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Pohozaev-type inequalities and their applications for elliptic equations

Bingyu Kou* and Tianqing An

*Correspondence: koubei@163.com
College of Science, Hohai University, Nanjing, 210098, China

Abstract

In this paper we derive the Pohozaev-type inequalities for p-Laplacian equations and weighted quasi-linear equations and then prove some non-existence results for the positive solutions of these equations in a class of domains that are more general than star-shaped ones.

Keywords: Pohozaev-type inequality; p-Laplacian; quasi-linear equation; positive solutions; non-star-shaped domain; non-existence

1 Introduction

This paper is mainly concerned with the elliptic equation

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $f(x,u) : R^N \times R^1 \to R^1$ is continuous, and $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary. We will establish the Pohozaev-type inequality for the solutions of (1.1) and then discuss the non-existence of positive solutions of the problem in the non-star-shaped domains. We also discuss a similar topic for the following weighted quasi-linear elliptic equation:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Recall that in the famous paper [1], Pohozaev considered the following elliptic boundary value problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in C(R^1, R^1)$, $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a domain with smooth boundary. Let $F(x) = \int_0^x f(s)\,ds$, and let $\nu(x)$ be the unit outward normal to $\partial\Omega$ at x. He proved the following famous identity.

Theorem A (Pohozaev identity, [1]) *Suppose that* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *is a solution of* (1.3). *Then*

$$\begin{aligned} &(2-n)\int_\Omega uf(u)\,dx + 2n\int_\Omega F(u)\,dx \\ &\quad = \int_{\partial\Omega} \langle x, \nu(x)\rangle \left|\frac{\partial u}{\partial \nu}\right|^2 ds. \end{aligned} \tag{1.4}$$

Based on this identity, Pohozaev obtained a remarkable non-existence result for the following elliptic boundary value problem under the conditions that Ω is star-shaped and $\alpha \geq \frac{n+2}{n-2}$, $\lambda \leq 0$.

$$\begin{cases} -\Delta u = |u|^{\alpha-1}u + \lambda u, \quad u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Since then, many new results on this topic have appeared. Some of them generalized Pohozaev's results to the more general equations, such as quasi-linear elliptic, polyharmonic equations, fractional differential equations. Others considered the case of domains more general than star-shaped ones. See [2–9] and the references therein. In [10], Bahri and Coron proved that if Ω is a smooth domain with non-trivial topology, equation (1.5) may have solutions. Dancer [11] and Ding [12] constructed the examples of contractible domains on which (1.5) has a solution. Therefore, it is interesting to discuss the problem on non-star-shaped contractible domains.

In 1989, Guedda and Veron [13] established the Pohozaev identity of the solutions of (1.1) and got the non-existence results. In [14], Isaia generalized the Pohozaev identity to a non-existence result of higher-order regular strong solutions of (1.1). In [15], Takáč and Il'yasov improved the well-known regularity results of the weak solutions of p-Laplacian equation from [16, 17] and, using the new regularity results for the Dirichlet and Neumann problem, established and proved the Pohozaev-type identity. In [18], Bartsch, Peng and Zhang generalized the non-existence result to the more general problem (1.2). It is also interesting to discuss some special cases of (1.1), such as $f(x,u) = \lambda u^{q-1} + u^{s-1}$, $f(x,u) = p(x)u^\alpha + q(x)u^\beta$, etc. See, for example, [19–22].

In this paper we also discuss the non-existence of the positive solution of (1.1) and (1.2). However, our method is different from all of the above work. Instead of the Pohozaev identities, we establish a kind of inequalities, named Pohozaev-type inequalities, which have the same effects as Pohozaev identities, and then prove some non-existence results for the positive solution of (1.1) and (1.2) on non-star-shaped domains.

2 The p-Laplacian equations

In this section, we consider the p-Laplacian equations (1.1). Firstly we give a lemma.

