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Multiple periodic solutions for second order Josephson-type differential systems

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Abstract

In this paper, some new existence theorems are obtained for multiple periodic solutions of second order Josephson-type differential systems with partially periodic potential by using the minimax methods in critical point theory, which generalize and improve some known results in the literature.

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1 Introduction

In this paper, we study the second order Josephson-type differential systems

$$\begin{cases} \ddot{u}(t) + Au(t) - \nabla F(t, u(t)) = h(t), & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \quad (1.1)$$

where A is an $(N \times N)$ -symmetric matrix, $h(t) \in L^1([0, T]; \mathbb{R}^N)$, $T > 0$, $\nabla F(t, x)$ denotes its gradient with respect to the second variable, that is,

$$\nabla F(t, x) = \nabla_x F(t, x) = (\partial F / \partial x_1, \dots, \partial F / \partial x_N), \quad x(t) = (x_1(t), \dots, x_N(t)),$$

and $F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following assumptions:

(H1) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

(H2) $\dim N(A) = m \geq 1$ and matrix A has no eigenvalue of the form $k^2 v^2$ ($k \in \mathbb{N} \setminus \{0\}$), where $v = 2\pi/T$.

(H3) There exist linearly independent vectors $e_j \in \mathbb{R}^N$ ($1 \leq j \leq m$) such that

$$N(A) = \text{span}\{e_1, e_2, \dots, e_m\}$$

and

$$\int_0^T (h(t), e_j) dt = 0.$$

This problem (1.1) occurs in various branches of mathematical physics, for example, when $A = N^2 D^2$ and $-\nabla F(t, u(t)) = f(u(t)) = (a_1 \sin u_1, \dots, a_N \sin u_N)$, problem (1.1) reduces to the nonlinear systems of the form

$$\begin{cases} \ddot{u}(t) + N^2 D^2 u(t) + f(u(t)) = h(t), & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.2}$$

where D is an $(N \times N)$ -symmetric matrix. This type of problem can be applied to describe the motion of forced linearly coupled pendulums.

During the past two decades, the existence of periodic solutions for second order differential systems have been studied extensively, and many solvability conditions have been obtained via variational methods and critical point theory. In this direction we mention the papers [1–15], and we refer the reader to [16–19] for a broad introduction to variational methods and critical point theory. It might be also interesting to study the above mentioned abstract equations with more general potentials, see the paper [20].

In the classical monograph [6], Mawhin and Willem proved that problem (1.1) has at least one solution by using the saddle point theorem under the following bounded condition: there exists $g \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| + |\nabla F(t, x)| \leq g(t) \tag{1.3}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. They obtained the following result.

Theorem A ([6]) *Suppose that F satisfies (H1)-(H3), (1.3) and*

(H4) *there exists $T_j > 0$ such that*

$$F(t, x + T_j e_j) = F(t, x), \quad 1 \leq j \leq m,$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Then Eq. (1.1) has at least one solution in H_T^1 , where the Sobolev space H_T^1 is defined by

$$H_T^1 = \left\{ u : [0, T] \rightarrow \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0) = u(T) \text{ and } \dot{u} \in L^2([0, T]; \mathbb{R}^N) \right\}$$

and H_T^1 is a Hilbert space with the norm

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}}, \quad u \in H_T^1.$$

When the nonlinearity $\nabla F(t, x)$ is sublinear, that is, there exist $f, g \in L^1([0, T]; \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that

$$|\nabla F(t, x)| \leq f(t)|x|^\alpha + g(t) \tag{1.4}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, Tang [12, 13] researched the existence of periodic solutions for system (1.1) in the case $A \equiv 0$.

Subsequently, Feng and Han [3] generalized Theorem A to the sublinear case, and they assume that the following assumptions hold:

(H5) There exist $T_j > 0, 1 \leq j \leq m$ such that

$$F(t, x + T_j e_j) = F(t, x), \quad 1 \leq j \leq r,$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

(H6) $\lim_{\|x\| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\|x\|^{2\alpha}} = -\infty$, as $x \in N(A) \ominus \text{span}\{e_1, e_2, \dots, e_r\}$.

Theorem B ([3]) *Suppose that F satisfies (H1)-(H3), (H5)-(H6) and (1.4). Then Eq. (1.1) has at least $r + 1$ distinct solutions in H_T^1 .*

In [15], the author obtained the following result.

