Zhang et al. *Boundary Value Problems* (2022) 2022:32
https://doi.org/10.1186/s13661-022-01613-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and multiplicity of nontrivial solutions for poly-Laplacian systems on finite graphs

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Abstract
In this paper, we investigate the existence and multiplicity of nontrivial solutions for poly-Laplacian system on a finite graph $G = (V, E)$, which is a generalization of the Yamabe equation on a finite graph. When the nonlinear term F satisfies the super-(p, q)-linear growth condition, by using the mountain pass theorem we obtain that the system has at least one nontrivial solution, and by using the symmetric mountain pass theorem, we obtain that the system has at least dim W nontrivial solutions, where W is the working space of the poly-Laplacian system. We also obtain the corresponding result for the poly-Laplacian equation. In some sense, our results improve some results in (Grigor'yan et al. in J. Differ. Equ. 261(9):4924–4943, 2016).

MSC: 34B15; 34B18

Keywords: Mountain pass theorem; Poly-Laplacian system; Finite graph; Super-(p, q)-linear growth condition

1 Introduction

In this paper, we mainly consider the following high-order Yamabe-type coupled system, which is called the poly-Laplacian system:

$$\begin{cases} \pounds_{m_1,p}u + h_1(x)|u|^{p-2}u = F_u(x,u,v), & x \in V, \\ \pounds_{m_2,q}v + h_2(x)|v|^{q-2}v = F_v(x,u,v), & x \in V, \end{cases} \tag{1.1}$$

where V is a finite graph, $m_i \geq 2$, $i = 1, 2$, $p, q > 1$ are integers, $h_i : V \to \mathbb{R}^+$, $i = 1, 2$, $F : V \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\pounds_{m,p}$ is defined as follows: for any function $\phi : V \to \mathbb{R}$,

$$\int_V (\pounds_{m,p}u)\phi \, d\mu = \begin{cases} \int_V |\nabla^m u|^{p-2}\Gamma(\Delta^{\frac{m-1}{2}}u, \Delta^{\frac{m-1}{2}}\phi)\, d\mu & \text{if } m \text{ is odd}, \\ \int_V |\nabla^m u|^{p-2}\Delta^{\frac{m}{2}}u\Delta^{\frac{m}{2}}\phi\, d\mu & \text{if } m \text{ is even}. \end{cases} \tag{1.2}$$

When $p = 2$, $\pounds_{m,p} = (-\Delta)^m u$ is called the poly-Laplacian operator of u, and when $m = 1$, $\pounds_{m,p} = -\Delta_p u$. A detailed definition is given in Sect. 2; see also [1].

When $m_1 = m_2 = m$, $p = q$, and $u = v$, system (1.1) becomes the scalar equation

$$£_{m,p}u + h(x)|u|^{p-2}u = f(x,u), \quad x \in V, \tag{1.3}$$

where $f(x,u) = F_u(x,u)$ for $x \in V$, and can be seen as a generalization of the following Yamabe equation on a finite graph:

$$\Delta u + h(x)|u|^{p-2}u = f(x,u), \quad x \in V. \tag{1.4}$$

In recent years, some scholars are devoted to studying the Yamabe equation on finite and infinite graphs. We refer the readers to [1–7]. Ge [2] studied the following Yamabe-type equations with p-Laplacian operator on finite graphs:

$$\Delta_p u(x) + h(x)u^m = \lambda f(x)u^{\alpha-1}, \quad x \in V, \tag{1.5}$$

where $1 < m - 1 \leq \alpha$, $f > 0$, $h > 0$, and Δ_p is defined by

$$\Delta_p f_i = \frac{1}{\mu_i}\sum_{j\sim i}\omega_{ij}|f_j - f_i|^{p-2}(f_j - f_i),$$

where ω_{xy} is the weight of the edge connecting x and y. When the nonlinear term $f > 0$, $m = p - 1$, and $\lambda \in \mathbb{R}$, Ge established the existence of a positive solution. When $1 \leq \alpha \leq p \leq q$, $h \leq 0$, and $f > 0$, Zhang [3], extended the case of $m = p - 1$ in (1.5) to $m = q - 1$ and proved the existence of a positive solution. Ge and Jiang [5] and Zhang and Lin [6] extended the existence results of solutions on finite graphs to infinite graphs for $p = 2$ and $p > 2$ and obtained the existence of one positive solution. Han and Shao [4] investigated the nonlinear p-Laplacian equation

$$-\Delta_p u + \big(\lambda a(x) + 1\big)|u|^{p-2}u = f(x,u), \quad x \in V, \tag{1.6}$$

where $p \geq 2$, where the definition of the p-Laplacian operator Δ_p is different:

$$\Delta_p u(x) = \frac{1}{2\mu(x)}\sum_{y\sim x}\big(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x)\big)\omega_{xy}\big(u(y) - u(x)\big).$$

Under appropriate conditions on the nonlinear terms $f(x,u)$ and $a(x)$, the author obtained the existence of a positive solution for equation (1.6) via the mountain pass theorem. Pinamonti and Stefani [7] studied the following equation with the (m,p)-Laplacian operator on locally finite weighted graphs:

$$\begin{cases} £_{m,p}u = \lambda f(x,u) & \text{in } \Omega^\circ, \\ |\nabla^j u| = 0, & \text{on } \partial\Omega, 0 \leq j \leq m-1, \end{cases}$$

where Ω° and $\partial\Omega$ are the interior and boundary of Ω, respectively. They established the existence of at least one nontrivial solution when $0 < \lambda < \Lambda$ for some $\Lambda > 0$ via the variational method. Besides, they also investigated the following Yamabe-type equations"

$$\begin{cases} -\Delta_p u + g(x,u) = f(x,u) & \text{in } \Omega^\circ, \\ u = h & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$, $h \in L^1(\partial\Omega)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $g(x,0) = 0$ and $t \mapsto g(x,t)$ is nondecreasing for all $x \in \Omega$. They obtained the uniqueness of weak solutions.

