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Solutions for a category of singular nonlinear fractional differential equations subject to integral boundary conditions

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Abstract

We concentrate on a category of singular boundary value problems of fractional differential equations with integral boundary conditions, in which the nonlinear function f is singular at $t = 0, 1$. We use Banach's fixed-point theorem and Hölder's inequality to verify the existence and uniqueness of a solution. Moreover, also we prove the existence of solutions by Krasnoselskii's and Schaefer's fixed point theorems.

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Keywords: Singular boundary value problem; Fractional differential equation; Integral boundary condition; Fixed point theorem

1 Introduction

The current work concentrates on the existence and uniqueness of solutions for a category of singular nonlinear fractional differential equations (NFDEs) subject to integral boundary conditions (BCs). Specifically, we discuss the problem

$$\begin{cases} {}^cD^\alpha_{0^+}x(t) = f(t,x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \\ x(1) = \int_\gamma^1 x(\tau)\,d\tau, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha_{0^+}$ stands for the Caputo derivative of order α, α and γ are real numbers satisfying $2 < \alpha \le 3$ and $0 < \gamma < 1$, respectively, and the function $f(t,x(t))$ has singular characteristics $\lim_{t\to 0^+} f(t,x(t)) = \lim_{t\to 1^-} f(t,x(t)) = \infty$.

In recent decades, great growth has been attained on the theory and applications of fractional calculus. There is a vast literature on this subject, where the basic concepts, properties, and applications of fractional-order operators are introduced [1–6], and the related initial and boundary value problems are studied [7–21]. Darwish and Ntouyas [16] verified the existence of solutions for the BVP

$$\begin{cases} {}^cD_{0^+}^q x(t) = f(t, x(t)), & 0 < t < 1, 0 < q \le 1, \\ x(0) + \alpha \int_\mu^\nu x(\tau)\,d\tau = x(1), & 0 < \mu < \nu < 1 (\mu \ne \nu), \end{cases}$$

where ${}^cD_{0^+}^\alpha$ stands for the Caputo derivative, and $f : [0,1] \times R \to R$ is a continuous function. Various fixed point theorems state the existence and uniqueness of solutions.

BVPs for singular NFDEs have become a hot research topic in recent years [22–28]. For example, Qiu and Bai [25] discussed the problem

$$\begin{cases} D_{0^+}^\alpha y(t) = f(t, y(t)), & 0 < t < 1, \\ y(0) = y'(1) = y''(0) = 0, \end{cases}$$

where $2 < \alpha \le 3$, $D_{0^+}^\alpha$ stands for the Caputo derivative, and $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ satisfies $\lim_{t\to 0^+} f(t,\cdot) = +\infty$. They hypothesized that $t^\sigma f(t,y(t))$ is continuous on $[0,1] \times [0,+\infty)$ and employed nonlinear alternative and Krasnoselskii's fixed point theorem to extract two positive solutions to this problem.

Several papers have dealt with problems for singular NFDEs containing integral boundary conditions [29–33].

He [29] discussed the existence and multiplicity of positive solutions for NFDEs with integral BCs

$$\begin{cases} {}^cD^\alpha y(t) + f(t, y(t)) = 0, & 0 < t < 1, \\ y''(0) = y'''(0) = 0, \\ y'(0) = y(1) = \eta \int_0^1 y(\tau)\,d\tau, \end{cases}$$

where ${}^cD^\alpha$ stands the Caputo's fractional derivative of order α, $3 < \alpha \le 4$, $0 < \eta < 2$, and f can have a singularity at $u = 0$.

Vong [32] verified the following nonlocal BVP for a class of singular NFDEs:

$$\begin{cases} {}^cD^\alpha y(t) + f(t, y(t)) = 0, & 0 < t < 1, \\ y'(0) = \cdots = y^{(n-1)}(0) = 0, \\ y(1) = \int_0^1 y(\tau)\,d\tau, \end{cases}$$

where $n \ge 2$, $\alpha \in (n-1, n)$, $\mu(s)$ denotes a bounded-variation function, which can be singular at $t = 0$.

Motivated by all the mentioned studies, we aim to demonstrate the existence and uniqueness of solutions to problem (1.1). We use some typical fixed point theorems and the generalized Hölder inequality to obtain fundamental results.

2 Preliminaries

This subsection contains the required concepts and features of the fractional calculus and some lemmas necessary to prove our essential results.

Definition 2.1 ([1]) Let $\Omega = [a,b]$ $(-\infty < a < b < +\infty)$ be a bounded interval on R. The Riemann–Liouville fractional integrals $I_{a^+}^\alpha f$ and $I_{b^-}^\alpha f$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ can be represented as

$$\left(I_{a^+}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad \left(x > a; \Re(\alpha) > 0\right)$$

and

$$\left(I_{b^-}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt \quad \left(x < b; \Re(\alpha) > 0\right),$$

respectively, where Γ is the gamma function.

Definition 2.2 ([1]) If $y(x) \in AC^n[a,b]$, the Caputo derivatives $({}^cD_{a^+}^{\alpha}y)(x)$ and $({}^cD_{b^-}^{\alpha}y)(x)$ exist almost everywhere on $[a,b]$.

(a) When $\alpha \notin N_0$, $({}^cD_{a^+}^{\alpha}y)(x)$ and $({}^cD_{b^-}^{\alpha}y)(x)$ are definedd as

$$\left({}^cD_{a^+}^{\alpha}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt$$

and

$$\left({}^cD_{b^-}^{\alpha}y\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{y^{(n)}(t)}{(t-x)^{\alpha-n+1}}\,dt,$$

respectively, where D stands for the derivative operator, and $n = [\Re(\alpha)] + 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$.

(b) If $\alpha \in N_0$, then $({}^cD_{a^+}^{n}y)(x) = y^{(n)}(x)$ and $({}^cD_{b^-}^{n}y)(x) = (-1)^{(n)}y^{(n)}(x)$.