Theorem C ([15]) *Suppose that F satisfies (H1)-(H3), (H5), (1.4) and the following generalized Ahmad-Lazer-Paul type coercive conditions:*

(H7) $\lim_{\|x\| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\|x\|^{2\alpha}} < -L$, as $x \in N(A) \ominus \text{span}\{e_1, e_2, \dots, e_r\}$, where L is a positive constant. Then Eq. (1.1) has at least $r + 1$ distinct solutions in H_T^1 .

In this paper, we use a more general control function instead of $|x|^\alpha$ in (1.4). By using the generalized saddle point theorem due to Liu [5], we can prove the existence of multiple periodic solutions for the second order Josephson-type differential systems for a new and large range of the nonlinear term.

2 Preliminaries

In [6], Mawhin and Willem established a variational structure which enables us to reduce the existence of solutions for problem (1.1) to the existence of critical points of the following energy functional. Define the energy functional associated with problem (1.1) on H_T^1

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} \int_0^T (Au(t), u(t)) dt + \int_0^T F(t, u(t)) dt \\ & + \int_0^T (h(t), u(t)) dt. \end{aligned}$$

It follows from assumption (H1) that the functional φ is continuously differentiable. Moreover, one has

$$\begin{aligned} \langle \varphi'(u), v \rangle = & \int_0^T (\dot{u}(t), \dot{v}(t)) dt - \int_0^T (Au(t), v(t)) dt + \int_0^T (\nabla F(t, u(t)), v(t)) dt \\ & + \int_0^T (h(t), v(t)) dt, \quad \forall u, v \in H_T^1. \end{aligned}$$

Then the solutions of problem (1.1) correspond to the critical points of φ (see [6]).

Let

$$q(u) = \int_0^T [|\dot{u}(t)|^2 - (Au(t), u(t))] dt.$$

Therefore, we can see that

$$\begin{aligned} q(u) &= \|u\|^2 - \int_0^T ((A + I)u(t), u(t)) dt \\ &= \langle (I - K)u, u \rangle, \end{aligned}$$

where I denotes the identity operator on H_T^1 and $K : H_T^1 \rightarrow H_T^1$ is the linear self-adjoint operator defined, using Riesz representation theorem, by

$$\int_0^T ((A + I)u(t), v(t)) dt = (Ku, v), \quad \forall u, v \in H_T^1.$$

It is easy to see that K is compact. By classical spectral theory, we can decompose H_T^1 into the orthogonal sum of invariant subspaces for $I - K$

$$H_T^1 = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = \text{Ker}(I - K) = N(A)$ and $\dim H^- < +\infty$, for some $\delta > 0$, we have

$$q(u) \leq -\delta \|u\|^2, \quad \forall u \in H^-, \tag{2.1}$$

$$q(u) \geq \delta \|u\|^2, \quad \forall u \in H^+. \tag{2.2}$$

Lemma 2.1 ([6]) *There is a continuous embedding $H_T^1 \hookrightarrow C([0, T], \mathbb{R}^N)$, and the embedding is compact. Then there exists $C_0 > 0$ such that*

$$\|u\|_\infty := \max_{0 \leq t \leq T} |u(t)| \leq C_0 \|u\|, \quad \forall u \in H_T^1. \tag{2.3}$$

Define

$$Y_0 = \text{span}\{e_1, e_2, \dots, e_r\}, \quad Y_1 = N(A) \ominus Y_0 = \text{span}\{e_{r+1}, e_{r+2}, \dots, e_m\},$$

then

$$u(t) = u^-(t) + u^+(t) + Pu^0 + Qu^0,$$

where $u^- \in H^-$, $u^+ \in H^+$, $Pu^0 \in Y_1$ and $Qu^0 = \sum_{j=1}^r c_j e_j$. Let

$$G = \left\{ \sum_{i=1}^r k_i T_i e_i \mid k_i \in \mathbb{Z}, 1 \leq i \leq r \right\}$$

be a discrete subgroup of H_T^1 , where \mathbb{Z} is the set of all integers, and let $\pi : H_T^1 \rightarrow H_T^1/G$ be the canonical surjection. Let

$$H_T^1/G = X \times V = (W \oplus Z) \times V,$$

where $W = H^+$, $Z = H^- \oplus Y_1$, $V = Y_0/G$, then $\dim Z < +\infty$, $\dim V < +\infty$, and V is isomorphic to the torus T^r . The element in V can be represented as

