Boundary Value Problems
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Boundary value problems for second-order causal differential equations

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Abstract

This paper focuses on second-order differential equations involving causal operators with nonlinear two-point boundary conditions. By applying the monotone iterative technique in the presence of upper and lower solutions, with a new comparison theorem, we obtain the existence of extremal solutions. This is an extension of classical theory of second-order differential equations. Finally, we present two examples to show the usefulness of our results.

Keywords: Causal differential equations; Two-point boundary conditions; Monotone iterative technique; Extremal solutions

1 Introduction

The theory of boundary value problem (BVP) for differential equations has highly significant applications in applied science and engineering, for example, many practical problems in the fields of engineering, mechanics, astronomy, economics, and biology are usually attributed to solving boundary value problems. Readers can refer to [2, 11–13] for details. Along this line, it is of great significance in mathematical theory and practical applications to find solutions to differential equations with boundary conditions. In recent years, driven by the theory of functional analysis and practical problems, the BVP for second-order differential equations has developed rapidly, many authors have made a profound study on this subject [1, 3, 10] and obtained many new research methods. One of them is the monotone iterative technique. It is worthwhile mentioning that this technique and the method of upper and lower solutions is an effective way to demonstrate existence results of nonlinear BVPs. There is an extensive literature on the applications of this method in differential equations; for details, see [8, 14–16]. This paper develops the monotone iterative technique to discuss the following second-order causal differential equation:

$$\begin{cases} -u''(\mathfrak{s}) = (\mathcal{Q}u)(\mathfrak{s}), & \mathfrak{s} \in J = [0,1], \\ \mathcal{P}u(\theta) = \chi_\theta, & \theta = 0,1, \end{cases} \tag{1}$$

where $\mathcal{P}u(\theta) = \alpha_\theta u(\theta) + (-1)^{\theta+1}\beta_\theta u'(\theta) = \chi_\theta$, $\alpha_0, \alpha_1 \geq 0$, $\beta_0, \beta_1 > 0$, $\chi_\theta \in \mathbb{R}$, $E = C([0,1],\mathbb{R})$, $\mathcal{Q} \in C(E,E)$, and $\mathcal{Q}$ is a causal operator.

A causal operator is adopted from the engineering literature, it was first used by Volterra implicitly in his work on integral equations. For more information, see [9]. In recent years, causal differential equations have been studied widely [5, 6, 17–19]. Compared with the traditional model, the framework of causal differential equations seems to be an excellent model for the real-world problems, and has a wider range of real-time applications in many disciplines. However, there is not so much work on second-order causal differential equations [4, 7]. This paper extends the notion of casual operators to second-order differential equations with nonlinear boundary conditions. Now, we will provide some sufficient conditions for the existence results of problem (1) and present two illustrative examples.

The paper is organized as follows: Sect. 2 presents a new comparison theorem while Sect. 3 proves the existence and uniqueness of solutions to second-order linear causal differential equations. Then, based on the monotone iterative technique, the existence of extremal solutions to (1) is obtained. Finally, in the last section, we provide two illustrative examples.

2 A new comparison theorem

Definition 2.1 An operator $\mathscr{Q} \in C(E,E)$ is called causal if $\mu(x) = \nu(x)$, $0 \le x \le y \le N$, N being arbitrary, where $\mu, \nu \in E$, and

$$(\mathscr{Q}\mu)(x) = (\mathscr{Q}\nu)(x), \quad 0 \le x \le y.$$

Lemma 2.1 *Let* $\lambda \in C^2([0,1],\mathbb{R})$ *satisfy*

$$\begin{cases} \lambda''(\mathfrak{s}) \ge \mathscr{X}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s}), & \mathfrak{s} \in J = [0,1], \\ \mathcal{P}\lambda(\theta) \le 0, & \theta = 0,1, \end{cases} \tag{2}$$

where $\mathscr{X} \in C(J,[0,+\infty])$, $\mathscr{X}(\mathfrak{s}) > 0$, $\mathfrak{s} \in (0,1)$, Γ *is a positive linear operator with* $\Gamma \in C(E,E)$, *and*

$$\int_0^1 \left(\int_s^1 \bigl[\mathscr{X}(\mathfrak{s}) + (\Gamma \mathbf{1})(\mathfrak{s}) \bigr] \, d\mathfrak{s} \right) ds \le 1, \qquad \mathbf{1}(\mathfrak{s}) = 1, \quad \mathfrak{s} \in [0,1]. \tag{3}$$

Then $\lambda(\mathfrak{s}) \le 0$ *for* $\mathfrak{s} \in J$.