Lemma 2.1 ([1]) *The general solution of the fractional-order equation* $({}^cD_{a^+}^{\alpha}y)(x) = 0$ *can be obtained as*

$$y(x) = \sum_{k=0}^{n-1}\frac{y^{(i)}(a)}{i!}(x-a)^i.$$

In particular, for $a = 0$, it can be presented as

$$y(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1},$$

where $c_i = \frac{y^{(i)}(0)}{i!}$ $(i = 0,1,\ldots n-1)$ stand for certain constants.

Lemma 2.2 *Let* $y(t) \in C[0,1]$. *Then the BVP*

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = y(t), \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \\ x(1) = \int_{\gamma}^1 x(\tau)\,d\tau, \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)\,d\tau + \frac{3t^2}{(2+\gamma^3)\Gamma(\alpha+1)}\int_0^1 (1-\tau)^{\alpha}y(\tau)\,d\tau$$

$$- \frac{3t^2}{(2+\gamma^3)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau - \frac{3t^2}{(2+\gamma^3)\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^{\alpha} y(\tau)\,d\tau.$$

where $2 < \alpha \le 3$ *and* $0 < \gamma < 1$.

Proof By Lemma 2.1 we easily get

$$x(t) = I_{0^+}^{\alpha} y(t) + c_0 + c_1 t + c_2 t^2 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\,d\tau + c_0 + c_1 t + c_2 t^2$$

and

$$x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} y(\tau)\,d\tau + c_1 + 2c_2 t$$

for some $c_0, c_1, c_2 \in R$. From the BCs in (2.1) we have $c_0 = c_1 = 0$ and

$$c_2 = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau + \int_\gamma^1 x(\tau)\,d\tau.$$

Hence

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\,d\tau - \frac{t^2}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau + t^2 \int_\gamma^1 x(\tau)\,d\tau. \tag{2.2}$$

Integrating both sides of (2.2) from γ to 1 yields

$$\begin{aligned}
&\int_\gamma^1 x(t)\,dt \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_\gamma^1 \left[\int_0^1 (t-\tau)^{\alpha-1} y(\tau)\,d\tau\right] dt - \frac{1}{\Gamma(\alpha)} \int_\gamma^1 t^2\,dt \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau \\
&\qquad + \int_\gamma^1 t^2\,dt \int_\gamma^1 x(\tau)\,d\tau \\
&\quad = \frac{1}{\alpha\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha} y(\tau)\,d\tau - \frac{1}{\alpha\Gamma(\alpha)} \int_0^\gamma (\gamma-\tau)^{\alpha} y(\tau)\,d\tau \\
&\qquad - \frac{1-\gamma^3}{3\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau + \frac{1-\gamma^3}{3} \int_\gamma^1 x(\tau)\,d\tau.
\end{aligned}$$

By switching and rearranging this equation we have

$$\begin{aligned}
\int_\gamma^1 x(t)\,dt = {}& \frac{3}{(2+\gamma^3\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha} y(\tau)\,d\tau\,dt \\
& - \frac{3}{(2+\gamma^3)\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^{\alpha} y(\tau)\,d\tau \\
& - \frac{1-\gamma^3}{(2+\gamma^3)\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,d\tau.
\end{aligned}$$

Substituting this equation into equation (2.2), we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\,d\tau + \frac{3t^2}{(2+\gamma^3)\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha} y(\tau)\,d\tau$$

$$-\frac{3t^2}{(2+\gamma^3)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}y(\tau)\,d\tau-\frac{3t^2}{(2+\gamma^3)\Gamma(\alpha+1)}\int_0^{\gamma}(\gamma-\tau)^{\alpha}y(\tau)\,d\tau.$$

The proof is finished. □

The conclusions of this paper are mainly derived from the following fixed point theorems.

Lemma 2.3 ([1] Banach's fixed point theorem) *Let (U,d) be a nonempty complete metric space, let $0\le\omega<1$, and let $T:U\to U$ be a mapping such*

$$d(Tu,Tv)\le\omega\,d(u,v)$$

for all $u,v\in U$. Then T contains a unique fixed point (FP) $u^*\in U$, that is, $Tu^*=u^*$.

Lemma 2.4 ([34] Krasnoselskii's fixed point theorem) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A and B are mappings satisfying the following conditions: (a) $Ax+By\in M$ for $x,y\in M$; (b) A is compact and continuous; (c) B is a contraction. Then there is $z\in M$ such that $z=Az+Bz$.*

Lemma 2.5 ([35] Schaefer's fixed point theorem) *Let X be a Banach space. Let $T:X\to X$ be a completely continuous operator, and let $V=\{u\in X\mid u=\mu Tu, 0<\mu<1\}$ be a bounded set. Then T has a fixed point in X.*

Finally, we introduce some basic knowledge of L^p space and present the Hölder inequality and its generalized form [36].

Let $\Omega\subset R^n$ be an open set (or a measurable set), let $f(x)$ be a real-valued measurable function on Ω. For $1\le p<\infty$, since $|f(x)|^p$ is also measurable on Ω, the integral $\int_\Omega|f(x)|^p\,dx$ makes sense. Then the function space $L^p(\Omega)$ is defined as follows:

$L^p(\Omega)=\{f(x)|f(x)$ is measurable on Ω, and $\int_\Omega|f(x)|^p\,dx<\infty\}$.

For $f\in L^p(\Omega)$, the following norm can be defined:

$$\|f\|_p=\left(\int_\Omega\big|f(x)\big|^p\,dx\right)^{1/p}.$$

We call $1<p,q<\infty$ conjugate exponentials of each other if $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 2.6 ([36] Hölder's inequality) *Let $\Omega\subset R^n$ be an open set, let p, q be conjugate exponentials, let $f(x)\in L^p(\Omega)$ and $g(x)\in L^q(\Omega)$. Then the function $f(x)g(x)$ is integrable on Ω, and*

$$\int_\Omega\big|f(x)g(x)\,dx\big|\,dx\le\|f\|_p\|g\|_q.$$

This inequality can be generalized as follows:

$$\int_\Omega\big|f_1(x)\cdots f_n(x)\,dx\big|\,dx\le\|f_1\|_{p_i}\cdots\|f_n\|_{p_n}.$$

provided that $f_i(x)\in L^{p_i}(\Omega)$, $1<p_i<\infty$, and $\sum_{k=1}^n\frac{1}{p_i}=1$.

