega\eta\,d\Omega = 0, & \forall\eta\in H_0^1(\Omega). \end{cases} \tag{5}$$

Now, we introduce a compact notation for the spectral problem (5). Let $\mathbb{A} : (H_0^1(\Omega) \times H_0^1(\Omega)) \times (H_0^1(\Omega) \times H_0^1(\Omega)) \longrightarrow \mathbb{R}$ and $\mathbb{B} : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ be the continuous and symmetric bilinear forms, respectively, defined by

$$\begin{aligned} &\mathbb{A}\big((\omega,\psi),(\nu,\eta)\big) := \int_\Omega \nabla\omega \cdot \nabla\nu \, d\Omega + \int_\Omega \nabla\psi \cdot \nabla\eta \, d\Omega + \int_\Omega \omega\eta \, d\Omega, \\ &\mathbb{B}(\psi,\eta) := \lambda \int_\Omega (\boldsymbol{\sigma}\nabla\psi) \cdot \nabla\nu \, d\Omega. \end{aligned} \tag{6}$$

Then problem (5) can be written as follows:

$$\mathbb{A}\big((\omega,\psi),(\nu,\eta)\big) = -\lambda\mathbb{B}(\psi,\eta), \quad \forall(\nu,\eta) \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{7}$$

Problem (7) has an eigenvalue sequence λ^j (see [8])

$$0 < \lambda^1 \le \lambda^2 \le \cdots \le \lambda^k \le \lambda^{k+1} \le \cdots, \quad \lim_{k\to\infty} \lambda^k = \infty$$

and the associated eigenfunctions

$$\varphi^1, \varphi^2, \ldots, \varphi^k, \varphi^{k+1}, \ldots.$$

Before introducing the spectral method, we introduce the following bounded linear operator which is called solution operator:

$$\begin{aligned} &T : H_0^1(\Omega) \to H_0^1(\Omega), \\ &f \mapsto \psi, \end{aligned}$$

with $(\omega,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ being the solution of the corresponding source problem

$$\mathbb{A}\big((\omega,\psi),(\nu,\eta)\big) = -\mathbb{B}(f,\eta), \quad \forall(\nu,\eta) \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{8}$$

This problem can be decomposed into the following well-posed problems:

- Find $\omega \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\omega \cdot \nabla\nu \, d\Omega = -\int_\Omega (\boldsymbol{\sigma}\nabla f) \cdot \nabla\nu \, d\Omega, \quad \forall\nu \in H_0^1(\Omega); \tag{9}$$

- Find $\psi \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\psi \cdot \nabla\eta \, d\Omega = -\int_\Omega \omega\eta \, d\Omega, \quad \forall\eta \in H_0^1(\Omega). \tag{10}$$

From (7) we know that an equivalent operator formulation of (8) is

$$T\psi = \lambda^{-1}\psi.$$

Problems (9) and (10) are equivalent to the source problem of equations (1) as follows:

Given $f \in H_0^1(\Omega)$, find $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\int_\Omega \Delta\psi \,\Delta v \, d\Omega = \int_\Omega (\boldsymbol{\sigma}\nabla f)\cdot\nabla v, \quad \forall v \in H_0^1(\Omega)\cap H^2(\Omega). \tag{11}$$

This equivalence was introduced by Zhang and Zhang [17] and was proved by Millar and Mora [12] for the sake of completeness.

In order to imply the spectral method on problem (8), we introduce the discrete space as follows.

Let

$$\mathbb{S}_N := \{\Phi \in L^2(\Omega); \Phi \in \mathbb{P}_N(\Omega)\},$$

where $\mathbb{P}_N(\Omega)$ denotes the space of all polynomials of degree at most N with respect to each variable in Ω. Now, let

$$V_N = \mathbb{S}_N \cap H_0^1(\Omega).$$

Then the corresponding $\|\cdot\|_1$ is defined by

$$\forall\varphi \in V_N, \quad \|\varphi\|_1^2 = (\varphi,\varphi) + (\nabla\varphi,\nabla\varphi).$$

We define below an orthogonal projector with respect to the global domain $\Omega : \Pi_N^{1,0} : V \to V_N$ such that for all $\psi \in V$,

$$\big(\nabla\big(\psi - \Pi_N^{1,0}\psi\big), \nabla\chi_N\big)_\Omega = 0, \quad \forall\chi_N \in V_N.$$

Then we have the following approximation results.