Proof Assume that $\lambda(\mathfrak{s}) \le 0$, $\mathfrak{s} \in J$ is not true. It means that $\lambda(\mathfrak{s})$ will have a positive maximum at some $\mathfrak{s}_0 \in J$, i.e., $\lambda(\mathfrak{s}_0) = \max_{\mathfrak{s}\in J} \lambda(\mathfrak{s}) = L > 0$.

Case 1: Suppose that $\lambda(\mathfrak{s}) \ge 0$ for all $\mathfrak{s} \in J$.

If $\mathfrak{s}_0 \in (0,1)$, then $\lambda''(\mathfrak{s}_0) \le 0$ and $\lambda'(\mathfrak{s}_0) = 0$. Then (2) implies that $0 \ge \lambda''(\mathfrak{s}_0) \ge \Phi(\mathfrak{s}_0)L > 0$, this is a contradiction. If $\mathfrak{s}_0 = 0$, then we only have $\lambda'(0) = \lim_{t\to 0^+} \frac{\lambda(\mathfrak{s})-\lambda(0)}{\mathfrak{s}} \le 0$. However, using the boundary conditions, we obtain $\alpha_0\lambda(0) - \beta_0\lambda'(0) \le 0$, which implies $0 \le \alpha_0 L = \alpha_0\lambda(0) \le \beta_0\lambda'(0) \le 0$, and since $\beta_0 > 0$, we get $\lambda'(0) = 0$. Using (2), we have $\lambda''(0) \ge \Phi(0)\lambda(0) + (\Gamma\lambda)(0) \ge 0$. That is a contradiction. A similar argument holds if $\mathfrak{s}_0 = 1$.

Case 2: Assume that there is an $\mathfrak{s}_*$ satisfying $\lambda(\mathfrak{s}_*) < 0$.

If $\mathfrak{s}_0 = 0$ or 1, utilizing the boundary conditions as before, a contradiction is also obtained. If $\mathfrak{s}_0 \in (0,1)$, then $\lambda'(\mathfrak{s}_0) = 0$, and we set $\min_{\mathfrak{s}\in J} \lambda(\mathfrak{s}) = -d$, $d > 0$. Without loss of generality, set $\lambda(\mathfrak{s}_*) = -d$.

It yields

$$\lambda''(\mathfrak{s}) \geq \mathscr{L}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s}) \geq \lambda(\mathfrak{s}_*)\big[\mathscr{L}(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\big] = -d\big[\mathscr{L}(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\big].$$

The latter formula is integrated from s to $\mathfrak{s}_0$ to obtain

$$-\lambda'(s) \geq -d\int_s^{\mathfrak{s}_0}\big[\Phi(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\big]\,d\mathfrak{s}.$$

Continuing to integrate the above formula from $\mathfrak{s}_*$ to $\mathfrak{s}_0$, we get

$$-d > \lambda(\mathfrak{s}_*) - z(\mathfrak{s}_0) \geq -d\int_{\mathfrak{s}_*}^{\mathfrak{s}_0}\left(\int_s^{\mathfrak{s}_0}\big[\Phi(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\big]\,d\mathfrak{s}\right)ds.$$

Thus one can get

$$\int_0^1\left(\int_s^1\big[\mathscr{L}(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\big]\,d\mathfrak{s}\right)ds > 1,$$

which is a contradiction due to (3), and the proof is completed. □

3 Extremal solutions

In this part, the existence of extremal solutions to (1) is shown.

The function $\phi \in C^2(J,\mathbb{R})$ is called a lower solution of (1) if

$$-\phi''(\mathfrak{s}) \leq (\mathscr{Q}\phi)(\mathfrak{s}), \qquad \mathcal{P}\phi(\theta) \leq \chi_\theta, \quad \theta = 0, 1,$$

and an upper solution of (1) if the reversed inequalities hold.