The research of this paper is mainly inspired by a recent work due to Grigor'yan, Lin, and Yang [1], who investigated the Yamabe equation and its generalization, that is, poly-Laplacian equation on locally finite and finite graphs. To be specific, in [1], for equation (1.3) on a finite graph V, they assumed that $h(x) > 0$ for all $x \in V$ and F satisfies the following conditions:

(V_1) $F(x,s) = \int_0^s f(x,t)\,dt$ for $x \in V$, $f(x,0) = 0$, and $f(x,t)$ is continuous with respect to $t \in \mathbb{R}$;

(V_2) $\limsup_{t\to 0} \frac{|f(x,t)|}{|t|^{p-1}} < \lambda_{mp}(V)$, where λ_{mp} is the first eigenvalue of the operator $\pounds_{m,p}$, and

$$\lambda_{mp}(V) = \inf_{u\not\equiv 0} \frac{\int_V (|\nabla^m u|^p + h|u|^p)\,d\mu}{\int_V |u|^p\,d\mu};$$

(V_3) there exist $\theta > p$ and $M > 0$ such that if $|s| \geq M$, then

$$0 < \theta F(x,s) \leq sf(x,s) \quad \forall x \in V.$$

They obtained the existence of a nontrivial solution via the mountain pass theorem.

In this paper, we would like to generalize and improve the above result in [1]. We use the mountain pass theorem to study the existence of a nontrivial solution and use the symmetric mountain pass theorem to study the multiplicity of nontrivial solutions for system (1.1) on a finite graph, where the nonlinear term F satisfies the super-(p,q)-linear growth condition. Our work is also inspired by Luo and Zhang [8], who considered the following nonlinear p-Laplacian difference system:

$$\Delta\big(\phi_p\big(\Delta u(n-1)\big)\big) - a(n)\big|u(n)\big|^{p-2}u(n) + \nabla F\big(n,u(n)\big) = 0, \quad n \in \mathbb{Z}, \tag{1.7}$$

where $p \geq 2$, $\phi_p(s) = |s|^{p-2}s$, $\Delta u(n) = u(n+1) - u(n)$, $F(n,x)$ is continuously differentiable in x for all $n \in \{1,\dots,M\}$, and $M > 1$ is a positive integer. By the linking theorem in [9] they obtained that the system has at least one nonconstant periodic solution when F satisfies super-p-linear growth condition.

Notations $h_{i,\min} := \min_{x\in V} h_i(x)$, $i = 1,2$; $h_{\min} := \min_{x\in V} h(x)$; $\mu_{\min} := \min_{x\in V}\mu(x)$, where $\mu : V \to \mathbb{R}^+$ is a finite measure; $W := W^{m_1,p}(V) \times W^{m_2,q}(V)$ with the norm $\|(u,v)\| = \|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)}$ defined in Sect. 2.

Next, we state our main results.

Theorem 1.1 *Assume that F satisfies the following conditions*:

(F_1) $F(x,0,0) = 0$, *and* $F(x,t,s)$ *is continuously differentiable in* $(t,s) \in \mathbb{R}^2$ *for all* $x \in V$;

(F_2) $\lim_{|(t,s)|\to 0}\frac{F(x,t,s)}{|t|^p+|s|^q} < \min\{\frac{1}{pK_1^p},\frac{1}{qK_2^q}\}$ *for all* $x\in V$, *where*

$$K_1 = \frac{(\sum_{x\in V}\mu(x))^{\frac{1}{p}}}{\mu_{\min}^{\frac{1}{p}}h_{1,\min}^{\frac{1}{p}}}, \qquad K_2 = \frac{(\sum_{x\in V}\mu(x))^{\frac{1}{q}}}{\mu_{\min}^{\frac{1}{q}}h_{2,\min}^{\frac{1}{q}}};$$

(F_3) $\lim_{|(t,s)|\to \infty}\frac{F(x,t,s)}{|t|^p+|s|^q} = +\infty$ *for all* $x\in V$;

(F_4) *there are constants* $\gamma_1>0$ *and* $\gamma_2>0$ *such that*

$$\liminf_{|(t,s)|\to\infty}\frac{F_t(x,t,s)t+F_s(x,t,s)s-\max\{p,q\}F(x,t,s)}{|t|^{\gamma_1}+|s|^{\gamma_2}} > 0 \quad \textit{for all } x\in V,$$

where $F_t(x,t,s)=\frac{\partial F(x,t,s)}{\partial t}$ *and* $F_s(x,t,s)=\frac{\partial F(x,t,s)}{\partial s}$. *Then system* (1.1) *has at least one nontrivial solution.*

Theorem 1.2 *Assume that* (F_1)–(F_4) *and the following condition hold*:

(F_5) $F(x,-t,-s)=F(x,t,s)$ *for* $(x,t,s)\in V\times\mathbb{R}^2$.