Lemma 1 *Let $r \geq 1$, then for all $\psi \in H^r(\Omega)\cap V$, we have the following estimate*:

$$\big\|\psi - \Pi_N^{1,0}\psi\big\|_{1;\Omega} \lesssim N^{1-r}\|\psi\|_{r;\Omega}.$$

Proof For details of the proof, one can refer to Theorem 7.3 in [18]. □

3 Error estimate for eigenvalues

The Legendre-Galerkin spectral method for (1) is as follows: find $(\omega_N, \psi_N) \in V_N \times V_N$ such that

$$\mathbb{A}\big((\omega_N,\psi_N),(v_N,\eta_N)\big) = -\lambda_N \mathbb{B}(\psi_N,\eta_N), \quad \forall(v_N,\eta_N) \in V_N \times V_N. \tag{12}$$

This problem decomposes into a sequence of two well-posed problems, which are the discretizations of problems (5):

- Find $\lambda_N, \omega_N \in V_N$ such that

$$\int_\Omega \nabla\omega_N \cdot \nabla v_N \, d\Omega = -\lambda_N \int_\Omega (\boldsymbol{\sigma}\nabla\psi_N)\cdot\nabla v_N \, d\Omega, \quad \forall v_N \in V_N; \tag{13}$$

- Find $\psi_N \in V_N$ such that

$$\int_\Omega \nabla\psi_N \cdot \nabla\eta_N \, d\Omega = -\int_\Omega \omega_N \eta_N \, d\Omega, \quad \forall \eta_N \in V_N. \tag{14}$$

Problem (12) has eigenvalues

$$0 < \lambda_N^1 \le \lambda_N^2 \le \cdots \le \lambda_N^k \le \lambda_N^{k+1} \le \cdots \le \lambda_N^K, \quad K = \dim(V_N)$$

and the associated eigenfunctions

$$\varphi_N^1, \varphi_N^2, \ldots, \varphi_N^k, \varphi_N^{k+1}, \ldots, \varphi_N^K.$$

As in the continuous case, we introduce for the analysis the discrete solution operator

$$T_N : V_N \to V_N,$$
$$f \mapsto \psi_N,$$

with $(\omega_N, \psi_N) \in V_N \times V_N$ being the solution of the corresponding source problem

$$\mathbb{A}\big((\omega_N, \psi_N), (\nu_N, \eta_N)\big) = -\mathbb{B}(f, \eta_N), \quad \forall (\nu_N, \eta_N) \in V_N \times V_N. \tag{15}$$

This problem can be rewritten in the following forms, which are the respective discretizations of problems (9) and (10):

- Find $\omega_N \in V_N$ such that

$$\int_\Omega \nabla\omega_N \cdot \nabla\nu_N \, d\Omega = -\int_\Omega (\boldsymbol{\sigma}\nabla f) \cdot \nabla\nu_N \, d\Omega, \quad \forall \nu_N \in V_N; \tag{16}$$

- Find $\psi_N \in V_N$ such that

$$\int_\Omega \nabla\psi_N \cdot \nabla\eta_N \, d\Omega = -\int_\Omega \omega_N \eta_N \, d\Omega, \quad \forall \eta_N \in V_N. \tag{17}$$

From (12) we know that an equivalent operator formulation of (15) is

$$T_N \psi = \lambda_N^{-1} \psi.$$

Theorem 1 *There exists $C > 0$, $\forall r > 1$ such that, for all $f \in H_0^1(\Omega)$,*

$$\big\|(T - T_N)f\big\|_{1,\Omega} \le CN^{1-r}.$$

Proof For a given $f \in H_0^1(\Omega)$, let (ω, ψ) and (ω_N, ψ_N) be the solutions of problems (8) and (15), respectively, so that $\psi = Tf$ and $\psi_N = T_N f$. From (9) and (17), and the first Strang lemma [19], we have

$$\|\psi - \psi_N\|_{1,\Omega} \le \left(\inf_{\varphi_N \in V_N} \|\psi - \varphi_N\|_{1,\Omega} + \sup_{\psi_N \in V_N} \frac{\int_\Omega (\omega - \omega_N)\eta_N \, d\Omega}{\|\eta_N\|_{1,\Omega}} \right).$$