Lemma 2.1 *Assume that* $V(x) = (V_1(x), \ldots, V_n(x))$ *is a* C^1 *vector field on* $\mathbb{R}^n$ *and* $u \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ *is a solution of* (1.1). *Then*

$$\int_\Omega u \operatorname{div}\bigl(|\nabla u|^{p-2} V(x)\bigr)\,dx = -\int_\Omega |\nabla u|^{p-2}\bigl\langle V(x), \nabla u\bigr\rangle\,dx,$$

and

$$\int_\Omega F(x,u)\operatorname{div} V(x)\,dx + \int_\Omega F_1(x,u)\,dx = -\int_\Omega f(u)\langle V(x),\nabla u\rangle\,dx,$$

where $F(x,t)=\int_0^t f(x,s)\,ds$, $F_1(x,t)=\sum_{i=1}^n V_i\frac{\partial F(x,t)}{\partial x_i}$.

Proof By the divergence theorem and the fact $u(x)=0$ and $F(x,u)=0$ for $x\in\partial\Omega$, we have the following results:

$$\begin{aligned}0 &= \int_{\partial\Omega}\langle u(x)|\nabla u|^{p-2}V(x),\nu(x)\rangle\,ds = \int_\Omega \operatorname{div}\bigl(u(x)|\nabla u|^{p-2}V(x)\bigr)\,dx\\ &= \int_\Omega u\operatorname{div}\bigl(|\nabla u|^{p-2}V(x)\bigr)\,dx + \int_\Omega\langle|\nabla u|^{p-2}V(x),\nabla u\rangle\,dx\\ &= \int_\Omega u\operatorname{div}\bigl(|\nabla u|^{p-2}V(x)\bigr)\,dx + \int_\Omega|\nabla u|^{p-2}\langle V(x),\nabla u\rangle\,dx,\end{aligned}$$

and because $V_if(x,u)\frac{\partial u}{\partial x_i} = V_i(\frac{\partial F}{\partial x_i} - \int_0^u\frac{\partial f(x,s)}{\partial x_i}\,ds)$, we have

$$\begin{aligned}&-\sum_{i=1}^n\int_\Omega V_if\frac{\partial u}{\partial x_i}\,dx\\ &\quad= -\sum_{i=1}^n\left[\int_\Omega\left(\frac{\partial}{\partial x_i}(V_iF)-\frac{\partial V_i}{\partial x_i}F\right)dx - \int_\Omega V_i\frac{\partial F}{\partial x_i}\,dx\right]\\ &\quad= -\int_{\partial\Omega}F(x,u)\langle V(x),\nu(x)\rangle\,ds + \int_\Omega F(x,u)\operatorname{div}\bigl(V(x)\bigr)\,dx + \int_\Omega F_1(x,u)x\\ &\quad= \int_\Omega F(x,u)\operatorname{div}\bigl(V(x)\bigr)\,dx + \int_\Omega F_1(x,u)x.\end{aligned}$$

The proof is complete. □

Based on Lemma 2.1, we can derive a Pohozaev-type inequality for the solutions of (1.1).

Theorem 2.2 (Pohozaev-type inequality) *Let* $V(x)$ *be a linear vector field on* $\mathbb{R}^n$ *with the form*

$$V(x)=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&\vdots&\vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix}x.$$

Suppose that $V(x)$ *satisfies* $\operatorname{div} V(x)=n$ *and* $\langle V(x),x\rangle>0$ *for* $\forall x\in\mathbb{R}^n\setminus\{0\}$. *If* $u\in W_0^{1,p}(\Omega)\cap C^1(\bar\Omega)$ *is a solution of* (1.1), *then*

$$\begin{aligned}&(p\mu-n)\int_\Omega uf(x,u)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx\\ &\quad\ge (p-1)\int_{\partial\Omega}\langle V(x),\nu(x)\rangle\left|\frac{\partial u}{\partial\nu}\right|^p ds,\end{aligned}\tag{2.1}$$

where $\mu=\sup_{|x|\neq0}\frac{\langle V(x),x\rangle}{|x|^2}$.