To demonstrate the existence of extremal solutions to (1), the corresponding linear problem is considered, given by

$$\begin{cases} -\varphi''(\mathfrak{s}) = -\mathscr{L}(\mathfrak{s})\varphi(\mathfrak{s}) - (\Gamma\varphi)(\mathfrak{s}) + \sigma_\delta(\mathfrak{s}), & \mathfrak{s} \in J, \\ \mathcal{P}\varphi(\theta) = \chi_\theta, \quad \theta = 0, 1, \end{cases} \tag{4}$$

where $\sigma_\delta(\mathfrak{s}) = (\mathscr{Q}\delta)(\mathfrak{s}) + \mathscr{L}(\mathfrak{s})\delta(\mathfrak{s}) + (\Gamma\delta)(\mathfrak{s})$.

Theorem 3.1 *Assume that*

(H_1) (3) *holds with* $\mathscr{L} \in C(J,[0,+\infty])$, $\Phi(\mathfrak{s}) > 0$, $\mathfrak{s} \in (0,1)$, *and* Γ *being a positive linear operator*;

(H_2) $\phi, \psi \in C^2(J,\mathbb{R})$ *are lower and upper solutions of problem* (1), *respectively, and* $\phi \leq \psi$;

(H_3) $\mathscr{Q} \in C(E,E)$ *satisfies*

$$(\mathscr{Q}\bar{g})(\mathfrak{s}) - (\mathscr{Q}g)(\mathfrak{s}) \geq -\mathscr{L}(\mathfrak{s})\big(\bar{g}(\mathfrak{s}) - g(\mathfrak{s})\big) - \big(\Gamma(\bar{g} - g)\big)(\mathfrak{s}),$$

for $\phi(\mathfrak{s}) \leq g(\mathfrak{s}) \leq \bar{g}(\mathfrak{s}) \leq \psi(\mathfrak{s})$, $\mathfrak{s} \in J$;

(H_4) $\delta \in C^2(J,\mathbb{R})$ *and* $\phi(\mathfrak{s}) \leq \delta(\mathfrak{s}) \leq \psi(\mathfrak{s})$, $\mathfrak{s} \in J$.

Then problem (4) *has only one solution in the sector*

$$[\phi,\psi] = \big\{\varphi \in C^2(J,\mathbb{R}) : \phi(\mathfrak{s}) \leq \varphi(\mathfrak{s}) \leq \psi(\mathfrak{s}), \mathfrak{s} \in J\big\}.$$

Proof We need four steps to complete the proof.

Step 1. We rewrite problem (4) in the following way:

$$\varphi(\mathfrak{s}) = \int_0^1 G(\mathfrak{s},s)\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds + \omega(\mathfrak{s}), \quad \mathfrak{s} \in J, \tag{5}$$

where $G(\mathfrak{s},s)$ is the Green's function defined by

$$G(\mathfrak{s},s) = \frac{1}{\frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + 1}\begin{cases} (s + \frac{\beta_0}{\alpha_0})(\mathfrak{s} - 1 - \frac{\beta_1}{\alpha_1}), & 0 \le s \le \mathfrak{s} \le 1, \\ (\mathfrak{s} + \frac{\beta_0}{\alpha_0})(s - 1 - \frac{\beta_1}{\alpha_1}), & 0 \le \mathfrak{s} \le s \le 1, \end{cases}$$

and $\omega(\mathfrak{s}) = k_1\mathfrak{s} + k_2$ is the solution of the associated boundary value problem:

$$\varphi'' = 0, \qquad \mathcal{P}\varphi(\theta) = \chi_\theta, \quad \theta = 0, 1, \tag{6}$$

where $k_1 = \frac{\alpha_0\chi_1 - \alpha_1\chi_0}{\alpha_0(\alpha_1+\beta_1)+\alpha_1\beta_0}$ and $k_2 = \frac{\chi_0(\alpha_1+\beta_1)+\chi_1\beta_0}{\alpha_0(\alpha_1+\beta_1)+\alpha_1\beta_0}$.

Apparently, if $\varphi(\mathfrak{s}) \in C^2(J,\mathbb{R})$ is a solution of (5), we have $\alpha_0\varphi(0) - \beta_0\varphi'(0) = \chi_0$, $\alpha_1\varphi(1) + \beta_1\varphi'(1) = \chi_1$, and $\varphi''(\mathfrak{s}) = \mathscr{Z}(\mathfrak{s})\varphi(\mathfrak{s}) + (\Gamma\varphi)(\mathfrak{s}) - \sigma_\delta(\mathfrak{s})$, so φ is a solution of problem (4).