$$Q\hat{u}^0 = \sum_{j=1}^r \hat{c}_j e_j,$$

where $\hat{c}_j = c_j - k_j T_j$, $0 \leq \hat{c}_j < T_j$. Let

$$\hat{u}(t) = u^-(t) + u^+(t) + Pu^0 + Q\hat{u}^0.$$

By (H3) and (H5), we have

$$F(t, u(t)) = F\left(t, \hat{u}(t) + \sum_{i=1}^r k_i T_i e_i\right) = F(t, \hat{u}(t)),$$

$$\nabla F(t, u(t)) = \nabla F\left(t, \hat{u}(t) + \sum_{i=1}^r k_i T_i e_i\right) = \nabla F(t, \hat{u}(t)),$$

and

$$\int_0^T (h(t), u(t)) dt = \int_0^T \left(h(t), \hat{u}(t) + \sum_{i=1}^r k_i T_i e_i\right) dt = \int_0^T (h(t), \hat{u}(t)) dt.$$

Thus, $\varphi(u) = \varphi(\hat{u})$, $\varphi'(u) = \varphi'(\hat{u})$. Define $\psi : X \times V \mapsto \mathbb{R}$: $\psi(\pi(u)) = \varphi(u)$, then ψ is well defined. Moreover, ψ is continuously differentiable and

$$\psi(\pi(u)) = \psi(\pi(\hat{u})), \quad \psi'(\pi(u)) = \psi'(\pi(\hat{u})).$$

Definition 2.1 ([6]) Suppose that ψ satisfies the (PS) condition, that is, every sequence $\{x_n\}$ of $X \times V$ such that $\psi\{x_n\}$ is bounded and $\psi'\{x_n\} \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence.

Lemma 2.2 (The generalized saddle point theorem [5]) *Let X be a Banach space with a decomposition $X = Z + W$, where Z and W are two subspaces of X with $\dim Z < +\infty$. Let V be a finite-dimensional, compact C^2 -manifold without boundary. Let $\psi : X \times V \rightarrow \mathbb{R}$ be a C^1 -function satisfying the (PS) condition. Suppose that there exist constants $\rho > 0$ and $\gamma < \beta$ such that*

- (a) $\inf_{x \in W \times V} \psi(x) \geq \beta$,
- (b) $\sup_{x \in S \times V} \psi(x) \leq \gamma$,

where $S = \partial D$, $D = \{z \in Z \mid |z| \leq \rho\}$. Then the functional ψ has at least $\text{cuplength}(V) + 1$ critical points.

3 Main results

Here are our main results.

Theorem 3.1 *Suppose that assumptions (H1)-(H3), (H5) hold and there exist constants $M_i > 0, i = 0, 1, 2$, and a nonnegative function $\omega \in C([0, \infty), [0, \infty))$ with the properties:*

- ($\omega 1$) $\omega(s) \leq \omega(t), \forall s \leq t, s, t \in [0, \infty)$;
- ($\omega 2$) $\omega(s + t) \leq M_0(\omega(s) + \omega(t)), \forall s, t \in [0, \infty)$;
- ($\omega 3$) $0 \leq \omega(s) \leq M_1s + M_2, \forall s, t \in [0, \infty)$;
- ($\omega 4$) $\omega(s) \rightarrow +\infty$ as $s \rightarrow +\infty$.

Moreover, there exist constant $a > 3$ and $f, g \in L^1([0, T]; \mathbb{R}^+)$ with

$$\int_0^T f(t) dt < \frac{\delta}{(2 + a)M_0M_1C_0^2} \tag{3.1}$$

such that

$$|\nabla F(t, x)| \leq f(t)\omega(|x|) + g(t) \tag{3.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and

$$\limsup_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\omega^2(|x|)} < -\left(\frac{15}{2} \frac{1}{2a - 1} + \frac{3}{2}\right) \frac{M_0}{M_1} \int_0^T f(t) dt \tag{3.3}$$

as $x \in N(A) \ominus \text{span}\{e_1, e_2, \dots, e_r\}$. Then Eq. (1.1) has at least $r + 1$ distinct solutions in H_T^1 .

Theorem 3.2 *Suppose that assumptions (H1)-(H3), (H5), (3.1), (3.2) hold and*

$$\liminf_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\omega^2(|x|)} > \left(\frac{15}{2} \frac{1}{2a - 1} + \frac{3}{2}\right) \frac{M_0}{M_1} \int_0^T f(t) dt$$

as $x \in N(A) \ominus \text{span}\{e_1, e_2, \dots, e_r\}$. Then Eq. (1.1) has at least $r + 1$ distinct solutions in H_T^1 .