Proof It is easy to see that

$$0 < \langle V(x), x\rangle < \mu|x|^2, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

We multiply the equation $-\Delta_p u = f(x,u)$ by $\langle V(x), \nabla u\rangle$, and then we integrate in Ω. By the divergence theorem and Lemma 2.1, the right-hand side is

$$\begin{aligned}
&-\int_\Omega f(x,u)\langle V(x), \nabla u\rangle\,dx \\
&\quad= -\int_\Omega \sum_{i=1}^n f(x,u)V_i(x)\frac{\partial u}{\partial x_i} \\
&\quad= \int_\Omega F(x,u)\operatorname{div} V(x)\,dx + \int_\Omega F_1(x,u)\,dx.
\end{aligned}$$

The left-hand side is

$$\begin{aligned}
&\int_\Omega \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)\langle V(x), \nabla u\rangle\,dx \\
&\quad= \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\right)\left(\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right)dx \\
&\quad= \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right) - |\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right)dx.
\end{aligned}$$

Following this, we have

$$\begin{aligned}
&= \int_{\partial\Omega}\bigl\langle V(x), \nu(x)\bigr\rangle|\nabla u|^p\,ds - \sum_{j=1}^n \int_\Omega |\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\sum_{i=1}^n a_{ij}\frac{\partial u}{\partial x_i} + \sum_{i=1}^n V_i(x)\frac{\partial^2 u}{\partial x_i\partial x_j}\right) \\
&= \int_{\partial\Omega}\bigl\langle V(x), \nu(x)\bigr\rangle|\nabla u|^p\,ds + \int_\Omega u\operatorname{div}\bigl(|\nabla u|^{p-2}V(\nabla u)\bigr) \\
&\quad - \sum_{j=1}^n \int_\Omega |\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\sum_{i=1}^n V_i\frac{\partial^2 u}{\partial x_i\partial x_j}\right) \\
&= \int_{\partial\Omega}\bigl\langle V(x), \nu(x)\bigr\rangle|\nabla u|^p\,ds - \int_\Omega |\nabla u|^{p-2}\bigl\langle V(\nabla u), \nabla u\bigr\rangle\,dx - \frac{1}{p}\sum_{i=1}^n \int_\Omega V_i(x)\frac{\partial}{\partial x_i}\bigl(|\nabla u|^p\bigr)\,dx \\
&= \int_{\partial\Omega}\bigl\langle V(x), \nu(x)\bigr\rangle|\nabla u|^p\,ds - \int_\Omega |\nabla u|^{p-2}\bigl\langle V(\nabla u), \nabla u\bigr\rangle\,dx \\
&\quad - \frac{1}{p}\int_{\partial\Omega}\bigl\langle V(x), \nu(x)\bigr\rangle|\nabla u|^p\,ds + \frac{1}{p}\sum_{i=1}^n a_{ii}\int_\Omega |\nabla u|^p\,dx.
\end{aligned}$$

Comparing the left- and right-hand sides, we get the following identity:

$$\begin{aligned}
&\left(1 - \frac{1}{p}\right)\int_{\partial\Omega}|\nabla u|^p\bigl\langle V(x), \nu(x)\bigr\rangle\,ds \\
&\quad= -\frac{1}{p}\sum_{i=1}^n a_{ii}\int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla u|^{p-2}\bigl\langle V(\nabla u), \nabla u\bigr\rangle\,dx + \int_\Omega F\operatorname{div} V(x)\,dx
\end{aligned}$$

$$= -\frac{1}{p}\sum_{i=1}^{n} a_{ii}\int_\Omega uf(x,u)\,dx + \int_\Omega |\nabla u|^{p-2}\langle V(\nabla u),\nabla u\rangle\,dx + \sum_{i=1}^{n} a_{ii}\int_\Omega F(x,u)\,dx$$
$$+ \int_\Omega F_1(x,u)\,dx.$$

Because $0 < \langle V(x), x\rangle \leq \mu|x|^2$, we know that

$$0 < \langle V(\nabla u),\nabla u\rangle \leq \mu|\nabla u|^2.$$

Thus,

$$\int_\Omega |\nabla u|^{p-2}\langle V(\nabla u),\nabla u\rangle\,dx \leq \mu\int_\Omega |\nabla u|^p\,dx = \mu\int_\Omega uf(u)\,dx.$$

By $\operatorname{div} V(x) = \sum_{i=1}^n a_{ii} = n$, we obtain the following inequality:

$$(p\mu - n)\int_\Omega uf(x,u)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx$$
$$\geq (p-1)\int_{\partial\Omega}\langle V(x),\nu(x)\rangle\left|\frac{\partial u}{\partial \nu}\right|^p ds.$$

The proof is complete. □

Based on the above Pohozaev-type inequality, we discuss the non-existence of a positive solution of the following boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda_1|u|^{r-2}u + \lambda_2|u|^{s-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $1 < r \leq s < +\infty$.