3 Fundamental results

Let $X = C([0,1],R)$ be the Banach space of real-valued continuous functions on $[0,1]$ endowed with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Throughout this paper, we make the following assumption on the singularity of nonlinear function $f(t,x(t))$ in (1.1):

(H1) $f(t,x(t))$ has a singularity at $t = 0$ and $t = 1$, that is,

$$\lim_{t\to 0^+} f(t,\cdot) = \infty, \qquad \lim_{t\to 1^-} f(t,\cdot) = \infty.$$

Moreover, there exist constants $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ such that $t^{\theta_1}(1-t)^{\theta_2} f(t,x(t))$ is continuous on $[0,1]$.

Based on condition (H1), we know that there is a positive constant M_0 such that

$$\bigl|t^{\theta_1}(1-t)^{\theta_2} f\bigl(t,x(t)\bigr)\bigr| \le M_0, \quad x\in X, t\in[0,1]. \tag{3.1}$$

Let $\lambda = \frac{3}{2+\gamma^3}$. By Lemma 2.2 the operator $A : X \to X$ can be represented as

$$\begin{aligned}(Ax)(t) = {}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\bigl(\tau,x(\tau)\bigr)\,d\tau + \frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^1 (1-\tau)^{\alpha} f\bigl(\tau,x(\tau)\bigr)\,d\tau \\ & - \frac{\lambda t^2}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f\bigl(\tau,x(\tau)\bigr)\,d\tau \\ & - \frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^{\gamma} (\gamma-\tau)^{\alpha} f\bigl(\tau,x(\tau)\bigr)\,d\tau. \end{aligned} \tag{3.2}$$

Then the solutions of problem (1.1) include the FPs of A.

Lemma 3.1 *Suppose* $0 < \theta_1 < 1$ *and* $0 < \theta_2 < 1$. *Then the integral operator* J *defined as*

$$J(t) = \int_0^t (t-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau, \quad t\in[0,1]$$

has the following specifications:

(1) $\lim_{t\to 0^+} J(t) = 0$;

(2) $|J(t) - J(t_0)| < (\alpha-1)B(1-\theta_1, \alpha-\theta_2-1)|t-t_0|$ *for all* $t, t_0 \in [0,1]$,

where $B(\cdot,\cdot)$ *denotes the beta function.*

Proof (1) By Lemma 2.6, for any $p_1 > 1, p_2 > 1, p_3 > 1$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$, $0 < p_1\theta_1 < 1$, and $0 < p_2\theta_2 < 1$, we have

$$\begin{aligned} J(t) &\le \biggl[\int_0^t \tau^{-p_1\theta_1}\,d\tau\biggr]^{1/p_1}\biggl[\int_0^t (1-\tau)^{-p_2\theta_2}\,d\tau\biggr]^{1/p_2}\biggl[\biggl[\int_0^t (t-\tau)^{p_3(\alpha-1)}\,d\tau\biggr]^{1/p_3} \\ &= \biggl[\frac{1}{1-p_1\theta_1}\tau^{1-p_1\theta_1}\bigg|_0^t\biggr]^{1/p_1}\biggl[-\frac{1}{1-p_2\theta_2}(1-\tau)^{1-p_2\theta_2}\bigg|_0^t\biggr]^{1/p_2} \\ &\quad\cdot\biggl[-\frac{1}{1+p_3(\alpha-1)}(t-\tau)^{1+p_3(\alpha-1)}\bigg|_0^t\biggr]^{1/p_3} \\ &= \frac{1}{\sqrt[p_1]{1-p_1\theta_1}}\frac{1}{\sqrt[p_2]{1-p_2\theta_2}}\frac{1}{\sqrt[p_3]{1+p_3(\alpha-1)}}\sqrt[p_1]{t^{1-p_1\theta_1}} \end{aligned}$$

$$\cdot \sqrt[p_2]{1-(1-t)^{1-p_2\theta_2}} \cdot \sqrt[p_3]{t^{1+p_3(\alpha-1)}}.$$

Since $J(t) \geq 0$, and $\lim_{t\to 0^+}(\sqrt[p_1]{t^{1-p_1\theta_1}} \cdot \sqrt[p_2]{1-(1-t)^{1-p_2\theta_2}} \cdot \sqrt[p_3]{t^{1+p_3(\alpha-1)}}) = 0$, we get

$$\lim_{t\to 0^+} J(t) = 0.$$

(2) By the expression of $J(t)$ we easily get

$$\begin{aligned} J'(t) &= (\alpha-1)\int_0^t (t-\tau)^{\alpha-2}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\ &\leq (\alpha-1)\int_0^1 (1-\tau)^{\alpha-2}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\ &= (\alpha-1)\int_0^1 (1-\tau)^{\alpha-\theta_2-2}\tau^{-\theta_1}\,d\tau \\ &= (\alpha-1)B(1-\theta_1,\alpha-\theta_2-1). \end{aligned}$$

Hence the mean value theorem gives us

$$\left|J(t)-J(t_0)\right| = J'(\xi)|t-t_0| < (\alpha-1)B(1-\theta_1,\alpha-\theta_2-1)|t-t_0|,$$

where the number ξ is between t and t_0. □

Lemma 3.2 *Let* $2 < \alpha \leq 3$, *and let* $g:(0,1)\to R$ *be a continuous function such that* $\lim_{t\to 0^+} g(t) = \infty$ *and* $\lim_{t\to 1^-} g(t) = \infty$. *Suppose that there exist two constants* $0 < \theta_1 < 1$ *and* $0 < \theta_2 < 1$ *such that* $t^{\theta_1}(1-t)^{\theta_2}g(t)$ *is continuous in* $[0,1]$. *Then the function*

$$\begin{aligned} G(t) := {}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g(\tau)\,d\tau + \frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^1 (1-\tau)^{\alpha}g(\tau)\,d\tau \\ & - \frac{\lambda t^2}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}g(\tau)\,d\tau - \frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^{\gamma} (\gamma-\tau)^{\alpha}g(\tau)\,d\tau \end{aligned}$$

is continuous in $[0,1]$.