Then system (1.1) *has at least* $\dim W$ *nontrivial solutions.*

Remark 1.1 In Theorems 1.1 and 1.2, we do not eliminate the case of seminontrivial solutions. Hence, in Theorems 1.1 and 1.2 the solutions have three possibilities: $(u_*,v_*)=(0,v_*)$, $(u_*,v_*)=(u_*,0)$, or $(u_*,v_*)\neq(0,0)$.

From Theorems 1.1 and 1.2 we easily obtain the following results corresponding to (1.3).

Theorem 1.3 *Assume that* F *satisfies the following conditions*:

(F_1') $F(x,0)=0$, *and* $F(x,t)$ *is continuously differentiable in* $t\in\mathbb{R}$ *for all* $x\in V$;

(F_2') $\lim_{|t|\to 0}\frac{F(x,t)}{|t|^p} < \frac{1}{pK^p}$ *for all* $x\in V$, *where* $K=\frac{(\sum_{x\in V}\mu(x))^{\frac{1}{p}}}{\mu_{\min}^{\frac{1}{p}}h_{\min}^{\frac{1}{p}}}$;

(F_3') $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^p}=+\infty$ *for all* $x\in V$;

(F_4') *there exists a constant* $\gamma>0$ *such that*

$$\liminf_{|t|\to\infty}\frac{F_t(x,t)t-pF(x,t)}{|t|^{\gamma}} > 0 \quad \textit{for all } x\in V,$$

where $F_t(x,t)=\frac{\partial F(x,t)}{\partial t}$. *Then equation* (1.3) *has at least one nontrivial solution.*

Theorem 1.4 *Assume that* (F_1')–(F_4') *and the following condition hold*:

(F_5') $F(x,-t)=F(x,t)$ *for* $(x,t)\in V\times\mathbb{R}$.

Then equation (1.3) *has at least* $\dim W^{m,p}(V)$ *nontrivial solutions.*

Remark 1.2 There are examples satisfying the conditions of Theorem 1.1, for example,

$$F(x,t,s)=\ln\bigl(1+|t|^p\bigr)|t|^{\max\{p,q\}}+\ln\bigl(1+|s|^q\bigr)|s|^{\max\{p,q\}}.$$

Remark 1.3 It is not difficult to verify that (V_3) implies (F_3') and (F_4'). There exist examples satisfying the conditions of Theorem 1.3 but not satisfying (V_1)–(V_3), for example, $F(x,t)=\ln(1+|t|^p)|t|^p$ for $x\in V$.

Remark 1.4 In some sense, (F_1')–(F_4') can be seen as a generalization of the assumptions in [8], where the difference equation (1.7) is studied, defined on the set $\mathbb{Z}$ of integers. However, in this paper, we study the high-order Yamabe-type coupled system involving the poly-Laplacian on a finite graph. Hence we generalize those conditions in [8] from $m = 1$ to $m \geq 2$ and from $n \in \mathbb{Z}$ to $x \in V$, which is a finite graph. Moreover, we also present the multiplicity results, that is, Theorems 1.2 and 1.4, which are not considered in [1].

2 Preliminaries

In this section, we state some useful properties of poly-Laplacian and Sobolev spaces on graphs. For details, we refer to [1].

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. For any edge $xy \in E$ with two vertexes of $x, y \in V$, assume that its weight $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$. For any $x \in V$, its degree is defined as $\deg(x) = \sum_{y\sim x} \omega_{xy}$, where we write $y \sim x$ if $xy \in E$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. Define

$$\Delta\psi(x) = \frac{1}{\mu(x)} \sum_{y\sim x} w_{xy}\big(\psi(y) - \psi(x)\big). \tag{2.1}$$

The corresponding gradient form is

$$\Gamma(\psi_1, \psi_2)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}\big(\psi_1(y) - \psi_1(x)\big)\big(\psi_2(y) - \psi_2(x)\big). \tag{2.2}$$

Write $\Gamma(\psi) = \Gamma(\psi, \psi)$. The length of the gradient is defined by

$$|\nabla\psi|(x) = \sqrt{\Gamma(\psi)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}\big(\psi(y) - \psi(x)\big)^2\right)^{\frac{1}{2}}. \tag{2.3}$$

Similarly to the case in Euclidean space, we use $|\nabla^m\psi|$ to represent the length of the mth-order gradient of ψ defined by

$$\big|\nabla^m\psi\big| = \begin{cases} |\nabla\Delta^{\frac{m-1}{2}}\psi| & \text{when } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}}\psi| & \text{when } m \text{ is even,} \end{cases} \tag{2.4}$$

where $|\nabla\Delta^{\frac{m-1}{2}}\psi|$ is defined as in (2.3) with ψ replaced by $\Delta^{\frac{m-1}{2}}\psi$, and $|\Delta^{\frac{m}{2}}\psi|$ denotes the absolute value of the function $\Delta^{\frac{m}{2}}\psi$. For any function $\psi : V \to \mathbb{R}$, we denote

$$\int_V \psi(x)\,d\mu = \sum_{x\in V} \mu(x)\psi(x) \tag{2.5}$$

and $|V| = \sum_{x\in V} \mu(x)$.