Step 2. We show that problem (5) has a solution.

Notice that E is a Banach space and $\|\varphi\| = \max_{\mathfrak{s}\in J}|\varphi(\mathfrak{s})|$. For the purpose of using Schauder's fixed point theorem, we consider the right-hand side of (5) and denote it using operator $\mathscr{P}: E \to E$. It follows that $\mathscr{Z}(\mathfrak{s})\varphi(\mathfrak{s}) + (\Gamma\varphi)(\mathfrak{s}) - \sigma_\delta(\mathfrak{s})$ is bounded on J, $\mathscr{P}$ is continuous and bounded.

Moveover, let $|\mathscr{Z}(\mathfrak{s})\varphi(\mathfrak{s}) + (\Gamma\varphi)(\mathfrak{s}) - \sigma_\delta(\mathfrak{s})| \le T$, $T > 0$, and take $\mathfrak{s}_1, \mathfrak{s}_2 \in J$, $\mathfrak{s}_1 < \mathfrak{s}_2$. Then we have

$$\begin{aligned}
&\big|(\mathscr{P}\varphi)(\mathfrak{s}_1) - (\mathscr{P}\varphi)(\mathfrak{s}_2)\big| \\
&\quad\le \left|\int_0^1 \big[G(\mathfrak{s}_1,s) - G(\mathfrak{s}_2,s)\big]\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds\right| + |k_1||\mathfrak{s}_1 - \mathfrak{s}_2| \\
&\quad\le \frac{1}{\frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + 1}\left|(\mathfrak{s}_1 - \mathfrak{s}_2)\int_0^{\mathfrak{s}_1}\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds\right. \\
&\qquad + \left(\mathfrak{s}_1 + \frac{\beta_0}{\alpha_0}\right)\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\left(s - 1 - \frac{\beta_1}{\alpha_1}\right)\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds \\
&\qquad - \left(\mathfrak{s}_2 - 1 - \frac{\beta_1}{\alpha_1}\right)\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\left(s + \frac{\beta_0}{\alpha_0}\right)\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds \\
&\qquad \left. + (\mathfrak{s}_1 - \mathfrak{s}_2)\int_{\mathfrak{s}_2}^1\left(s - 1 - \frac{\beta_1}{\alpha_1}\right)\big[\mathscr{Z}(s)\varphi(s) + (\Gamma\varphi)(s) - \sigma_\delta(s)\big]\,ds\right| + |k_1||\mathfrak{s}_1 - \mathfrak{s}_2| \\
&\quad\le \left(\frac{1}{4}QT + k\right)|\mathfrak{s}_1 - \mathfrak{s}_2|,
\end{aligned}$$

where $Q = \frac{1}{\frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + 1}$ and $k = |k_1|$. As $\mathfrak{s}_2 \to \mathfrak{s}_1$, the right-hand side of the above inequality tends to zero. Thus operator $\mathscr{P}$ is compact. It then follows from Schauder's fixed point theorem that $\mathscr{P}$ has a fixed point. Apparently, this fixed point is the solution of (4).

Step 3. We show that problem (4) has at most one solution.

We suppose that problem (4) has two different solutions $\varphi_1, \varphi_2 \in C^2(J, R)$. Set $\lambda = \varphi_1 - \varphi_2$, then $\lambda''(\mathfrak{s}) = \mathscr{L}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s})$ and $\mathcal{P}\lambda(\theta) = 0$, $\theta = 0, 1$ on J. From (H_1) and Lemma 2.1, we obtain $\varphi_1(\mathfrak{s}) \geq \varphi_2(\mathfrak{s})$, $\mathfrak{s} \in J$. Now letting $\lambda = \varphi_2 - \varphi_1$, we get $\varphi_2(\mathfrak{s}) \geq \varphi_1(\mathfrak{s})$, $\mathfrak{s} \in J$, based on Lemma 2.1. Hence $\varphi_1(\mathfrak{s}) = \varphi_2(\mathfrak{s})$, $\mathfrak{s} \in J$.

By the above steps, we obtain that problem (4) has a unique solution. Denote it as $\varphi = \varphi(\mathfrak{s})$.