Theorem 2.3 *Suppose that there exists a vector field $V(x)$ which satisfies the conditions of Theorem* 2.2 *and is transverse to $\partial\Omega$. Then* (2.2) *has no positive solution in the following cases respectively*:

(D_1) $p_1^* > r$, $\lambda_1 > 0$, $\lambda_2 > 0$, $p_1^* > s$;
(D_2) $p_1^* > r$, $\lambda_1 > 0$, $\lambda_2 < 0$, $p_1^* < s$;
(D_3) $p_1^* < r$, $\lambda_1 < 0$, $\lambda_2 < 0$, $p_1^* < s$,

where $p_1^* = \frac{np}{n-p\mu}$.

Proof It is easy to see that

$$f(x,u) = \lambda_1|u|^{r-2}u + \lambda_2|u|^{s-2}u, \qquad F(x,u) = \int_0^u f(x,s)\,ds = \frac{\lambda_1}{r}|u|^r + \frac{\lambda_2}{s}|u|^s,$$

and then we have

$$uf(x,u) = \lambda_1|u|^r + \lambda_2|u|^s, \qquad \frac{\partial F(x,u)}{\partial x_i} = 0, \qquad F_1 = 0.$$

Suppose that u is a positive solution of (2.2), according to Theorem 2.2, inequality (2.1) holds, then we have

$$(p\mu - n)\int_\Omega \left(\lambda_1|u|^r + \lambda_2|u|^s\right)dx + pn\int_\Omega \frac{\lambda_1}{r}|u|^r + \frac{\lambda_2}{s}|u|^s\,dx \geq 0.$$

This yields

$$\lambda_1\|u\|_r^r\left(\frac{r-p_1^*}{r}\right) \geq \lambda_2\|u\|_s^s\left(\frac{p_1^*-s}{s}\right). \tag{2.3}$$

By (D_1)-(D_3), the left-hand side of (2.3) is negative, and the right-hand side is positive, which is a contradiction. Then the proof is complete. □

Especially, we consider the problem in the case of $p = 2$:

$$\begin{cases} -\Delta u = \lambda_1|u|^{r-2}u + \lambda_2|u|^{s-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Corollary 2.4 *Suppose that there exists a vector field $V(x)$ which satisfies the conditions of Theorem* 2.2 *and is transverse to $\partial\Omega$. Then* (2.4) *has no positive solution in the following cases respectively*:

(D_1') $2_1^* > r$, $\lambda_1 > 0$, $\lambda_2 > 0$, $2_1^* > s$;
(D_2') $2_1^* > r$, $\lambda_1 > 0$, $\lambda_2 < 0$, $2_1^* < s$;
(D_3') $2_1^* < r$, $\lambda_1 < 0$, $\lambda_2 < 0$, $2_1^* < s$,

where $2_1^* = \frac{2n}{n-2\mu}$.

3 The weighted quasi-linear elliptic boundary value problem

This section is devoted to the weighted quasi-linear elliptic problem (1.2).

Theorem 3.1 (Pohozaev-type inequality) *Let $V(x)$ be a linear vector field on $\mathbb{R}^n$ with the form*

$$V(x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} x.$$

Suppose that $V(x)$ satisfies $\operatorname{div} V(x) = n$ *and* $\langle V(x), x\rangle > 0$ *for* $\forall x \in \mathbb{R}^n \setminus \{0\}$. *If* $u \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ *is a solution of* (1.2), *then*

$$\begin{aligned} &\left((a+1)p\mu - n\right)\int_\Omega uf(x,u)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx \\ &\quad \geq (p-1)\int_{\partial\Omega}\left\langle V(x), \nu(x)\right\rangle\left|\frac{\partial u}{\partial \nu}\right|^p ds, \end{aligned} \tag{3.1}$$

where $\mu = \sup_{|x|\neq 0}\frac{\langle V(x),x\rangle}{|x|^2}$.