When $p \geq 2$, we define the p-Laplacian operator by $\Delta_p\psi$ by

$$\Delta_p\psi(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \big(|\nabla\psi|^{p-2}(y) + |\nabla\psi|^{p-2}(x)\big)\omega_{xy}\big(\psi(y) - \psi(x)\big). \tag{2.6}$$

In the distributional sense, $\Delta_p\psi$ can be written as follows. For any $\phi \in \mathcal{C}_c(V)$,

$$\int_V (\Delta_p\psi)\phi \, d\mu = -\int_V |\nabla\psi|^{p-2}\Gamma(\psi,\phi)\, d\mu, \tag{2.7}$$

where $\mathcal{C}_c(V)$ is the set of all real functions with compact support. It is easy to see that $\pounds_{m,p}$ defined by (1.2) is a generalization of $\Delta_p\psi$.

Define the space

$$W^{m,p}(V) = \left\{\psi : V \to \mathbb{R} \,\middle|\, \int_V \left(\left|\nabla^m\psi(x)\right|^p + h(x)\left|\psi(x)\right|^p\right) d\mu < \infty\right\}$$

endowed with the norm

$$\|\psi\|_{W^{m,p}(V)} = \left(\int_V \left(\left|\nabla^m\psi(x)\right|^p + h(x)\left|\psi(x)\right|^p\right) d\mu\right)^{\frac{1}{p}}, \tag{2.8}$$

where $m \geq 2$, $p > 1$, and $h(x) > 0$ for all $x \in V$. Then $W^{m,p}(V)$ is a Banach space of finite dimension. Let $1 < r < +\infty$. Define

$$L^r(V) = \left\{\psi : V \to \mathbb{R} \,\middle|\, \int_V \left|\psi(x)\right|^r d\mu < \infty\right\}$$

with the norm

$$\|\psi\|_{L^r(V)} = \left(\int_V \left|\psi(x)\right|^r d\mu\right)^{\frac{1}{r}}. \tag{2.9}$$

Let X be a Banach space, and let $\varphi \in C^1(X,\mathbb{R})$. We say that the functional φ satisfies the Palais–Smale (PS) condition if $\{u_n\}$ has a convergent subsequence in X whenever $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$. We call that φ satisfies the Cerami (C) condition if $\{u_n\}$ has a convergent subsequence in X whenever $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\| \times (1 + \|u_n\|) \to 0$.

Lemma 2.1 (Mountain pass theorem [10]) *Let X be a real Banach space, and let $\varphi \in C^1(X,\mathbb{R})$, $\varphi(0) = 0$ satisfy the (PS)-condition. Suppose that φ satisfies the following conditions*:

(i) *there exists a constant $\rho > 0$ such that $\varphi|_{\partial B_\rho(0)} > 0$, where $B_\rho = \{w \in X : \|w\|_X < \rho\}$*;
(ii) *there exists $w \in X\backslash\bar{B}_\rho(0)$ such that $\varphi(w) \leq 0$.*

Then φ has a critical value c with

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \varphi\bigl(\gamma(t)\bigr),$$

where

$$\Gamma := \bigl\{\gamma \in C\bigl([0,1],X\bigr) : \gamma(0) = 0, \gamma(1) = w\bigr\}.$$

Lemma 2.2 (Symmetric mountain pass theorem [10]) *Let X be an infinite-dimensional Banach space, let $X = Y \oplus Z$, where Y is finite-dimensional, and let $\varphi \in C^1(X,\mathbb{R})$, $\varphi(0) = 0$, satisfy the (PS)-condition. Suppose that φ satisfies the following conditions*:

(i) $\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ *for all* $u \in X$;
(ii) *there exists a constant* $\rho, \alpha > 0$, *such that* $\varphi|_{\partial B_\rho(0)\cap Z} \geq \alpha$;
(iii) *for any finite-dimensional subspace* $\widetilde{X} \subset X$, *there is* $R = R(\widetilde{X}) > 0$ *such that* $\varphi(u) \leq 0$ *on* $\widetilde{X} \backslash B_R(0)$.

Then φ *possesses an unbounded sequence of critical values.*

Remark 2.1 As shown in [11], the deformation lemma can be proved with the weaker (C)-condition instead of the (PS)-condition, so that Lemmas 2.1 and 2.2 also hold under the (C)-condition.

Remark 2.2 If X is finite-dimensional, the result of Lemma 2.2 can also be obtained with the conclusion that φ possesses at least dim Z critical values (see [10], Remark 9.36).

Lemma 2.3 *Let* $p > 1$. *For all* $\psi \in W^{m,p}(V)$, *we have*

$$\|\psi\|_\infty \leq d\|\psi\|_{W^{m,p}(V)},$$

where $\|\psi\|_\infty = \max_{x\in V} |\psi(x)|$ *and* $d = (\frac{1}{\mu_{\min} h_{\min}})^{\frac{1}{p}}$.