Step 4. We can prove $\varphi \in [\phi, \psi]$.

Setting $\lambda(\mathfrak{s}) = \phi(\mathfrak{s}) - \varphi(\mathfrak{s})$, due to (H_2), (H_3), and (H_4), we acquire

$$\begin{aligned}\lambda''(\mathfrak{s}) &= \phi''(\mathfrak{s}) - \varphi''(\mathfrak{s}) \\ &\geq -(\mathscr{Q}\nu)(\mathfrak{s}) + (\mathscr{Q}\delta)(\mathfrak{s}) + \mathscr{L}(\mathfrak{s})\big(\delta(\mathfrak{s}) - \varphi(\mathfrak{s})\big) + \big(\Gamma(\delta - \varphi)\big)(\mathfrak{s}) \\ &\geq \mathscr{L}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s}).\end{aligned}$$

Noticing $\mathcal{P}\lambda(\theta) \leq 0$, $\theta = 0, 1$, it then follows from Lemma 2.1 that $\phi \leq \varphi$. Analogously, we can prove that $\varphi \leq \psi$, and we have $\varphi \in [\phi, \psi]$. The proof is then completed. □

Theorem 3.2 *Let the assumptions* (H_1)–(H_3) *be satisfied. Then problem* (1) *has extremal solutions in the sector* $[\phi, \psi]$.

Proof For each $\delta \in [\phi, \psi]$, consider the boundary value problem (4). By Theorem 3.1, problem (4) possesses a unique solution $\varphi \in C^2(J, \mathbb{R})$, and we define the mapping $\mathcal{F}$ by $\mathcal{F}\delta = \varphi$. We shall use this mapping to construct two sequences $\{\phi_n\}$, $\{\psi_n\}$. For this purpose, we shall prove that

(1) $\phi \leq \mathcal{F}\phi$, $\psi \geq \mathcal{F}\psi$;

(2) $\mathcal{F}$ is a monotone mapping in $[\phi, \psi]$.

In order to prove (1), set $\lambda(\mathfrak{s}) = \phi(\mathfrak{s}) - \phi_1(\mathfrak{s})$, where $\phi_1 = \mathcal{F}\phi$. Then we have

$$\begin{aligned}\lambda''(\mathfrak{s}) &= \phi''(\mathfrak{s}) - \phi_1''(\mathfrak{s}) \\ &\geq -(\mathscr{Q}\phi)(\mathfrak{s}) - \mathscr{L}(\mathfrak{s})\phi_1(\mathfrak{s}) - (\Gamma\phi_1)(\mathfrak{s}) + \big[(\mathscr{Q}\phi)(\mathfrak{s}) + \mathscr{L}(\mathfrak{s})\phi(\mathfrak{s}) + (\Gamma\phi)(\mathfrak{s})\big] \\ &= \mathscr{L}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s}), \quad \mathfrak{s} \in J,\end{aligned}$$

and

$$\begin{aligned}\alpha_0\lambda(0) - \beta_0\lambda'(0) &= \big[\alpha_0\phi(0) - \beta_0\phi'(0)\big] - \big[\alpha_0\phi_1(0) - \beta_0\phi_1'(0)\big] \leq 0, \\ \alpha_1\lambda(1) + \beta_1\lambda'(1) &= \big[\alpha_1\phi(1) + \beta_1\phi'(1)\big] - \big[\alpha_1\phi_1(1) + \beta_1\phi_1'(1)\big] \leq 0.\end{aligned}$$

By virtue of Lemma 2.1, one arrives at $\lambda \leq 0$, so $\phi \leq \phi_1$. Analogously, one attains $\mathcal{F}\psi \leq \psi$.