Proof Based on the expression of $G(t)$, we easily find $G(0) = 0$. As $t^{\theta_1}(1-t)^{\theta_2}g(t)$ is continuous in $[0,1]$, there is a positive constant M_1 such that $|t^{\theta_1}(1-t)^{\theta_2}g(t)| \leq M_1$ for all $t \in [0,1]$. For all $t_0 \in [0,1]$, we will prove the continuity of $G(t)$ in three cases.

(a) $t_0 = 0$, $t \in [0,1]$. We have

$$\begin{aligned} &\left|G(t)-G(0)\right| \\ &\quad= \bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\tau^{\theta_1}(1-\tau)^{\theta_2}g(\tau)\,d\tau \\ &\qquad+ \frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^1 (1-\tau)^{\alpha}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\tau^{\theta_1}(1-\tau)^{\theta_2}g(\tau)\,d\tau \\ &\qquad- \frac{\lambda t^2}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\tau^{\theta_1}(1-\tau)^{\theta_2}g(\tau)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda t^2}{\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^\alpha \tau^{-\theta_1}(1-\tau)^{-\theta_2}\tau^{\theta_1}(1-\tau)^{\theta_2} g(\tau)\,d\tau \bigg| \\
&\le \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau + \frac{\lambda M_1 t^2}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^\alpha \tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\quad + \frac{\lambda M_1 t^2}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau + \frac{\lambda M_1 t^2}{\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^\alpha \tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\le \frac{M_1}{\Gamma(\alpha)} J(t) + \frac{\lambda M_1 t^2}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau + \frac{\lambda M_1 t^2}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau \\
&\quad + \frac{\lambda M_1 t^2}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau \\
&= \frac{M_1}{\Gamma(\alpha)} J(t) + \frac{2\lambda M_1 t^2}{\Gamma(\alpha+1)} B(1-\theta_1, \alpha-\theta_2+1) + \frac{\lambda M_1 t^2}{\Gamma(\alpha)} B(1-\theta_1, \alpha-\theta_2) \\
&\to 0 \quad (t \to t_0 = 0).
\end{aligned}
$$

(b) $t_0 \in (0,1]$, $t \in [0, t_0)$. Then

$$
\begin{aligned}
&\big|G(t) - G(t_0)\big| \\
&= \bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0-\tau)^{\alpha-1} g(\tau)\,d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\,d\tau \\
&\quad + \frac{\lambda(t_0^2-t^2)}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^\alpha g(\tau)\,d\tau + \frac{\lambda(t_0^2-t^2)}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} g(\tau)\,d\tau \\
&\quad + \frac{\lambda(t_0^2-t^2)}{\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^\alpha g(\tau)\,d\tau \bigg| \\
&\le \bigg| \frac{1}{\Gamma(\alpha)} \int_0^t \big[(t_0-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\big] g(\tau)\,d\tau + \frac{1}{\Gamma(\alpha)} \int_t^{t_0} (t_0-\tau)^{\alpha-1} g(\tau)\,d\tau \bigg| \\
&\quad + \frac{\lambda M_1 (t_0+t)(t_0-t)}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau \\
&\quad + \frac{\lambda M_1 (t_0+t)(t_0-t)}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau \\
&\quad + \frac{\lambda M_1 (t_0+t)(t_0-t)}{\Gamma(\alpha+1)} \int_0^\gamma (\gamma-\tau)^\alpha \tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\le \frac{M_1}{\Gamma(\alpha)} \int_0^t \big[(t_0-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\big] \tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\quad + \frac{M_1}{\Gamma(\alpha)} \int_t^{t_0} (t_0-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\quad + \frac{2\lambda M_1 (t_0-t)}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau + \frac{2\lambda M_1 (t_0-t)}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau \\
&\quad + \frac{2\lambda M_1 (t_0-t)}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^\alpha \tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&= \frac{M_1}{\Gamma(\alpha)} \int_0^{t_0} (t_0-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau - \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau \\
&\quad + \frac{2\lambda M_1 (t_0-t)}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau + \frac{2\lambda M_1 (t_0-t)}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\lambda M_1(t_0-t)}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\
&=\frac{M_1}{\Gamma(\alpha)}\bigl[J(t_0)-J(t)\bigr]+\frac{4\lambda M_1(t_0-t)}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1)\\
&+\frac{2\lambda M_1(t_0-t)}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2+1).
\end{aligned}
$$

By the second result of Lemma 3.1 we have

$$
\bigl|G(t)-G(t_0)\bigr|\le M_1\frac{\alpha(\alpha-1)+4\lambda+2\lambda\alpha}{\Gamma(\alpha+1)}\cdot B(1-\theta_1,\alpha-\theta_2+1)(t_0-t)\to 0(t\to t_0).
$$

(c) $t_0\in(0,1)$, $t\in(t_0,1]$. Since the proof for this case is the same as that in case (b), we omit it. □

Lemma 3.3 *Let $2<\alpha\le 3$, and let $f:(0,1)\times R\to R$ be a continuous function satisfying the singularity condition* (H1). *Then the operator $A:X\to X$ is completely continuous.*

Proof According to Lemma 3.2, $A:X\to X$ is continuous. Let $D\subset X=C([0,1],R)$ be a bounded set, that is, there is a positive constant L_1 such that $\|x\|\le L_1$ for all $x\in D$.