The first term on the right-hand side of the above inequality can be estimated by (6.2.18) in [20]

$$\inf_{\varphi_N \in V_N} \|\psi - \varphi_N\|_{1,\Omega} \le \left\|\psi - \Pi_N^{1,0}\psi\right\|_{1,\Omega} \le CN^{1-r}\|\psi\|_r.$$

The estimate of the second term can be obtained by the Cauchy-Schwarz inequality and by (6.9.12) in [20]

$$\sup_{\psi_N \in V_N} \frac{\int_\Omega (\omega - \omega_N)\eta_N \, d\Omega}{\|\eta_N\|_{1,\Omega}} \le \|\omega - \omega_N\|_{0,\Omega} \le \left\|\omega - \Pi_N^{1,0}\omega\right\|_{0,\Omega} \le CN^{-r}\|\omega\|_r.$$

Combining the above three inequalities gives the desired result. □

Let $E(\lambda)$ be the eigenfunctions space of (3) corresponding to the eigenvalue λ, and let C stand for a generic positive constant independent of any functions and of any discretization parameters. Then we give theoretical analysis of the eigenvalue problem by assuming here that all eigenvalues have ascent.

Theorem 2 *Let* (λ, φ) *and* (λ_N, φ_N) *be an eigenpair of* (7) *and* (12), *respectively. If* $\varphi \in V \cap H^r(\Omega)$ *with* $r \ge 2$, *then for all* $N \ge 2$,

$$|\varphi - \varphi_N|_1 \le CN^{1-r}, \tag{18a}$$

$$\lambda - \lambda_N \le CN^{2(1-r)}. \tag{18b}$$

Proof Thanks to $\lim_{N\to\infty} |T - T_N| = 0$, Theorem 1 and Theorem 7.4 in [8], we have

$$\begin{aligned} |\varphi - \varphi_N|_1 &\le C\left|(T - T_N)|_{E(\lambda)}\right|_1 \\ &= C \sup_{\varphi \in E(\lambda), |\varphi|_1 = 1} \left|(T - T_N)\varphi\right|_1 \\ &\le CN^{1-r}. \end{aligned}$$

From (7) and (12), we can obtain

$$\begin{aligned} &\mathbb{A}\big((\omega,\psi) - (\omega_N,\psi_N), (\omega,\psi) - (\omega_N,\psi_N)\big) + \lambda\mathbb{B}(\psi - \psi_N, \psi - \psi_N) \\ &\quad = \mathbb{A}\big((\omega,\psi),(\omega,\psi)\big) - 2\mathbb{A}\big((\omega,\psi),(\omega_N,\psi_N)\big) + \mathbb{A}\big((\omega_N,\psi_N),(\omega_N,\psi_N)\big) \\ &\qquad + \lambda\mathbb{B}(\psi,\psi) - 2\lambda\mathbb{B}(\psi,\psi_N) + \lambda\mathbb{B}(\psi_N,\psi_N) \\ &\quad = \mathbb{A}\big((\omega_N,\psi_N),(\omega_N,\psi_N)\big) + \lambda\mathbb{B}(\psi_N,\psi_N) \\ &\quad = -(\lambda_N - \lambda)\mathbb{B}(\psi_N,\psi_N). \end{aligned}$$

Dividing by $\mathbb{B}(\psi_N,\psi_N)$ both sides of the above equation together with (12) yields

$$\begin{aligned} \lambda_N - \lambda &= \frac{|\omega_N - \omega|_1^2 + |\psi_N - \psi|_1^2}{\mathbb{B}(\psi_N,\psi_N)} - \lambda\frac{|\psi_N - \psi|_1^2}{\mathbb{B}(\varphi_N,\varphi_N)} \\ &\le \frac{|\omega_N - \omega|_1^2 + |\psi_N - \psi|_1^2}{\mathbb{B}(\psi_N,\psi_N)} \le CN^{2(1-r)}, \end{aligned} \tag{19}$$

which concludes the result (18b). □

4 Implementation

In this subsection, we start with some implementation details in the basis function construction.