In order to prove (2), let $\delta_1 \leq \delta_2$ be such that $\delta_1, \delta_2 \in [\phi, \psi]$. Assume that $\varphi_1 = \mathcal{F}\delta_1$, $\varphi_2 = \mathcal{F}\delta_2$, and $\lambda(\mathfrak{s}) = \varphi_1(\mathfrak{s}) - \varphi_2(\mathfrak{s})$. We obtain

$$\begin{aligned}\lambda''(\mathfrak{s}) &= \varphi_1''(\mathfrak{s}) - \varphi_2''(\mathfrak{s}) \\ &= \mathscr{L}(\mathfrak{s})\big(\varphi_1(\mathfrak{s}) - \varphi_2(\mathfrak{s})\big) + \big(\Gamma(\varphi_1 - \varphi_2)\big)(\mathfrak{s}) \\ &\quad + \big[(\mathscr{Q}\delta_2)(\mathfrak{s}) - (\mathscr{Q}\delta_1)(\mathfrak{s}) + \mathscr{L}(\mathfrak{s})\big(\delta_2(\mathfrak{s}) - \delta_1(\mathfrak{s})\big) + \big(\Gamma(\delta_2 - \delta_1)\big)(\mathfrak{s})\big] \\ &\geq \mathscr{L}(\mathfrak{s})\lambda(\mathfrak{s}) + (\Gamma\lambda)(\mathfrak{s}), \quad \mathfrak{s} \in J,\end{aligned}$$

and

$$\alpha_0\lambda(0) - \beta_0\lambda'(0) = \left[\alpha_0\varphi_1(0) - \beta_0\varphi_1'(0)\right] - \left[\alpha_0\varphi_2(0) - \beta_0\varphi_2'(0)\right] = 0,$$
$$\alpha_1\lambda(1) + \beta_1\lambda'(1) = \left[\alpha_1\varphi_1(1) + \beta_1\varphi_1'(1)\right] - \left[\alpha_1\varphi_2(1) + \beta_1\varphi_2'(1)\right] = 0.$$

In view of Lemma 2.1, we derive $\lambda \le 0$, which implies $\mathcal{F}\delta_1 \le \mathcal{F}\delta_2$.

Now, define the sequences $\{\phi_n(\mathfrak{s})\}$, $\{\varphi_n(\mathfrak{s})\}$ by $\phi_n = \mathcal{F}\phi_{n-1}$, $\psi_n = \mathcal{F}\psi_{n-1}$, and $\phi_0 = \phi$, $\psi_0 = \psi$, and conclude from previous arguments that

$$\phi_0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_n \le \cdots \le \psi_n \le \cdots \le \psi_2 \le \psi_1 \le \psi_0.$$

By means of standard arguments, we derive that $\lim_{n\to\infty}\phi_n(\mathfrak{s}) = \xi(\mathfrak{s})$ and $\lim_{n\to\infty}\psi_n(\mathfrak{s}) = \zeta(\mathfrak{s})$ uniformly and monotonically on J. It is easy to see that ξ, ζ are solutions of problem (1).

To demonstrate ξ and ζ are extremal solutions to (1), we set u be an arbitrarily solution to (1) with $\phi \le u \le \psi$. Suppose that for some $n \in \mathbb{N}$, $\phi_n \le u \le \psi_n$. Employing the monotonic nondecreasingness property of $\mathcal{F}$, we acquire $\phi_{n+1} = \mathcal{F}\phi_n \le \mathcal{F}u = u$, hence $\phi_{n+1} \le u$ on J. Analogously, we have $u \le \psi_{n+1}$ on J. Note that $\phi_0 \le u \le \psi_0$, so by induction we see that $\phi_n \le u \le \psi_n$ for every n. Taking the limit as $n \to \infty$, one concludes $\xi \le u \le \zeta$, and the proof is complete. □

4 Examples

Example 4.1 Consider the problem below

$$\begin{cases} -\varphi''(\mathfrak{s}) = -\mathfrak{s}\varphi(\mathfrak{s}) - \int_0^{\mathfrak{s}} s\varphi(s)\,ds + \mathfrak{s}^3 + \frac{1}{4}\mathfrak{s}^4 - 2, & \mathfrak{s} \in J = [0,1], \\ \varphi(0) - \varphi'(0) = 0, \qquad \varphi(1) + \varphi'(1) = 3. \end{cases} \tag{7}$$

Put $\rho(\mathfrak{s}) = \mathfrak{s}^2 - 1$, $\mathfrak{r}(\mathfrak{s}) = 1 + \mathfrak{s}$. We can easily check that $\rho(\mathfrak{s}) \le \mathfrak{r}(\mathfrak{s})$, and $\rho(\mathfrak{s})$, $\mathfrak{r}(\mathfrak{s})$ are also lower and upper solutions, respectively.