By Theorems 3.1 and 3.2, it is easy to obtain the following corollary.

Corollary 3.1 *Suppose that assumptions (H1)-(H3), (H5), (3.1), (3.2) hold and*

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\omega^2(|x|)} = +\infty \quad (\text{or } -\infty)$$

as $x \in N(A) \ominus \text{span}\{e_1, e_2, \dots, e_r\}$. Then Eq. (1.1) has at least $r + 1$ distinct solutions in H_T^1 .

Remark 3.1 (i) When $A \equiv 0$, assumptions ($\omega 1$)-($\omega 4$) and condition (3.2) were introduced in [10]. Comparing with the results in [10], the periodicity and coercivity conditions in our Theorem 3.1 are only in a part of variables of potentials, and we obtained multiplicity of periodic solutions for problem (1.1).

(ii) To show that our Theorem 3.1 is new, we give an example to illustrate our result. For example, let $1 \leq r \leq m$, $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$, and

$$F(t, x) = \left(\frac{2}{5}T - t\right) \ln^2 \left[1 + \left(r + 1 + \sum_{j=1}^r \sin^2 x_j + \frac{1}{2} \sum_{j=r+1}^N x_j^2 \right) \right] + (T^3 - t) \ln \left[1 + \left(r + 1 + \sum_{j=1}^r \sin^2 x_j + \frac{1}{2} \sum_{j=r+1}^N x_j^2 \right) \right],$$

let $\omega(|x|) = \ln[1 + (r + 1 + |x|^2)]$. Then F satisfies all the conditions of Theorem 3.1, but not covered by the results of [1–15].

For example, let $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$, $\omega(|x|) = |x|$ and

$$F(t, x) = \left(\frac{5}{6}T - t\right) \left(r + 1 + \sum_{j=1}^r \sin^2 x_j + \frac{1}{2} \sum_{j=r+1}^N x_j^2 \right) + (T^3 - t) \left(r + 1 + \sum_{j=1}^r \sin^2 x_j + \frac{1}{2} \sum_{j=r+1}^N x_j^2 \right)^{\frac{1}{2}},$$

where $1 \leq r \leq m$. Then F satisfies all the conditions of Theorem 3.2, but not covered by the results of [1–15].

For the sake of convenience, we denote by C_i ($i = 1, 2, 3, \dots, 33$) various positive constants.

Proof of Theorem 3.1 First, we prove that ψ satisfies the (PS) condition. Let $\pi : W_T^{1,p(t)} \rightarrow W_T^{1,p(t)}/G$ be the canonical surjection. Define $\psi : X \times V \mapsto \mathbb{R}$ by $\psi(\pi(u)) = \varphi(u)$. Assume that $(\pi(u_n))$ is a (PS) sequence for ψ , that is, $\psi(\pi(u_n))$ is bounded and $\psi'(\pi(u_n)) \rightarrow 0$. Then $\varphi(u_n)$ is bounded and $\varphi'(u_n) \rightarrow 0$.

We can get from $(\omega 1)$, $(\omega 2)$, and $(\omega 3)$ that

$$\begin{aligned} \omega(|\hat{u}(t)|) &= \omega(|u^+(t) + u^-(t) + Q\hat{u}^0 + Pu^0|) \\ &\leq \omega(|u^+| + |u^-| + |Q\hat{u}^0| + |Pu^0|) \\ &\leq M_0[\omega(|u^+| + |u^-| + |Q\hat{u}^0|) + \omega(|Pu^0|)] \\ &\leq M_0[M_1(|u^+| + |u^-| + |Q\hat{u}^0|) + M_2] + M_0\omega(|Pu^0|) \\ &\leq M_0M_1(\|u^+\|_\infty + \|u^-\|_\infty) + M_0M_1|Q\hat{u}^0| + M_0M_2 \\ &\quad + M_0\omega(|Pu^0|). \end{aligned} \tag{3.4}$$