Firstly, construct one-dimensional basis functions similar to [21] as follows: for $0 \le i \le N-2$,

$$\phi_i(x) = \frac{1}{\sqrt{4i+6}}\big(L_i(x) - L_{i+2}(x)\big), \quad x \in \Lambda = (-1,1).$$

We have

$$V_N = \operatorname{span}\big\{\phi_0(x), \phi_1(x), \ldots, \phi_{N-2}(x)\big\}.$$

Express $\psi_N(x,y)$ as a combination of the above basis functions

$$\omega_N(x,y) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \hat{\omega}_{ij}\phi_i(x)\phi_j(y), \qquad \psi_N(x,y) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \hat{\psi}_{ij}\phi_i(x)\phi_j(y), \tag{20}$$

where $\{\phi_i(x)\}_{i=0}^{N-2}$ and $\{\phi_i(y)\}_{i=0}^{N-2}$ are the sets of one-dimensional basis functions in the x- and y-direction, respectively (i.e., the basis functions of V_N).

By inserting the expansions of (20) into (12) and taking the test functions as $v_N = \phi_m\phi_n$, $\eta_N = \phi_p\phi_q$, we set

$$B_{ij} = (\phi_j, \phi_i)_\Lambda, \qquad C_{ij} = (\partial_x\phi_j, \phi_i)_\Lambda, \qquad S_{ij} = (\partial_x\phi_i, \partial_x\phi_j)_\Lambda,$$
$$B = (B_{ij})_{0\le i,j\le N-2}, \qquad C = (C_{ij})_{0\le i,j\le N-2}, \qquad S = (S_{ij})_{0\le i,j\le N-2}.$$

Denote

$$\mathbf{S} = S\otimes B + B\otimes S, \qquad \mathbf{M} = B\otimes B, \qquad \mathbf{E} = \sigma_{11}S\otimes B - (\sigma_{12}+\sigma_{21})C\otimes C + \sigma_{22}B\otimes S,$$

where

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Then problem (12) can be written in the following matrix form:

$$\begin{pmatrix} \mathbf{M} & \mathbf{S} \\ \mathbf{S}^t & \mathbf{0} \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \overline{\psi} \end{pmatrix} = \lambda_N \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \overline{\psi} \end{pmatrix}, \tag{21}$$

where $\otimes$ is the tensor product operator, $\overline{\omega}$, $\overline{\psi}$ is the coefficient vector of a numerical solution of ω_N, ψ_N.

The nonzero entries of B and C can be easily determined from the properties of Legendre polynomials as follows.

Lemma 2 *The stiffness matrix S is a diagonal matrix with*

$$S_{i,i} = 1, \quad i = 0,1,\ldots,N-2.$$

The mass matrix B is symmetric penta-diagonal whose nonzero elements are

$$B_{i,i} = \frac{2}{(2i+1)(2i+5)}, \qquad B_{i,i+2} = B_{i+2,i} = -\frac{2}{(2i+5)\sqrt{4i+6}\sqrt{4i+10}}.$$

Nonzero elements of the mass matrix C are

$$C_{ij} = \begin{cases} \frac{1}{\sqrt{(2i+3)(2i+5)}} & \textit{if } i-j=-1, \\ -\frac{1}{\sqrt{(2i+1)(2i+3)}} & \textit{if } i-j=1. \end{cases}$$

It is obvious that B is a symmetric positive definite matrix, and it is easy to show that

$$b_{kj} = 0, \quad \text{if } k \neq j, j \pm 2.$$

Hence, the above linear system (21) can be solved efficiently by LAPACK routine dggev. The dggev is based on the generalized Schur decomposition (the QZ decomposition). The computational complexity is $O(N^6)$ for the solution of the eigenvalue problem (by using the QZ decomposition), where N is a total number of collocation points for one direction of the unknowns.

5 Numerical results

In this section, we will show some numerical results which demonstrate the accuracy and efficiency of the Legendre-Galerkin spectral method for the buckling problem of simply supported Kirchhoff plates on the reference square. We have taken the unit square $\Omega = (0,1) \times (0,1)$ as an example of a convex domain.

Example 1 (Uniformly compressed square plate) We consider the following problem associated with the vibration problem of a simply supported Kirchhoff plate:

$$\Delta^2 u = -\lambda \Delta u, \quad \text{in } \Omega,$$
$$u = \Delta u = 0, \quad \text{on } \Gamma.$$

For more information on the exact eigenvalues and eigenfunctions of the last problem, one can refer to [8, 22].