Relations (3.1) and (3.2) give

$$
\begin{aligned}
|Ax|&\le\frac{1}{\Gamma(\alpha)}\left|\int_0^t(t-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|+\frac{\lambda}{\Gamma(\alpha+1)}\left|\int_0^1(1-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\left|\int_0^1(1-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|+\frac{\lambda}{\Gamma(\alpha+1)}\left|\int_0^\gamma(\gamma-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|\\
&\le\frac{M_0}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
&\quad+\frac{\lambda M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
&\quad+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^\gamma(\gamma-\tau)^{\alpha}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
&\le\frac{M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\
&\quad+\frac{\lambda M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\
&=\frac{(1+\lambda)M_0}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2)+\frac{2\lambda M_0}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1):=L_2,
\end{aligned}
$$

that is, $\|Ax\|\le L_2$, for all $x\in D$. Thus the operator A is bounded on D. This yields the compactness of A. For every $t\in[0,1]$, we have

$$
\begin{aligned}
\bigl|(Ax)'(t)\bigr|&=\left|\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}f\bigl(\tau,x(\tau)\bigr)\,d\tau+\frac{2\lambda t}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right.\\
&\quad\left.-\frac{2\lambda t}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau-\frac{2\lambda t}{\Gamma(\alpha+1)}\int_0^\gamma(\gamma-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|\\
&\le\frac{M_0}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
&+\frac{2\lambda M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
&+\frac{2\lambda M_0}{\Gamma(\alpha+1)}\int_0^{\gamma}(\gamma-\tau)^{\alpha}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\\
\le{}&\frac{M_0}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-\theta_2-2}\tau^{-\theta_1}\,d\tau+\frac{2\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\
&+\frac{2\lambda M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{2\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\
={}&\frac{M_0}{\Gamma(\alpha-1)}B(1-\theta_1,\alpha-\theta_2-1)+\frac{4\lambda M_0}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1)\\
&+\frac{2\lambda M_0}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2):=L_3.
\end{aligned}
$$

Now the following inequality holds for $t_1,t_2\in[0,1]$ and $t_1<t_2$:

$$
\bigl|(Ax)(t_2)-(Ax)(t_1)\bigr|=\left|\int_{t_1}^{t_2}(Ax)'(s)\,ds\right|\le L_3(t_2-t_1).
$$

Therefore A is equicontinuous on D. Thus, by the Arzelà–Ascoli theorem the operator A is completely continuous on X. □

Now we present and demonstrate our fundamental results. The first result deals with the existence and uniqueness of the solution to problem (1.1).

Theorem 3.1 *Let $2<\alpha\le 3$ and $0<\theta_1,\theta_2<1$ be constants, and let $f(t,x(t))$ satisfy condition* (H1) *and the following conditions*:

(H2) *There is a function $m(t)\in L^p([0,1],R^+)$ $(p>1)$ such that*

$$
t^{\theta_1}(1-t)^{\theta_2}\bigl|f(t,x)-f(t,y)\bigr|\le m(t)|x-y|.
$$

(H3) *There exist three constants p_1,p_2,p_3 satisfying $p_1>1$, $p_2>1$, $p_3>1$, $0<p_1\theta_1<1$, and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. If*

$$
\begin{aligned}
&\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}\biggl[\frac{1+\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}}\\
&\quad+\frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}<1,
\end{aligned}
\tag{3.3}
$$

then the solution to problem (1.1) *is unique.*

Proof For $x,y\in X=C([0,1])$ and $t\in[0,1]$, by (H2) we have

$$
\begin{aligned}
\bigl|(Ax)(t)-(Ay)(t)\bigr|\le{}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|\,d\tau\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\big|f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)\big|\,d\tau\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\bigg|\int_0^{\gamma}(\gamma-\tau)^{\alpha}\big|f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)\big|\,d\tau\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}m(\tau)\big|x(\tau)-y(\tau)\big|\,d\tau\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}m(\tau)\big|x(\tau)-y(\tau)\big|\,d\tau\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}m(\tau)\big|x(\tau)-y(\tau)\big|\,d\tau\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}m(\tau)\big|x(\tau)-y(\tau)\big|\,d\tau.
\end{aligned}
$$

By (H3) and the Hölder inequality we have

$$
\begin{aligned}
&\big|(Ax)(t)-(Ay)(t)\big|\\
&\le\big\|x(\tau)-y(\tau)\big\|\cdot\Bigg\{\frac{1}{\Gamma(\alpha)}\bigg[\int_0^1\tau^{-\theta_1p_1}\,d\tau\bigg]^{1/p_1}\bigg[\int_0^1(1-\tau)^{(\alpha-\theta_2-1)p_2}\,d\tau\bigg]^{1/p_2}\\
&\quad\times\bigg[\int_0^1 m^{p_3}(\tau)\,d\tau\bigg]^{1/p_3}\\
&\quad+\frac{\lambda}{\Gamma(\alpha+1)}\bigg[\int_0^1\tau^{-\theta_1p_1}\,d\tau\bigg]^{1/p_1}\bigg[\int_0^1(1-\tau)^{(\alpha-\theta_2)p_2}\,d\tau\bigg]^{1/p_2}\bigg[\int_0^1 m^{p_3}(\tau)\,d\tau\bigg]^{1/p_3}\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\bigg[\int_0^1\tau^{-\theta_1p_1}\,d\tau\bigg]^{1/p_1}\bigg[\int_0^1(1-\tau)^{(\alpha-\theta_2-1)p_2}\,d\tau\bigg]^{1/p_2}\bigg[\int_0^1 m^{p_3}(\tau)\,d\tau\bigg]^{1/p_3}\\
&\quad+\frac{\lambda}{\Gamma(\alpha+1)}\bigg[\int_0^1\tau^{-\theta_1p_1}\,d\tau\bigg]^{1/p_1}\bigg[\int_0^1(1-\tau)^{(\alpha-\theta_2)p_2}\,d\tau\bigg]^{1/p_2}\bigg[\int_0^1 m^{p_3}(\tau)\,d\tau\bigg]^{1/p_3}\Bigg\}\\
&=\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}\bigg[\frac{1+\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}}+\frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}\bigg]\\
&\quad\times\big\|x(\tau)-y(\tau)\big\|.
\end{aligned}
$$

Noticing (3.3), we conclude that A is a contraction mapping. Thus by Lemma 2.3 it has a unique FP, which is also the unique solution to problem (1.1). □

The second result states the existence of the solution to the BVP (1.1) derived from Lemma 2.4.