Proof We multiply equation (1.2) by $\langle V(x), \nabla u\rangle$, and then we integrate in Ω. Similar to the proof of Theorem 2.2, by the divergence theorem and Lemma 2.1, we have

$$\begin{aligned} -\int_\Omega f(x,u)\big\langle V(x),\nabla u\big\rangle\,dx &= -\int_\Omega \sum_{i=1}^n f(x,u)V_i(x)\frac{\partial u}{\partial x_i} \\ &= \int_\Omega F(x,u)\operatorname{div} V(x)\,dx + \int_\Omega F_1(x,u)\,dx, \end{aligned}$$

then

$$\begin{aligned} &= \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial x_j}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\right)\left(\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right)dx \\ &= \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial x_j}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right) \\ &\quad - |x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^n V_i(x)\frac{\partial u}{\partial x_i}\right)dx \\ &= \int_{\partial\Omega} |x|^{-ap}|\nabla u|^p\big\langle V(x),\nu(x)\big\rangle\,ds \\ &\quad - \sum_{j=1}^n \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\sum_{i=1}^n a_{ij}\frac{\partial u}{\partial x_i} + \sum_{i=1}^n V_i(x)\frac{\partial^2 u}{\partial x_i\partial x_j}\right)dx \\ &= \int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}||\nabla u|^p\,ds + \int_\Omega u\operatorname{div}\big(|x|^{-ap}|\nabla u|^{p-2}V(\nabla u)\big)\,dx \\ &\quad - \sum_{j=1}^n \int_\Omega |x|^{-ap}||\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\sum_{i=1}^n V_i(x)\frac{\partial^2 u}{\partial x_i\partial x_j}\right)dx \\ &= \int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}|\nabla u|^p\,ds - \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\big\langle V(\nabla u),\nabla u\big\rangle\,dx \\ &\quad - \frac{1}{p}\sum_{i=1}^n \int_\Omega |x|^{-ap}V_i(x)\frac{\partial}{\partial x_i}\big(|\nabla u|^p\big)\,dx \\ &= \int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}|\nabla u|^p\,ds - \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\big\langle V(\nabla u),\nabla u\big\rangle\,dx \\ &\quad - \frac{1}{p}\int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}|\nabla u|^p\,ds + \frac{1}{p}\sum_{i=1}^n a_{ii}\int_\Omega |x|^{-ap}|\nabla u|^p\,dx \\ &\quad + \frac{1}{p}\sum_{i=1}^n \int_\Omega |\nabla u|^p V_i\frac{\partial}{\partial x_i}|x|^{-ap}\,dx \\ &= \int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}|\nabla u|^p\,ds - \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\big\langle V(\nabla u),\nabla u\big\rangle\,dx \\ &\quad - \frac{1}{p}\int_{\partial\Omega}\big\langle V(x),\nu(x)\big\rangle|x|^{-ap}|\nabla u|^p\,ds + \frac{n}{p}\int_\Omega |x|^{-ap}|\nabla u|^p\,dx \\ &\quad - a\int_\Omega |x|^{-ap-2}|\nabla u|^p\big\langle V(x),x\big\rangle\,dx. \end{aligned}$$

Since $0 < \langle V(x), x\rangle \leq \mu|x|^2$, and $0 < \langle V(\nabla u), \nabla u\rangle \leq \mu|\nabla u|^2$, we have

$$\int_\Omega |\nabla u|^{p-2}\langle V(\nabla u), \nabla u\rangle\,dx \leq \mu \int_\Omega |\nabla u|^p\,dx = \mu \int_\Omega uf(u)\,dx.$$

We obtain the following inequality:

$$\begin{aligned}&\bigl((a+1)p\mu - n\bigr)\int_\Omega uf(x,u)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx\\&\quad\geq (p-1)\int_{\partial\Omega}\langle V(x), \nu(x)\rangle\left|\frac{\partial u}{\partial \nu}\right|^p ds.\end{aligned}$$

The proof is complete. □

Theorem 3.2 *Suppose that there exists a vector field $V(x)$ which satisfies the conditions of Theorem* 3.1 *and is transverse to $\partial\Omega$. Then* (1.2) *has no positive solution if*

$$\bigl((a+1)p\mu - n\bigr)\int_\Omega uf(x,u)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx < 0.$$

Now we consider two special but important cases as follows.

1. A weighted quasi-linear problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda_1|u|^{r-2}u + \lambda_2|u|^{s-2u}, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega.\end{cases} \tag{3.2}$$

By Theorem 3.2, problem (3.2) has no solution in the following cases.