By (2.3) and the boundedness of $|Q\hat{u}^0|$, we have

$$\begin{aligned} &\left| \int_0^T (\nabla F(t, \hat{u}(t)), u^+(t)) dt \right| \\ &\leq \int_0^T f(t)\omega(|\hat{u}(t)|)|u^+(t)| dt + \int_0^T g(t)|u^+(t)| dt \end{aligned}$$

$$\begin{aligned}
 &\leq M_0 M_1 \int_0^T f(t) dt (\|u^+\|_\infty^2 + \|u^-\|_\infty \|u^+\|_\infty) \\
 &\quad + M_0 \int_0^T f(t) dt \omega(|Pu^0|) \|u^+\|_\infty + C_1 \|u^+\|_\infty \\
 &\leq M_0 M_1 C_0^2 \int_0^T f(t) dt (\|u^+\|^2 + \|u^-\| \|u^+\|) \\
 &\quad + M_0 C_0 \int_0^T f(t) dt \omega(|Pu^0|) \|u^+\| + C_2 \|u^+\| \\
 &\leq M_0 M_1 C_0^2 \int_0^T f(t) dt \left(\|u^+\|^2 + \frac{\|u^-\|^2 + \|u^+\|^2}{2} \right) \\
 &\quad + M_0 C_0 \int_0^T f(t) dt \frac{\frac{\omega^2(|Pu^0|)}{M_1 C_0} + M_1 C_0 \|u^+\|^2}{2} + C_2 \|u^+\| \\
 &= M_0 M_1 C_0^2 \int_0^T f(t) dt \left(2\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 \right) \\
 &\quad + \frac{1}{2} \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu^0|) + C_2 \|u^+\|. \tag{3.5}
 \end{aligned}$$

From (H3) and (2.3), we obtain that

$$\begin{aligned}
 \left| \int_0^T (h(t), u^+(t)) dt \right| &\leq \|u^+\|_\infty \int_0^T |h(t)| dt \\
 &\leq C_0 \|u^+\| \int_0^T |h(t)| dt. \tag{3.6}
 \end{aligned}$$

It follows from (2.2), (3.5), and (3.6) that

$$\begin{aligned}
 \|u_n^+\| &\geq \langle \varphi'(u_n), u_n^+ \rangle \\
 &= \langle \varphi'(\hat{u}_n), u_n^+ \rangle \\
 &= \int_0^T |\dot{u}_n^+(t)|^2 dt - \int_0^T (Au_n(t), u_n^+(t)) dt \\
 &\quad + \int_0^T (\nabla F(t, \hat{u}_n(t)), u_n^+(t)) dt + \int_0^T (h(t), u_n^+(t)) dt \\
 &\geq \left(\delta - 2M_0 M_1 C_0^2 \int_0^T f(t) dt \right) \|u_n^+\|^2 \\
 &\quad - \frac{1}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt \|u_n^-\|^2 \\
 &\quad - \frac{1}{2} \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu_n^0|) + C_3 \|u_n^+\| \tag{3.7}
 \end{aligned}$$

for large n . So we have

$$\begin{aligned}
 &\frac{1}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt \left(\|u_n^-\|^2 + \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} \right) \\
 &\geq \left(\delta - 2M_0 M_1 C_0^2 \int_0^T f(t) dt \right) \|u_n^+\|^2 - C_4 \|u_n^+\|
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\delta - (2 + a)M_0M_1C_0^2 \int_0^T f(t) dt \right] \|u_n^+\|^2 \\
 &\quad + aM_0M_1C_0^2 \int_0^T f(t) dt \|u_n^+\|^2 - C_4 \|u_n^+\| \\
 &\geq aM_0M_1C_0^2 \int_0^T f(t) dt \|u_n^+\|^2 + C_5,
 \end{aligned}$$

where $C_5 = \min_{s \in [0, +\infty)} \{[\delta - (2 + a)M_0M_1C_0^2 \int_0^T f(t) dt]s^2 - C_4s\}$.

From (3.1), one has that $(2 + a)M_0M_1C_0^2 \int_0^T f(t) dt < \delta$, then $C_5 < 0$. Hence

$$\|u_n^+\|^2 \leq \frac{1}{2a} \left(\|u_n^-\|^2 + \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} \right) + C_6.$$

In a similar way, we have

$$\|u_n^-\|^2 \leq \frac{1}{2a} \left(\|u_n^+\|^2 + \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} \right) + C_7.$$

Combining the above two inequalities, one has that

$$\begin{aligned}
 \|u_n^+\|^2 &\leq \frac{1}{2a} \left(\|u_n^-\|^2 + \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} \right) + C_6 \\
 &\leq \frac{1}{4a^2} \|u_n^+\|^2 + \frac{1}{2a} \left(\frac{1}{2a} + 1 \right) \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} + \frac{C_7}{2a} + C_6.
 \end{aligned}$$

Consequently,

$$\|u_n^+\|^2 \leq \frac{1}{2a-1} \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} + C_8. \tag{3.8}$$