In Table 1 we list the lowest four buckling coefficients. The table also includes the results computed from [12] with $N = 40$ and $N = 80$ by the finite element method. The last column of the table shows the exact buckling coefficient. In Figure 1 we plot the error of the first four eigenvalues in a semi-log scale for the buckling problem of simply supported

Table 1 The lowest four buckling coefficients of a uniformly compressed simply supported square plate

	$N = 10$	Ref. [12] $N = 40$	Ref. [12] $N = 80$	Exact
λ_N^1	19.7392088021788	19.7614	19.7448	19.739208802178716
λ_N^2	49.3480220530284	49.5155	49.3899	49.348022005446794
λ_N^3	49.3480220530284	49.5155	49.3899	49.348022005446794
λ_N^4	78.9568353038783	79.4444	79.0786	78.956835208714864

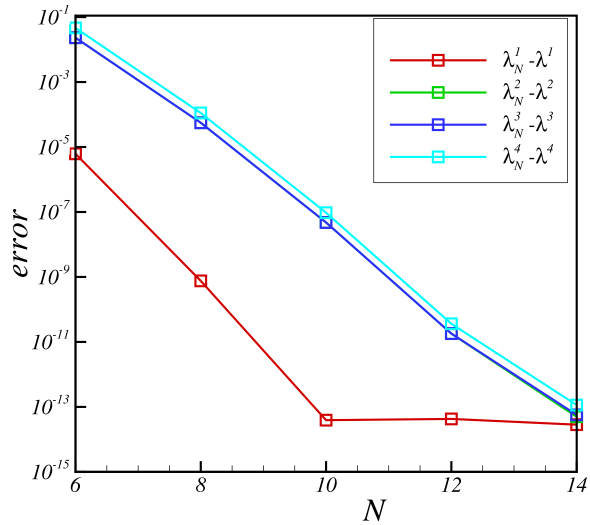
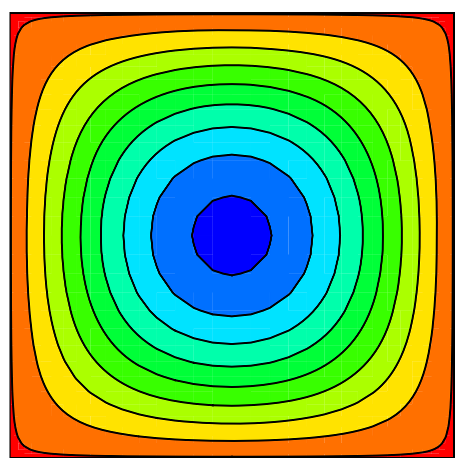


Figure 1 The error of the first four eigenvalues in a semi-log scale for the buckling problem of simply supported Kirchhoff plates.

Figure 2 The contour of eigenfunction corresponding to the first eigenvalue for the buckling problem of simply supported Kirchhoff plates, with $N = 36$.



Kirchhoff plates. These results demonstrate that our approach can achieve an exponential convergence rate which is in good agreement with the theoretical result. Moreover, Figure 2 shows the contour line of eigenfunction corresponding to the lowest buckling coefficient of the buckling problem.

Example 2 (Square plate under combined bending and compression in one direction) In this paper, we define a non-dimensional buckling intensity as follows:

$$\kappa_i := \frac{\lambda_N^i}{\pi^2}$$

and choose a plane stress field

$$\boldsymbol{\sigma} = \begin{pmatrix} 1-\alpha y & 0 \\ 0 & 0 \end{pmatrix}.$$

Table 2 The lowest four buckling coefficients of a uniformly compressed simply supported square plate

α	$N = 10$	$N = 20$	$N = 30$	Ref. [12] $N = 80$
2.0	25.5283500058494	25.5283479481006	25.5283479481008	25.5619
4/3	11.0117810710886	11.0117810716443	11.0117810716443	11.0152
1	7.81195727517932	7.81195727522176	7.81195727522178	7.8142
4/5	6.59506010731459	6.59506010732159	6.59506010732162	6.6026
2/3	5.96338451327805	5.96338451327973	5.96338451327975	5.9701