Theorem 3.2 *Let* $2<\alpha\le 3$ *and* $0<\theta_1,\theta_2<1$ *be constants, and let* $f(t,x(t))$ *satisfy conditions* (H1)–(H3) *and the following condition*:

$$
\begin{aligned}
&\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}\bigg[\frac{\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}}\\
&\quad+\frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}\bigg]<1.
\end{aligned}
\tag{3.4}
$$

Then problem (1.1) *has a solution*.

Proof We fix a constant

$$r \geq M_0\left[\frac{1+\lambda}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2)+\frac{2\lambda}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1)\right].$$

Consider a ball $B_r=\{x\in X=C([0,1],R):\|x\|\leq r\}$. Define two operators A_1 and A_1 on B_r as

$$\begin{aligned}(A_1x)(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau,\\ (A_2x)(t)&=\frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau\\ &\quad-\frac{\lambda t^2}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau\\ &\quad-\frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^{\gamma}(\gamma-\tau)^{\alpha}f\bigl(\tau,x(\tau)\bigr)\,d\tau.\end{aligned}$$

For $x,y\in B_r$, by (3.1) we can check that

$$\begin{aligned}\|A_1x+A_2y\|&\leq\frac{M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\ &\quad+\frac{\lambda M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda M_0}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\\ &=M_0\left[\frac{(1+\lambda)}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2)+\frac{2\lambda}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1)\right]\\ &\leq r.\end{aligned}$$

So $A_1x+A_2y\in B_r$. Like in the proof of Theorem 3.1, from (H2), (H3), and (3.4) we can conclude that the operator A_2 is also a contraction mapping. Lemma 3.2 and (H1) ensure the continuity of the operator A_1. For any $x\in B_r$, we have

$$\begin{aligned}\|A_1x\|&\leq\frac{M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-}\theta_2\,d\tau\\ &\leq\frac{M_0}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau\\ &=\frac{M_0}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2).\end{aligned}$$

Thus A_1 is uniformly bounded on B_r. For all $t_1,t_2\in[0,1]$ such that $t_1<t_2$, we obtain

$$\begin{aligned}&\bigl|(A_1x)(t_2)-(A_1x)(t_1)\bigr|\\ &\quad=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau-\int_0^{t_1}(t_1-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|\\ &\quad\leq\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\bigl[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\bigr]f\bigl(\tau,x(\tau)\bigr)\,d\tau+\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)\,d\tau\right|\\ &\quad\leq\frac{M_0}{\Gamma(\alpha)}\left[\int_0^{t_1}\bigl[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\bigr]\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\right.\end{aligned}$$

$$\begin{aligned}&+\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\tau^{-\theta_1}(1-\tau)^{-\theta_2}\,d\tau\Bigg]\\&=\frac{M_0}{\Gamma(\alpha)}\big[J(t_2)-J(t_1)\big].\end{aligned}$$

By Lemma 3.1 we have

$$\big|(A_1x)(t_2)-(A_1x)(t_1)\big|=\frac{(\alpha-1)M_0}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2-1)(t_2-t_1).$$

This means that A_1 is equicontinuous and relatively compact on B_r. Accordingly, by the Arzelà–Ascoli theorem A_1 is compact on B_r. Accordingly, Lemma 2.4 ensures the existence of a solution for problem (1.1) in $[0,1]$. □

The Schaefer fixed point theorem gives the last result.

Theorem 3.3 *Let $2<\alpha\le 3$ and $0<\theta_1,\theta_2<1$ be constants, and let $f(t,x(t))$ satisfy conditions* (H1) *and* (3.1). *Then problem* (1.1) *has a solution in* $[0,1]$.

Proof By Lemma 3.3 we know that the operator $A:X\to X$ is completely continuous.

Next, we prove that the set $V=\{x\in C([0,1],R):x=\mu Ax,0<\mu<1\}$ is bounded.

Let $x\in V$. Then $x=\mu(Ax)$. Thus, for each $t\in[0,1]$, we have

$$\begin{aligned}|x|&=\mu\big|(Ax)(t)\big|\\&=\mu\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\big(\tau,x(\tau)\big)\,d\tau+\frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha}f\big(\tau,x(\tau)\big)\,d\tau\\&\quad-\frac{\lambda t^2}{\Gamma(\alpha)}\int_0^t(1-\tau)^{\alpha-1}f\big(\tau,x(\tau)\big)\,d\tau-\frac{\lambda t^2}{\Gamma(\alpha+1)}\int_0^{\gamma}(\gamma-\tau)^{\alpha}f\big(\tau,x(\tau)\big)\,d\tau\Bigg|\\&\le M_0\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau\Bigg]\\&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-\theta_2-1}\tau^{-\theta_1}\,d\tau+\frac{\lambda}{\Gamma(\alpha+1)}\int_0^1(1-\tau)^{\alpha-\theta_2}\tau^{-\theta_1}\,d\tau]\\&=M_0\Bigg[\frac{1+\lambda}{\Gamma(\alpha)}B(1-\theta_1,\alpha-\theta_2)+\frac{2\lambda}{\Gamma(\alpha+1)}B(1-\theta_1,\alpha-\theta_2+1)\Bigg]=L_2.\end{aligned}$$

Hence we have

$$\|x\|\le L_2.$$

This shows that the set V is bounded. Lemma 2.5 ensures the existence of fixed points of A. Accordingly, there is at least one solution to problem (1.1) in $[0,1]$. □

4 Examples

We introduce three examples to clarify the performed work.