Using similar arguments, we can prove that

$$\|u_n^-\|^2 \leq \frac{1}{2a-1} \frac{\omega^2(|Pu_n^0|)}{M_1^2 C_0^2} + C_9. \tag{3.9}$$

By (3.5), (3.6), (3.7), (3.8), and (3.9), we have

$$\begin{aligned}
 &\int_0^T |\dot{u}_n^+(t)|^2 dt - \int_0^T (Au_n(t), u_n^+(t)) dt \\
 &\leq \|u_n^+\| - \int_0^T (\nabla F(t, \hat{u}_n(t)), u_n^+(t)) dt \\
 &\quad - \int_0^T (h(t), u_n^+(t)) dt \\
 &\leq M_0M_1C_0^2 \int_0^T f(t) dt \left(2 \|u_n^+\|^2 + \frac{1}{2} \|u_n^-\|^2 \right) \\
 &\quad + \frac{1}{2} \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu^0|) + C_{10} \|u_n^+\|
 \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{5}{2} \frac{1}{2a-1} + \frac{1}{2}\right) \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu_n^0|) \\ &\quad + C_{11}\omega(|Pu_n^0|) + C_{12}. \end{aligned} \tag{3.10}$$

In a similar way, we can obtain

$$\begin{aligned} &\int_0^T |\dot{u}_n^-(t)|^2 dt - \int_0^T (Au_n(t), u_n^-(t)) dt \\ &\leq \left(\frac{5}{2} \frac{1}{2a-1} + \frac{1}{2}\right) \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu_n^0|) \\ &\quad + C_{13}\omega(|Pu_n^0|) + C_{14}. \end{aligned} \tag{3.11}$$

By (3.2) and (ω_1) , one has

$$\begin{aligned} &\left| \int_0^T F(t, \hat{u}_n(t)) dt - \int_0^T F(t, Pu_n^0) dt \right| \\ &= \left| \int_0^T \int_0^1 (\nabla F(t, Pu_n^0 + s(Q\hat{u}_n^0 + u_n^+ + u_n^-)), Q\hat{u}_n^0 + u_n^+ + u_n^-) ds dt \right| \\ &\leq \int_0^T \int_0^1 f(t) \omega(|Pu_n^0| + |Q\hat{u}_n^0| + |u_n^+| + |u_n^-|) (|Q\hat{u}_n^0| + |u_n^+| + |u_n^-|) ds dt \\ &\quad + \int_0^T \int_0^1 g(t) (|Q\hat{u}_n^0| + |u_n^+| + |u_n^-|) ds dt. \end{aligned}$$

From (3.4), (2.3), and the boundedness of $|Q\hat{u}^0|$, we have

$$\begin{aligned} &\left| \int_0^T F(t, \hat{u}_n(t)) dt - \int_0^T F(t, Pu_n^0) dt \right| \\ &\leq \int_0^T f(t) dt M_0 M_1 (|Q\hat{u}_n^0| + \|u_n^+\|_\infty + \|u_n^-\|_\infty)^2 \\ &\quad + \int_0^T f(t) dt M_0 M_2 (|Q\hat{u}_n^0| + \|u_n^+\|_\infty + \|u_n^-\|_\infty) \\ &\quad + \int_0^T f(t) dt M_0 \omega(|Pu_n^0|) (|Q\hat{u}_n^0| + \|u_n^+\|_\infty + \|u_n^-\|_\infty) \\ &\quad + \int_0^T g(t) dt (|Q\hat{u}_n^0| + \|u_n^+\|_\infty + \|u_n^-\|_\infty) \\ &\leq M_0 M_1 C_0^2 \int_0^T f(t) dt (\|u_n^+\| + \|u_n^-\|)^2 \\ &\quad + M_0 C_0 \int_0^T f(t) dt \omega(|Pu_n^0|) (\|u_n^+\| + \|u_n^-\|) \\ &\quad + C_{17} (\|u_n^+\| + \|u_n^-\|) + C_{18} \omega(|Pu_n^0|) + C_{19}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left| \int_0^T F(t, \hat{u}_n(t)) dt - \int_0^T F(t, Pu_n^0) dt \right| \\ &\leq M_0 M_1 C_0^2 \int_0^T f(t) dt 2(\|u_n^+\|^2 + \|u_n^-\|^2) \end{aligned}$$