Theorem 3.3 *Suppose that the vector field $V(x)$ of Theorem* 3.1 *is transverse to $\partial\Omega$. Then* (3.2) *has no positive solution in the following cases respectively*:

(E_1) $p_2^* > r$, $\lambda_1 > 0$, $\lambda_2 > 0$, $p_2^* > s$;
(E_2) $p_2^* > r$, $\lambda_1 > 0$, $\lambda_2 < 0$, $p_2^* < s$;
(E_3) $p_2^* < r$, $\lambda_1 < 0$, $\lambda_2 < 0$, $p_2^* < s$,

where $p_2^* = \frac{np}{n-(a+1)p\mu}$.

2. A non-autonomous weighted quasi-linear problem:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda_1|x|^{-\alpha}|u|^{r-2}u + \lambda_2|x|^{-\beta}|u|^{s-2}u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega.\end{cases} \tag{3.3}$$

By Theorem 3.2, we have the next theorem of problem (3.3).

Theorem 3.4 *Suppose that there exists a vector field $V(x)$ which satisfies the conditions of Theorem* 2.3 *and is transverse to $\partial\Omega$. If*

$$\left(\alpha_0 - \frac{\alpha - n}{r}\right)\lambda_1 < 0, \quad \textit{and} \quad \left(\frac{\beta - n}{s} - \alpha_0\right)\lambda_2 > 0, \tag{3.4}$$

where $\alpha_0 = (a+1)\mu - \frac{n}{p}$, *then* (3.3) *has no positive solution.*

Proof It is easy to see that

$$
\begin{aligned}
f(x,u) &= \lambda_1|x|^{-\alpha}|u|^{r-2}u + \lambda_2|x|^{-\beta}|u|^{s-2}u, \\
F(x,u) &= \int_0^u f(x,s)\,ds = \frac{\lambda_1}{r}|x|^{-\alpha}|u|^r + \frac{\lambda_2}{s}|x|^{-\beta}|u|^s,
\end{aligned}
$$

and then we have

$$
uf(x,u) = \lambda_1|x|^{-\alpha}|u|^r + \lambda_2|x|^{-\beta}|u|^s,
$$

and

$$
F_1(x,u) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}F(x,u) = -\frac{\alpha\lambda_1}{r}|x|^{-\alpha}|u|^r - \frac{\beta\lambda_2}{s}|x|^{-\beta}|u|^s.
$$

Suppose that u is a positive solution of (3.3), then inequality (3.1) holds, that is,

$$
\begin{aligned}
&\bigl((a+1)p\mu - n\bigr)\int_\Omega \bigl(\lambda_1|x|^{-\alpha}|u|^r + \lambda_2|x|^{-\beta}|u|^s\bigr)\,dx + pn\int_\Omega F(x,u)\,dx + p\int_\Omega F_1(x,u)\,dx \\
&\quad \geq (p-1)\int_{\partial\Omega}\bigl\langle V(x),\nu(x)\bigr\rangle \left|\frac{\partial u}{\partial \nu}\right|^p ds \geq 0.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&(p\mu - n)\int_\Omega \bigl(\lambda_1|x|^{-\alpha}|u|^r + \lambda_2|x|^{-\beta}|u|^s\bigr)\,dx + pn\int_\Omega \frac{\lambda_1}{r}|x|^{-\alpha}|u|^r + \frac{\lambda_2}{s}|x|^{-\beta}|u|^s\,dx \\
&\quad \geq p\int_\Omega \frac{\alpha\lambda_1}{r}|x|^{-\alpha}|u|^r + \frac{\beta\lambda_2}{s}|x|^{-\beta}|u|^s\,dx.
\end{aligned}
$$

Hence, we get

$$
\lambda_1\left(\alpha_0 - \frac{\alpha - n}{r}\right)\int_\Omega |x|^{-\alpha}|u|^r\,dx \geq \lambda_2\left(\frac{\beta - n}{s} - \alpha_0\right)\int_\Omega |x|^{-\beta}|u|^s, \tag{3.5}
$$

which leads to a contradiction, because of (3.4). □

Remark Let $\lambda_1 = \lambda_2 = -1$, $\alpha = br$, $\beta = cs$, $r = \frac{np}{n-p(a+1-b)}$, $s = \frac{np}{n-p(a+1-c)}$, problem (3.3) is the equation that is discussed in [23].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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