Proof Indeed,

$$\begin{aligned}
\|\psi\|^p_{W^{m,p}(V)} &= \int_V \left(\left|\nabla^m \psi\right|^p + h(x)\left|\psi(x)\right|^p\right) d\mu \\
&= \sum_{x\in V} \mu(x)\left(\left|\nabla^m \psi\right|^p + h(x)\left|\psi(x)\right|^p\right) \\
&\geq \sum_{x\in V} \mu(x)h(x)\left|\psi(x)\right|^p \\
&\geq \mu_{\min} h_{\min} \sum_{x\in V}\left|\psi(x)\right|^p \\
&\geq \mu_{\min} h_{\min} \|\psi\|^p_\infty.
\end{aligned}$$

□

Lemma 2.4 *Let* $G = (V, E)$ *be a finite graph. Let* m *be any positive integer, and let* $q > 1$. *Then* $W^{m,p}(V) \hookrightarrow L^q(V)$ *for all* $1 \leq q \leq +\infty$. *In particular, if* $1 < q < +\infty$, *then for all* $\psi \in W^{m,p}(V)$,

$$\|\psi\|_{L^q(V)} \leq K\|\psi\|_{W^{m,p}(V)}, \tag{2.10}$$

where

$$K = \frac{(\sum_{x\in V} \mu(x))^{\frac{1}{q}}}{\mu_{\min}^{\frac{1}{p}} h_{\min}^{\frac{1}{p}}}.$$

In addition, $W^{m,p}(V)$ *is precompact, that is, if* $\{\psi_k\}$ *is bounded in* $W^{m,p}(V)$, *then up to a subsequence, there exists* $\psi \in W^{m,p}(V)$ *such that* $\psi_k \to \psi$ *in* $W^{m,p}(V)$.

Proof Note that V is a bounded set. Then $W^{m,p}(V)$ is a finite-dimensional space. Hence it is precompact. According to Lemma 2.3, we have

$$\begin{aligned}\|\psi\|^p_{L^q(V)} &= \int_V |\psi|^q \, d\mu \\ &= \sum_{x\in V} \mu(x)\bigl|\psi(x)\bigr|^q \\ &\le \sum_{x\in V} \mu(x)\|\psi\|^q_\infty \\ &\le \frac{\sum_{x\in V}\mu(x)}{\mu_{\min}^{\frac{q}{p}} h_{\min}^{\frac{q}{p}}}\|\psi\|^q_{W^{m,p}(V)}. \end{aligned}$$

□

Remark 2.3 The proofs of Lemmas 2.3 and 2.4 are given in [1]. However, the precise values of d and K are not given. In Lemmas 2.3 and 2.4, we specify their values.

3 Proofs of main results

Note that the space $W := W^{m_1,p}(V) \times W^{m_2,q}(V)$ with the norm $\|(u,v)\| = \|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)}$ is a finite-dimensional Banach space. Consider the functional $\varphi : W \to \mathbb{R}$ defined as

$$\begin{aligned}\varphi(u,v) = {}& \frac{1}{p}\int_V \bigl(\bigl|\nabla^{m_1}u\bigr|^p + h_1(x)|u|^p\bigr)\, d\mu + \frac{1}{q}\int_V \bigl(\bigl|\nabla^{m_2}v\bigr|^q + h_2(x)|v|^q\bigr)\, d\mu \\ & - \int_V F(x,u,v)\, d\mu. \end{aligned} \tag{3.1}$$

Then $\varphi \in C^1(W,\mathbb{R})$, and

$$\begin{aligned}\bigl\langle \varphi'(u,v),(\phi_1,\phi_2)\bigr\rangle = {}& \int_V \bigl[(£_{m_1,p}u,\phi_1) + \bigl(h_1(x)|u|^{p-2}u,\phi_1\bigr) - \bigl(F_u(x,u,v),\phi_1\bigr)\bigr]\, d\mu \\ & + \int_V \bigl[(£_{m_2,q}v,\phi_2) + \bigl(h_2(x)|v|^{q-2}v,\phi_2\bigr) - \bigl(F_v(x,u,v),\phi_2\bigr)\bigr]\, d\mu \end{aligned} \tag{3.2}$$

for all $(u,v),(\phi_1,\phi_2) \in W$. Then $(u,v) \in W$ is a critical point of φ if and only if

$$\int_V \bigl(\bigl(£_{m_1,p}u + h_1(x)|u|^{p-2}u - F_u(x,u,v)\bigr),\phi_1\bigr)\, d\mu = 0$$

and

$$\int_V \bigl(\bigl(£_{m_2,q}v + h_2(x)|v|^{q-2}v - F_v(x,u,v)\bigr),\phi_2\bigr)\, d\mu = 0.$$

By the arbitrariness of ϕ_1 and ϕ_2 we conclude that

$$\begin{aligned} & £_{m_1,p}u + h_1(x)|u|^{p-2}u = F_u(x,u,v), \\ & £_{m_2,q}v + h_2(x)|v|^{q-2}v = F_v(x,u,v). \end{aligned}$$

Thus the problem of finding the solutions of system (1.1) is reduced to finding the critical points of the functional φ on W.

Lemma 3.1 *Assume that* (F_4) *holds. Then the functional* φ *satisfies condition* (C), *that is,* $\{(u_k, v_k)\}$ *has a convergent subsequence in* W *whenever* $\varphi(u_k, v_k)$ *is bounded and* $\|\varphi'(u_k, v_k)\| \times (1 + \|(u_k, v_k)\|) \to 0$ *as* $k \to \infty$.