$$\begin{aligned}
 &+ M_0 C_0 \int_0^T f(t) dt \frac{\frac{1}{M_1 C_0} \omega^2(|Pu_n^0|) + M_1 C_0 \|u_n^+\|^2}{2} \\
 &+ M_0 C_0 \int_0^T f(t) dt \frac{\frac{1}{M_1 C_0} \omega^2(|Pu_n^0|) + M_1 C_0 \|u_n^-\|^2}{2} \\
 &+ C_{17}(\|u_n^+\| + \|u_n^-\|) + C_{18}\omega(|Pu_n^0|) + C_{19} \\
 \leq &\frac{5}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt (\|u_n^+\|^2 + \|u_n^-\|^2) \\
 &+ \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu_n^0|) \\
 &+ C_{17}(\|u_n^+\| + \|u_n^-\|) + C_{18}\omega(|Pu_n^0|) + C_{19}.
 \end{aligned} \tag{3.12}$$

By (3.8), (3.9), and (3.12), we have

$$\begin{aligned}
 &\left| \int_0^T F(t, \hat{u}_n(t)) dt - \int_0^T F(t, Pu_n^0) dt \right| \\
 &\leq \left(\frac{5}{2a-1} + 1 \right) \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu_n^0|) \\
 &\quad + C_{20}\omega(|Pu_n^0|) + C_{21}.
 \end{aligned} \tag{3.13}$$

From (H3), (2.3), (3.8), and (3.9), one has

$$\begin{aligned}
 \left| \int_0^T (h(t), \hat{u}_n(t)) dt \right| &\leq \int_0^T |h(t)| dt (\|u_n^+\|_\infty + \|u_n^-\|_\infty) \\
 &\leq C_0 \int_0^T |h(t)| dt (\|u_n^+\| + \|u_n^-\|) \\
 &\leq C_{15}\omega(|Pu_n^0|) + C_{16}.
 \end{aligned} \tag{3.14}$$

It follows from (3.10), (3.11), (3.13), and (3.14) that

$$\begin{aligned}
 \varphi(u_n) &= \varphi(\hat{u}_n) \\
 &= \frac{1}{2} \left[\int_0^T |\dot{u}_n^+(t)|^2 dt - \int_0^T (Au_n(t), u_n^+(t)) dt \right] \\
 &\quad + \frac{1}{2} \left[\int_0^T |\dot{u}_n^-(t)|^2 dt - \int_0^T (Au_n(t), u_n^-(t)) dt \right] \\
 &\quad + \left[\int_0^T F(t, \hat{u}_n(t)) dt - \int_0^T F(t, Pu_n^0) dt \right] \\
 &\quad + \int_0^T F(t, Pu_n^0) dt + \int_0^T (h(t), \hat{u}_n(t)) dt \\
 &\leq \left[\left(\frac{15}{2} \frac{1}{2a-1} + \frac{3}{2} \right) \frac{M_0}{M_1} \int_0^T f(t) dt + \frac{\int_0^T F(t, Pu_n^0) dt}{\omega^2(|Pu_n^0|)} \right] \omega^2(|Pu_n^0|) \\
 &\quad + C_{22}\omega(|Pu_n^0|) + C_{23},
 \end{aligned}$$

which implies $|Pu_n^0|$ is bounded. Otherwise, we assume $|Pu_n^0| \rightarrow \infty$ as $n \rightarrow \infty$. From (ω4), we obtain that

$$\omega(|Pu_n^0|) \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

By (3.3), we conclude that

$$\varphi(u_n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty,$$

this contradicts the boundedness of $\{\varphi(u_n)\}$, so $|Pu_n^0|$ is bounded. Combining (3.8) and (3.9), we obtain that $\|u_n^+\|$ and $\|u_n^-\|$ are bounded. Furthermore, $|Q\hat{u}^0|$ is bounded, so $\{\hat{u}_n\}$ is bounded in H_T^1 . Arguing then as in Proposition 4.1 in [6], $\{\hat{u}_n\}$ has a convergent subsequence. By $\pi(\hat{u}_n) = \pi(u_n)$, we conclude that ψ satisfies the (PS) condition.