Example 4.1 Consider the following fractional BVP:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{9}{4}}x(t) = \frac{\sin x}{\sqrt[46]{t}\cdot\sqrt[5]{1-t}}, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \\ x(1) = \int_{0.5}^{1} x(\tau)\,d\tau. \end{cases} \tag{4.1}$$

Thus $f(t,x) = \frac{\sin x}{\sqrt[46]{t}\cdot\sqrt[5]{1-t}}$, $\alpha = \frac{9}{4}$, $\gamma = 0.5$. Take $\theta_1 = \frac{1}{23}$, $\theta_2 = \frac{5}{6}$, and $p_1 = p_3 = 22$, $p_2 = 1.1$. Since

$$\begin{aligned} t^{\frac{1}{23}}(1-t)^{\frac{5}{6}}\left|f(t,x)-f(t,y)\right| &= t^{\frac{1}{46}}(1-t)^{\frac{19}{30}}|\sin x - \sin y| \\ &= 2t^{\frac{1}{46}}(1-t)^{\frac{19}{30}}\left|\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right| \\ &\le t^{\frac{1}{46}}(1-t)^{\frac{19}{30}}|x-y|. \end{aligned}$$

Accordingly, $m(t) = t^{\frac{1}{46}}(1-t)^{\frac{19}{30}}$. We can calculate the following: $\lambda = \frac{3}{2+\gamma^3} \approx 1.4118$, $\Gamma(\alpha) = \Gamma(\frac{9}{4}) \approx 1.128$, $\Gamma(\alpha+1) = \Gamma(1+\frac{9}{4}) \approx 2.5493$, $\frac{1+\lambda}{\Gamma(\alpha)} \approx 2.1381$, $\frac{2\lambda}{\Gamma(\alpha+1)} \approx 1.1076$, $\|m\|_{p_3} = \{\int_0^1 [t^{\frac{1}{46}}(1-t)^{\frac{19}{30}}]^{22}\,ds\}^{1/22} \approx 0.1521$, $\frac{1}{\sqrt[p_1]{1-p_1\theta_1}} \approx 1.1532$, $\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}} \approx 0.7097$, $\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}} \approx 0.4258$, $\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}[\frac{1+\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}} + \frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}] \approx 0.3489 < 1$. Since conditions (H1)–(H3) and (3.3) are all satisfied, Theorem 3.1 ensures a unique solution $x(t)$ in $[0,1]$ for this example.

Example 4.2 Consider the following fractional BVP:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{7}{3}}x(t) = \frac{\sin(tx)}{\sqrt[8]{t}\cdot\sqrt[3]{1-t}}, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \\ x(1) = \int_{0.2}^{1} x(\tau)\,d\tau. \end{cases} \tag{4.2}$$

Thus $f(t,x) = \frac{\sin(tx)}{\sqrt[8]{t}\cdot\sqrt[3]{1-t}}$, $\alpha = \frac{7}{3}$, $\gamma = 0.2$. Take $\theta_1 = \frac{1}{7}$, $\theta_2 = \frac{2}{3}$, and $p_1 = 6$, $p_2 = 30$, $p_3 = 1.25$. Since

$$\begin{aligned} t^{\frac{1}{7}}(1-t)^{\frac{2}{3}}\left|f(t,x)-f(t,y)\right| &= t^{\frac{1}{56}}(1-t)^{\frac{1}{3}}\left|\sin(tx) - \sin(ty)\right| \\ &= 2t^{\frac{1}{56}}(1-t)^{\frac{1}{3}}\left|\cos\frac{t(x+y)}{2}\sin\frac{t(x-y)}{2}\right| \\ &\le t^{1+\frac{1}{56}}(1-t)^{\frac{1}{3}}|x-y|. \end{aligned}$$

Therefore $m(t) = t^{1+\frac{1}{56}}(1-t)^{\frac{1}{3}}$. We can obtain the following: $\lambda = \frac{3}{2+\gamma^3} \approx 1.4940$, $\Gamma(\alpha) = \Gamma(\frac{7}{3}) \approx 1.1960$, $\Gamma(\alpha+1) = \Gamma(1+\frac{7}{3}) \approx 2.7907$, $\frac{1+\lambda}{\Gamma(\alpha)} \approx 2.0853$, $\frac{2\lambda}{\Gamma(\alpha+1)} \approx 1.0707$, $\frac{\lambda}{\Gamma(\alpha)} \approx 1.2492$, $\|m\|_{p_3} = \{\int_0^1 [t^{1+\frac{1}{56}}(1-t)^{\frac{1}{3}}]^{5/4}\,ds\}^{4/5} \approx 0.3268$, $\frac{1}{\sqrt[p_1]{1-p_1\theta_1}} \approx 1.3831$, $\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}} \approx 0.9035$, $\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}} \approx 0.8772$, $\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}[\frac{1+\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}} + \frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}] \approx 1.2762 > 1$, $\|m\|_{p_3}\frac{1}{\sqrt[p_1]{1-p_1\theta_1}}[\frac{\lambda}{\Gamma(\alpha)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2-1)}} + \frac{2\lambda}{\Gamma(\alpha+1)}\frac{1}{\sqrt[p_2]{1+p_2(\alpha-\theta_2)}}] \approx 0.9318 < 1$.

Accordingly, conditions (H1)–(H3) and (3.4) are all satisfied for this example, which means that this problem has at least a solution $x(t)$ in $[0,1]$ by Theorem 3.2.