Proof Let $\{(u_k, v_k)\}$ be a sequence in W such that $\varphi(u_k, v_k)$ is bounded and $\|\varphi'(u_k, v_k)\|(1 + \|(u_k, v_k)\|) \to 0$ as $k \to \infty$. Then there exists a positive constant L such that

$$\left|\varphi(u_k, v_k)\right| \le L, \left\|\varphi'(u_k, v_k)\right\| \left(1 + \left\|(u_k, v_k)\right\|\right) \le L$$

for every $k \in \mathbb{N}$. By (F_4),there are constants $C_1 > 0$ and $\delta_1 > 0$ such that

$$F_t(x, t, s)t + F_s(x, t, s)s - \max\{p, q\}F(x, t, s) \ge C_1\left(|t|^{\gamma_1} + |s|^{\gamma_2}\right) > 0$$

for all $|(t, s)| > \delta_1$ and $x \in V$. Therefore

$$F_t(x, t, s)t + F_s(x, t, s)s - \max\{p, q\}F(x, t, s) \ge C_1\left(|t|^{\gamma_1} + |s|^{\gamma_2}\right) - C_2$$

for all $(t, s) \in \mathbb{R}^2$ and $x \in V$, where

$$\begin{aligned} C_2 = {} & C_1 \max\left\{|t|^{\gamma_1} + |s|^{\gamma_2} \mid \left|(t, s)\right| \le \delta_1\right\} \\ & + \max\left\{F_t(x, t, s)t + F_s(x, t, s)s - \max\{p, q\}F(x, t, s) \mid \left|(t, s)\right| \le \delta_1\right\}. \end{aligned}$$

Then for all large k, we have

$$\begin{aligned} & \left(\max\{p, q\} + 1\right)L \\ & \quad \ge \max\{p, q\}\varphi(u_k, v_k) - \left(\varphi'(u_k, v_k), (u_k, v_k)\right) \\ & \quad = \max\{p, q\}\left[\frac{1}{p}\int_V \left(\left|\nabla^{m_1} u_k\right|^p + h_1(x)|u_k|^p\right) d\mu \right. \\ & \qquad \left. + \frac{1}{q}\int_V \left(\left|\nabla^{m_2} v_k\right|^q + h_2(x)|v_k|^q\right) d\mu - \int_V F(x, u_k, v_k)\, d\mu\right] \\ & \qquad - \int_V (£_{m_1,p} u_k, u_k)\, d\mu - \int_V h_1(x)|u_k|^p\, d\mu - \int_V (£_{m_2,q} v_k, v_k)\, d\mu \\ & \qquad - \int_V h_2(x)|v_k|^p\, d\mu + \int_V F_{u_k}(x, u_k, v_k)u_k\, d\mu + \int_V F_{v_k}(x, u_k, v_k)v_k\, d\mu. \end{aligned} \tag{3.3}$$

When $\max\{p, q\} = p$,

$$\begin{aligned} (p + 1)L \ge {} & \left(\frac{p}{q} - 1\right)\int_V \left(\left|\nabla^{m_2} v_k\right|^q + h_2(x)|v_k|^q\right) d\mu \\ & + \int_V \left[\left(F_{u_k}(x, u_k, v_k), u_k\right) + \left(F_{v_k}(x, u_k, v_k), v_k\right) - pF(x, u_k, v_k)\right] d\mu \\ \ge {} & \left(\frac{p}{q} - 1\right)\int_V \left(\left|\nabla^{m_2} v_k\right|^q + h_2(x)|v_k|^q\right) d\mu \end{aligned}$$

$$
\begin{aligned}
&+ \int_V C_1\big(|u_k|^{\gamma_1} + |v_k|^{\gamma_2}\big)\,d\mu - C_2\sum_{x\in V}\mu(x) \\
&= \left(\frac{p}{q} - 1\right)\|v_k\|^q_{W^{m_2,q}(V)} + C_1\int_V \big(|u_k|^{\gamma_1} + |v_k|^{\gamma_2}\big)\,d\mu - C_2\sum_{x\in V}\mu(x).
\end{aligned}
$$

Therefore $\|v_k\|_{W^{m_2,q}(V)}$, $\|u_k\|_{L^{\gamma_1}(V)}$, and $\|v_k\|_{L^{\gamma_2}(V)}$ are bounded. Since $(W, \|\cdot\|)$ is a finite-dimensional space, there exist positive constants D_1 and D_2 such that

$$
\|u_k\|_{W^{m_1,p}(V)} \le D_1\|u_k\|_{L^{\gamma_1}(V)}, \qquad \|v_k\|_{W^{m_2,q}(V)} \le D_2\|v_k\|_{L^{\gamma_2}(V)}. \tag{3.4}
$$

Thus $\|u_k\|_{W^{m_1,p}(V)}$ and $\|v_k\|_{W^{m_2,q}(V)}$ are bounded. So $\{(u_k, v_k)\}$ is bounded in W. Similarly, when $\max\{p,q\} = q$, we can also prove that $\{(u_k, v_k)\}$ is bounded in W. To sum up, $\{(u_k, v_k)\}$ is bounded in W. Since W is of finite dimension, there is a convergent subsequence of $\{(u_k, v_k)\}$. Hence φ satisfies the (C)-condition. □

Lemma 3.2 *There exists a constant $\rho > 0$ such that $\varphi|_{\partial B_\rho(0)} > 0$, where $B_\rho = \{(u,v) \in W : \|(u,v)\|_W < \rho\}$.*

Proof By (F_2) there are $0 < C_4 < \min\{\frac{1}{pK_1^p}, \frac{1}{qK_2^q}\}$ and a positive constant $\delta_2 < C_3$, where $C_3 = \max\{\frac{1}{\mu_{\min}h_{1,\min}}, \frac{1}{\mu_{\min}h_{2,\min}}\}$, such that

$$
\big|F(x,t,s)\big| \le C_4\big(|t|^p + |s|^q\big) \tag{3.5}
$$

for all $|(t,s)| \le \delta_2$. By Lemma 2.4 we have

$$
\|u\|_{L^p(V)} \le K_1\|u\|_{W^{m_1,p}(V)}, \qquad \|v\|_{L^q(V)} \le K_2\|v\|_{W^{m_2,q}(V)}, \tag{3.6}
$$

where K_1, K_2 is defined in (F_2). For every $(u,v) \in W$ with $\|(u,v)\| = \rho = \delta_2 C_3^{-1} < 1$, by Lemma 2.3 we have

$$
\big\|(u,v)\big\|_\infty \le \|u\|_\infty + \|v\|_\infty \le C_3\big(\|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)}\big) = \delta_2.
$$