Next, we only need to verify the linking conditions of the generalized saddle point theorem:

(a) For $\pi(u) \in W \times V$, $u(t) = u^+(t) + Qu^0$. By the proof of (3.12), we have

$$\begin{aligned} & \left| \int_0^T F(t, u^+(t) + Qu^0) dt - \int_0^T F(t, 0) dt \right| \\ & \leq \frac{5}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt \|u^+\|^2 + C_{24} \|u^+\| + C_{25}. \end{aligned}$$

Hence

$$\begin{aligned} \psi(\pi(u)) &= \psi(\pi(u^+ + Qu^0)) \\ &= \varphi(u^+ + Qu^0) \\ &= \frac{1}{2} \int_0^T |\dot{u}^+(t)|^2 dt + \frac{1}{2} \int_0^T (Au(t), u^+(t)) dt \\ &\quad + \left[\int_0^T F(t, u^+(t) + Qu^0) dt - \int_0^T F(t, 0) dt \right] \\ &\quad + \int_0^T (h(t), u^+(t) + Qu^0) dt + \int_0^T F(t, 0) dt \\ &\geq \left(\frac{\delta}{2} - \frac{5}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt \right) \|u^+\|^2 - C_{26} \|u^+\| - C_{27}. \end{aligned}$$

Note the boundedness of $|Qu^0|$ and (3.1), we obtain that $\psi(\pi(u)) \rightarrow +\infty$ as $\|u\| \rightarrow -\infty$ for all $\pi(u) \in W \times V$, which implies that there exists $\beta \in \mathbb{R}$ such that $\psi(\pi(u)) \geq \beta$ on $W \times V$.

(b) In a way similar to the proof of (3.12), we have

$$\begin{aligned} & \left| \int_0^T F(t, u^- + Pu^0 + Qu^0) dt - \int_0^T F(t, Pu^0) dt \right| \\ & \leq \frac{5}{2} M_0 M_1 C_0^2 \int_0^T f(t) dt \|u^-\|^2 + \frac{M_0}{M_1} \int_0^T f(t) dt \omega^2(|Pu^0|) \\ & \quad + C_{28} \|u^-\| + C_{29} \omega(|Pu^0|) + C_{30}. \end{aligned}$$

Consequently,

$$\begin{aligned}
 \psi(\pi(u)) &= \psi(\pi(u^- + Pu^0 + Qu^0)) \\
 &= \varphi(u^- + Pu^0 + Qu^0) \\
 &= \frac{1}{2} \int_0^T |\dot{u}^-|^2 dt + \frac{1}{2} \int_0^T (Au, u^-) dt + \int_0^T F(t, Pu^0) dt \\
 &\quad + \left[\int_0^T F(t, u^- + Pu^0 + Qu^0) dt - \int_0^T F(t, Pu^0) dt \right] \\
 &\quad + \int_0^T (h(t), u^- + Pu^0 + Qu^0) dt \\
 &\leq \frac{1}{2} \left[-\delta + 5M_0M_1C_0^2 \int_0^T f(t) dt \right] \|u^-\|^2 + C_{31} \|u^-\| \\
 &\quad + \left[\frac{M_0}{M_1} \int_0^T f(t) dt + \frac{\int_0^T F(t, Pu^0) dt}{\omega^2(|Pu^0|)} \right] \omega^2(|Pu^0|) \\
 &\quad + C_{32}\omega(|Pu^0|) + C_{33}.
 \end{aligned}$$

From $a > 3$ and (3.1), one has that $-\delta + 5M_0M_1C_0^2 \int_0^T f(t) dt < 0$.

By (3.3), we deduce that

$$\limsup_{|Pu^0| \rightarrow \infty} \frac{\int_0^T F(t, Pu^0) dt}{\omega^2(|Pu^0|)} < -\frac{M_0}{M_1} \int_0^T f(t) dt,$$

so we obtain that $\psi(\pi(u)) \rightarrow +\infty$ as $\|u\| \rightarrow -\infty$ for all $\pi(u) \in Z \times V$, which implies that there exists $\gamma < \beta$ such that $\psi(\pi(u)) \leq \gamma$ on $Z \times V$.

The functional ψ satisfies all the assumptions of Lemma 2.2, so it has at least $\text{cuplength}(V) + 1$ critical points, and since V is the torus T^r , then $\text{cuplength}(V) = r$. Hence φ has at least $r + 1$ critical points. Therefore, problem (1.1) has at least $r + 1$ distinct solutions in H_T^1 . The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2 The proof of Theorem 3.2 is similar to the proof of Theorem 3.1, so we omit the discussions here. □

Remark 3.2 Using the parallel arguments with little change as in the proofs of Theorems 3.1 and 3.2, the conclusions of Theorems 3.1 and 3.2 hold if we replace $(\omega 3)$ with

$$(\omega 3)' \quad 0 \leq \omega(s) \leq M_1s^\alpha + M_2, \forall s, t \in [0, \infty), \text{ where } 0 \leq \alpha < 1.$$

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Author's contributions

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