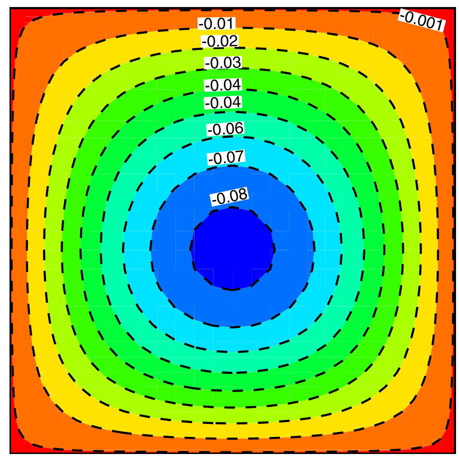


Figure 3 Principal buckling mode corresponding to $\alpha = 2$, with $N = 36$.

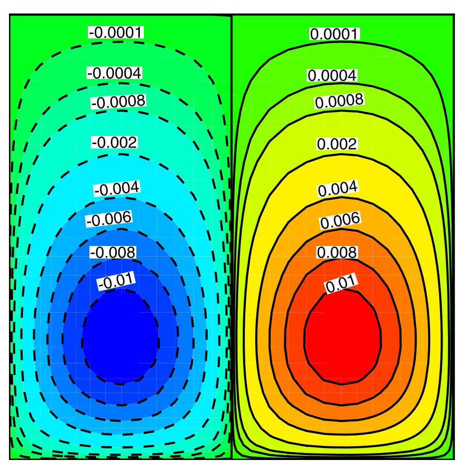


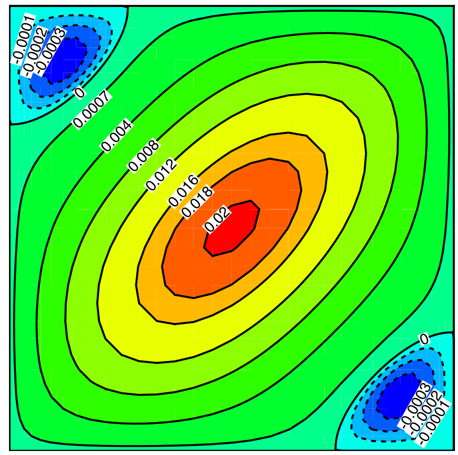
Figure 4 Principal buckling mode corresponding to $\alpha = 1$, with $N = 36$.

For $0 < \alpha < 2$, the linearly varying load represents an eccentric bending which can be regarded as a combination of pure bending and uniform compression. It is the case of pure in-plane bending when $\alpha = 2$. The non-dimensional buckling intensity is listed in Table 2. The last columns show the results from [12]. It can be seen from Table 2 that the results obtained with our method present an excellent agreement with those in [12] for all linearly varying loading cases. Figures 3 and 4 show the transverse displacements of the principal buckling mode.

Table 3 The lowest four buckling coefficients of a uniformly compressed simply supported square plate with ten convergence digits

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	Ref. [12]
κ_1	9.3245119409	9.3245202591	9.3245202616	9.3245202616	9.3236

Figure 5 Principal buckling mode of the shear loaded square plate, with $N = 36$.



Example 3 (Shear loaded square plate) In this example, we chose the plane stress field

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have computed the non-dimensional buckling intensity of the same plate as in the previous example, subjected to a uniform shear load.

In Table 3 we list the lowest non-dimensional buckling intensity. Also we compare our results with those obtained in [12] which are listed in the last column. Figure 5 shows the transverse displacements of the principal buckling mode for the shear loaded square plate computed with the computational parameter $N = 36$.

6 Conclusion

We have proposed an efficient mixed Legendre-Galerkin spectral method for the solution of the buckling problem of simply supported Kirchhoff plates. The optimal error estimates for eigenvalues and eigenfunctions are also provided. Finally, the efficiency and accuracy of the spectral method for the buckling problem of simply supported Kirchhoff plates have been illustrated by numerical results. In the future, we will consider dealing with more complicated domains by the mixed Legendre-Galerkin spectral element method. Firstly, we will divide the domain into lots of subdomains by domain decomposition, then we will construct the same basis functions in each subdomain similar to the square plate examples and construct the hat basis functions at each interface between the two domains. Here we want to mention that if the subdomain is irregular, the expressions of the mass matrix and the stiffness matrix should be computed by numerical integration.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors discussed, read and approved the final version of the manuscript.

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