Then by (3.5) and (3.6), for all $(u,v) \in W$ with $\|(u,v)\| = \rho$, we have

$$
\begin{aligned}
&\varphi(u,v) \\
&\quad= \frac{1}{p}\int_V \big(\big|\nabla^{m_1}u\big|^p + h_1(x)|u|^p\big)\,d\mu + \frac{1}{q}\int_V \big(\big|\nabla^{m_2}v\big|^q + h_2(x)|v|^q\big)\,d\mu - \int_V F(x,u,v)\,d\mu \\
&\quad\ge \frac{1}{p}\int_V \big(\big|\nabla^{m_1}u\big|^p + h_1(x)|u|^p\big)\,d\mu + \frac{1}{q}\int_V \big(\big|\nabla^{m_2}v\big|^q + h_2(x)|v|^q\big)\,d\mu \\
&\qquad - C_4\int_V \big(|u|^p + |v|^q\big)\,d\mu \\
&\quad\ge \left(\frac{1}{p} - K_1^p C_4\right)\int_V \big(\big|\nabla^{m_1}u\big|^p + h_1(x)|u|^p\big)\,d\mu \\
&\qquad + \left(\frac{1}{q} - K_2^q C_4\right)\int_V \big(\big|\nabla^{m_2}v\big|^q + h_2(x)|v|^q\big)\,d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{p} - K_1^p C_4\right)\|u\|^p_{W^{m_1,p}(V)} + \left(\frac{1}{q} - K_2^q C_4\right)\|v\|^q_{W^{m_2,q}(V)} \\
&\geq \min\left\{\left(\frac{1}{p} - K_1^p C_4\right), \left(\frac{1}{q} - K_2^q C_4\right)\right\} \cdot \begin{cases} \frac{1}{2^{p-1}}(\|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)})^p & \text{if } p \geq q, \\ \frac{1}{2^{q-1}}(\|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)})^q & \text{if } p < q \end{cases} \\
&\geq \min\left\{\left(\frac{1}{p} - K_1^p C_4\right), \left(\frac{1}{q} - K_2^q C_4\right)\right\} \cdot \begin{cases} \frac{\rho^p}{2^{p-1}} & \text{if } p \geq q, \\ \frac{\rho^q}{2^{q-1}} & \text{if } p < q \end{cases} \\
&:= \alpha > 0.
\end{aligned}
$$

The proof is completed. □

Lemma 3.3 *Assume that* (F_1) *and* (F_3) *hold. Then there exists* $(u_0, v_0) \in W \backslash \bar{B}_\rho(0)$ *such that* $\varphi(u_0, v_0) \leq 0$.

Proof Choose $e = (e_1, e_2) \in W$ such that $\|e_1\|_{L^p(V)} \neq 0$ and $\|e_2\|_{L^q(V)} \neq 0$. By (F_3) there exist $\varepsilon_1 > 0$ and $\delta_3 > 0$ such that

$$
F(x,t,s) \geq \left(\frac{1}{p}\frac{\|e_1\|^p_{W^{m_1,p}(V)}}{\|e_1\|^p_{L^p(V)}} + \frac{1}{q}\frac{\|e_2\|^q_{W^{m_2,q}(V)}}{\|e_2\|^q_{L^q(V)}} + \frac{\varepsilon_1}{2}\right)\left(|t|^p + |s|^q\right)
$$

for all $|(t,s)| > \delta_3$ and $x \in V$. Thus by (F_1) there exists $C_5 > 0$ such that for all $(t,s) \in \mathbb{R}^2$ and all $x \in V$,

$$
F(x,t,s) \geq \left(\frac{1}{p}\frac{\|e_1\|^p_{W^{m_1,p}(V)}}{\|e_1\|^p_{L^p(V)}} + \frac{1}{q}\frac{\|e_2\|^q_{W^{m_2,q}(V)}}{\|e_2\|^q_{L^q(V)}} + \frac{\varepsilon_1}{2}\right)\left(|t|^p + |s|^q\right) - C_5.
$$