Example 4.3 Consider the following fractional BVP:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{5}{2}}x(t) = \frac{\sqrt[5]{\tan t}}{\sqrt[3]{t}\cdot\sqrt[4]{1-t}}[\sin(x-t)+\cos(tx)], \quad 0<t<1, \\ x(0)=x'(0)=0, \\ x(1)=\int_{0.4}^{1}x(\tau)\,d\tau. \end{cases} \tag{4.3}$$

We have $f(t,x(t)) = \frac{\sqrt[5]{\tan t}}{\sqrt[3]{t}\cdot\sqrt[4]{1-t}}[\sin(x-t)+\cos(tx)]$, $\alpha=\frac{5}{2}$, $\gamma=0.4$. Take $\theta_1=\frac{2}{3}$, $\theta_2=\frac{2}{3}$. Then $t^{\frac{2}{3}}(1-t)^{\frac{2}{3}}f(t,x) = \sqrt[6]{t}\sqrt[3]{1-t}\sqrt[5]{\tan t}[\sin(x-t)+\cos(tx)]$ is continuous in $[0,1]$, and $|t^{\frac{2}{3}}(1-t)^{\frac{2}{3}}f(t,x)| \le 2\sqrt[6]{\frac{4}{27}}\cdot\sqrt[10]{3}$.

Since conditions (H1) and (3.1) are all satisfied for this example, by Theorem 3.3 this problem has at least a solution $x(t)$ in $[0,1]$.

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References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory of Fractional Differential Equations. North Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
3. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
4. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. **69**(8), 2677–2682 (2008)
5. Lakshmikantham, V., Vatsala, A.S.: Theory of fractional differential inequalities and applications. Commun. Appl. **11**(3–4), 395–402 (2007)
6. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives, Theory and Applications. Gordon & Breach, Yverdon (1993)
7. Zhang, S.: Positive solutions for boundary value problems of nonlinear fractional differential equations. Electron. J. Differ. Equ. **2006**, 36 (2006)
8. Agarwal, R.P., Benchora, M., Hamani, S.: Boundary value problems for fractional differential equations. Georgian Math. J. **16**(3), 401–411 (2009)
9. Bai, Z., Liu, H.: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**(2), 495–505 (2005)
10. Wu, J., Liu, Y.: Existence and uniqueness of solutions for the fractional integro-differential equations in Banach spaces. Electron. J. Differ. Equ. **2009**, 129 (2009)
11. Jiang, D., Yuan, C.: The positive properties of the Green function for Dirichlet type boundary value problems of nonlinear fractional differential equations and its application. Nonlinear Anal. **72**, 710–719 (2010)
12. Delbosco, D., Rodino, L.: Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. **204**, 609–625 (1996)
13. Xu, X., Jiang, D., Yuan, C.: Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equations. Nonlinear Anal. **71**, 4676–4688 (2009)

14. Ahmad, B., Sivasundaram, S.: On four point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl. Math. Comput. **217**, 480–487 (2010)
15. Tariboon, J., Sitthiwirattham, T., Ntouyas, S.K.: Boundary value problems for a new class of three-point nonlocal Riemann–Liouville integral boundary conditions. Adv. Differ. Equ. **2013**, 213 (2013)
16. Darwish, M.A., Ntouyas, S.K.: Existence results for first order boundary value problems for fractional differential equations with four-point integral boundary conditions. Miskolc Math. Notes **15**(1), 51–61 (2014)
17. Ahmad, B., Ntouyas, S.K., Assolami, A.: Caputo type fractional differential equations with nonlocal Riemann–Liouville integral boundary conditions. J. Appl. Math. Comput. **41**, 339–350 (2013)
18. Wang, Y., Liu, L.: Positive properties of the Green function for two-term fractional differential equations and its applications. J. Nonlinear Sci. Appl. **10**, 2094–2102 (2017)
19. Wang, Y.: The Green function of a class of two-term fractional differential equation boundary value problem and its application. Adv. Differ. Equ. **2020**, 80 (2020)
20. Matar, M.M.: On existence of positive solution for initial value problem of nonlinear fractional differential equations of order. Acta Math. Univ. Comen. **LXXXIV**(1), 51–57 (2015)
21. Zhao, Y., Sun, S., Han, Z., Zhang, M.: Positive solutions for boundary value problems of nonlinear fractional differential equations. Appl. Math. Comput. **217**, 6950–6958 (2011)
22. Wang, Y., Liu, L., Wu, Y.: Existence and uniqueness of a positive solution to singular fractional differential equations. Bound. Value Probl. **2012**, 81 (2012)
23. Zhang, X., Zhong, Q.: Multiple positive solutions for nonlocal boundary value problems of singular fractional differential equations. Bound. Value Probl. **2016**, 65 (2016)
24. Agarwal, R.P., O'regan, D., Staněk, S.: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57–68 (2010)
25. Qiu, T., Bai, Z.: Existence of positive solutions for singular fractional differential equations. Electron. J. Differ. Equ. **146**, 1 (2008)
26. Jleli, M., Samet, B.: On positive solutions for a class of singular nonlinear fractional differential equations. Bound. Value Probl. **2012**, 73 (2012)
27. Guo, L., Zhang, X.: Existence of positive solutions for the singular fractional differential equations. J. Appl. Math. Comput. **44**, 215–228 (2014)
28. Yan, D.: Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with singular boundary value conditions. Math. Probl. Eng. **2021**, 11 (2021)
29. He, Y.: Existence and multiplicity of positive solutions for singular fractional differential equations with integral boundary value conditions. Adv. Differ. Equ. **2016**, 31 (2016)
30. Min, D., Liu, L., Wu, Y.: Uniqueness of positive solutions for the singular nonlinear fractional differential equations involving integral boundary value conditions. Bound. Value Probl. **2018**, 23 (2018)
31. Liu, L., Min, D., Wu, Y.: Existence and multiplicity of positive solutions for a new class of singular higher-order fractional differential equations with Riemann–Stieltjes integral boundary value conditions. Adv. Differ. Equ. **2020**, 154 (2020)
32. Vong, S.: Positive solutions of singular fractional differential equation with integral boundary conditions. Math. Comput. Model. **57**, 1053–1059 (2013)
33. Wang, Y.: Existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems. Bound. Value Probl. **2019**, 92 (2019)
34. Krasnoselskii, M.A.: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123–127 (1955)
35. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
36. Wang, Y., Xu, J.: Sobolev Space, Southeast University Press (2003) (in Chinese)