Then for every $\lambda > 0$, we have

$$
\begin{aligned}
\varphi(\lambda e_1, \lambda e_2) \ &= \ \frac{1}{p}\int_V \left(\left|\nabla^{m_1}\lambda e_1\right|^p + h_1(x)|\lambda e_1|^p\right) d\mu + \frac{1}{q}\int_V \left(\left|\nabla^{m_2}\lambda e_2\right|^q + h_2(x)|\lambda e_2|^q\right) d\mu \\
&\quad - \int_V F(x, \lambda e_1, \lambda e_2) \\
&\leq \ \frac{1}{p}\lambda^p\|e_1\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\lambda^q\|e_2\|^q_{W^{m_2,q}(V)} \\
&\quad - \left(\frac{1}{p}\frac{\|e_1\|^p_{W^{m_1,p}(V)}}{\|e_1\|^p_{L^p(V)}} + \frac{1}{q}\frac{\|e_2\|^q_{W^{m_2,q}(V)}}{\|e_2\|^q_{L^q(V)}} + \frac{\varepsilon_1}{2}\right)\left(\lambda^p\|e_1\|^p_{L^p(V)} + \lambda^q\|e_2\|^q_{L^q(V)}\right) \\
&\quad + C_5\sum_{x\in V}\mu(x) \\
&\leq \ -\frac{\varepsilon_1}{2}\lambda^p\|e_1\|^p_{L^p(V)}\, d\mu - \frac{\varepsilon_1}{2}\lambda^q\|e_2\|^q_{L^q(V)} + C_5\sum_{x\in V}\mu(x) \\
&\to -\infty, \quad \text{as } \lambda \to \infty.
\end{aligned}
$$

Hence there exists a sufficiently large $\lambda^* > 1$ such that $\varphi(\lambda^* e_1, \lambda^* e_2) < 0$. Let $\lambda^* e_1 = u_0$ and $\lambda^* e_2 = v_0$. Then $\varphi(u_0, v_0) \leq 0$. □

Proof of Theorem 1.1 It is easy to see that $\varphi(0,0)=0$. It follows from Lemmas 2.1 and 3.1–3.3, φ possesses a critical value $c\geq\alpha>0$, that is, there exists a point $(u_*,v_*)\in W$ such that

$$\varphi(u_*,v_*)=c \quad \text{and} \quad \varphi'(u_*,v_*)=0.$$

Hence the associated point $(u_*,v_*)\in W$ is a nontrivial weak solution of system (1.1). □

Lemma 3.4 *Assume that* (F_1) *and* (F_3) *hold. Then for any finite-dimensional subspace* $\widetilde{X}\subset W$, *there is* $R=R(\widetilde{X})>0$ *such that* $\varphi(u)\leq 0$ *on* $\widetilde{X}\backslash B_R(0)$.

Proof Let $\dim\widetilde{X}=m$. Then there exist positive constants $C_6(m)$ and $C_7(m)$ such that

$$\|u\|_{W^{m_1,p}(V)}\leq C_6(m)\|u\|_{L^p(V)}, \qquad \|v\|_{W^{m_2,q}(V)}\leq C_7(m)\|v\|_{L^q(V)} \tag{3.7}$$

for all $(u,v)\in\widetilde{X}$. By (F_3) we know that there exist constants $\beta>\frac{C_6(m)^p}{p}+\frac{C_7(m)^q}{q}$ and $r>0$ such that

$$F(x,t,s)\geq\beta\bigl(|t|^p+|s|^q\bigr) \quad \text{for all } \bigl|(t,s)\bigr|\geq r \text{ and } x\in V. \tag{3.8}$$

It follows from (F_1) and (3.8) that there exists $C_8>0$ such that

$$F(x,t,s)\geq\beta\bigl(|t|^p+|s|^q\bigr)-C_8 \quad \text{for all } (t,s)\in\mathbb{R}^2 \text{ and } x\in V. \tag{3.9}$$

Then by (3.7) and (3.9) we have

$$\begin{aligned}
&\varphi(u,v)\\
&\quad=\frac{1}{p}\int_V\bigl(\bigl|\nabla^{m_1}u\bigr|^p+h_1(x)|u|^p\bigr)\,d\mu+\frac{1}{q}\int_V\bigl(\bigl|\nabla^{m_2}v\bigr|^q+h_2(x)|v|^q\bigr)\,d\mu-\int_V F(x,u,v)\,d\mu\\
&\quad\leq\frac{1}{p}\|u\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v\|^q_{W^{m_2,q}(V)}-\beta\bigl(\|u\|^p_{L^p}+\|v\|^q_{L^q}\bigr)+C_8\sum_{x\in V}\mu(x)\\
&\quad\leq\frac{1}{p}\|u\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v\|^q_{W^{m_2,q}(V)}-\beta\biggl(\frac{1}{C_6^p(m)}\|u\|^p_{W^{m_1,p}(V)}+\frac{1}{C_7^q(m)}\|v\|^q_{W^{m_2,q}(V)}\biggr)\\
&\qquad+C_8\sum_{x\in V}\mu(x),
\end{aligned}$$

for all $(u,v)\in\widetilde{X}$. Note that $\beta>\frac{C_6(m)^p}{p}+\frac{C_7(m)^q}{q}$. So $\varphi(u,v)\to-\infty$ as $\|(u,v)\|\to\infty$. Thus we complete the proof. □

Proof of Theorem 1.2 By (F_1) and (F_5) we know that φ is even and $\varphi(0,0)=0$. Let $X=W$, $Y=\{0\}$ and $Z=W$. Then by Lemma 3.1, Lemma 3.2, Lemma 3.4, Remark 2.1, Remark 2.2, and Lemma 2.2 we obtain that φ possesses at least $\dim W$ critical values. Thus we complete the proof. □

Acknowledgements
Not applicable.

Funding
This project is supported by Yunnan Ten Thousand Talents Plan Young & Elite Talents Project and Candidate Talents Training Fund of Yunnan Province, China (No: 2017HB016).

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XZ and XZ wrote the main manuscript text, and JX and XY participated in the proofs of theorems. All authors read and approved the final manuscript.

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Publisher's Note

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Received: 18 March 2022 Accepted: 22 April 2022 Published online: 10 May 2022

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