Set $(\mathscr{Q}\varphi)(\mathfrak{s}) = -\mathfrak{s}\varphi(\mathfrak{s}) - \int_0^{\mathfrak{s}} s\varphi(s)\,ds + \mathfrak{s}^3 + \frac{1}{4}\mathfrak{s}^4 - 2$ and $(\Gamma\varphi)(\mathfrak{s}) = \int_0^{\mathfrak{s}} s\varphi(s)\,ds$. By computing, one achieves

$$(\mathscr{Q}\bar{g})(\mathfrak{s}) - (\mathscr{Q}g)(\mathfrak{s}) \ge -\mathfrak{s}\big(\bar{g}(\mathfrak{s}) - g(\mathfrak{s})\big) - \big(\Gamma(\bar{g} - g)\big)(\mathfrak{s}),$$

where $\rho(\mathfrak{s}) \le g(\mathfrak{s}) \le \bar{g}(\mathfrak{s}) \le \mathfrak{r}(\mathfrak{s})$ on $\mathfrak{s} \in J$, $\mathscr{L}(\mathfrak{s}) = \mathfrak{s}$.

We may easily prove that $\int_0^1(\int_s^1[\mathscr{L}(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})]\,d\mathfrak{s})\,ds = \frac{11}{24} < 1$. Applying Theorem 3.2, one arrives at the existence of monotone sequences that approximate the extremal solutions to (7) in $[\rho, \mathfrak{r}]$.

Example 4.2 Consider the following problem:

$$\begin{cases} -\varphi''(\mathfrak{s}) = -\mathscr{A}(\mathfrak{s})\cos\varphi(\mathfrak{s}) - \mathscr{B}(\mathfrak{s})\sin\varphi(\frac{1}{2}\mathfrak{s}) - \frac{2}{3\pi}(\mathscr{B}(\mathfrak{s}) + \varepsilon)\varphi(\frac{1}{3}\mathfrak{s}), & \mathfrak{s} \in J, \\ \alpha_0\varphi(0) - \beta_0\varphi'(0) = \chi_0, \qquad \alpha_1\varphi(1) + \beta_1\varphi'(1) = \chi_1. \end{cases} \tag{8}$$

Suppose that $J = [0,1]$, $\mathscr{A}, \mathscr{B} \in C(J, [0, +\infty))$, $\alpha_0, \alpha_1 \ge 0$, $\beta_0, \beta_1 > 0$, and $\varepsilon > 0$ is sufficiently small. Also assume that $-\alpha_0 \le \frac{2}{3\pi}\chi_0 \le \alpha_0$, $-\alpha_1 \le \frac{2}{3\pi}\chi_1 \le \alpha_1$, and $\int_0^1(\int_s^1(\mathscr{A}(\mathfrak{s}) + 2\mathscr{B}(\mathfrak{s}))\,d\mathfrak{s})\,ds \le 1$.

Let $\phi(\mathfrak{s}) = -\frac{3\pi}{2}$, $\varphi(\mathfrak{s}) = \frac{3\pi}{2}$. It is easy to prove that $\phi(\mathfrak{s})$ is a lower solution, $\psi(\mathfrak{s})$ is an upper solution, and $\phi(\mathfrak{s}) \leq \psi(\mathfrak{s})$, $\mathfrak{s} \in J$.

Take $(\mathscr{Q}\varphi)(\mathfrak{s}) = -\mathscr{A}(\mathfrak{s})\cos\varphi(\mathfrak{s}) - \mathscr{B}(\mathfrak{s})\sin\varphi(\frac{1}{2}\mathfrak{s}) - \frac{2}{3\pi}(\mathscr{B}(\mathfrak{s}) + \varepsilon)\varphi(\frac{1}{3}\mathfrak{s})$ and $(\Gamma\varphi)(\mathfrak{s}) = \mathscr{B}(\mathfrak{s})\sin\varphi(\frac{1}{2}\mathfrak{s}) + \frac{2}{3\pi}(\mathscr{B}(\mathfrak{s}) + \varepsilon)\varphi(\frac{1}{3}\mathfrak{s})$. By computing, one attains

$$\begin{aligned}
&(\mathscr{Q}\bar{g})(\mathfrak{s}) - (\mathscr{Q}g)(\mathfrak{s}) \\
&\quad = -\mathscr{A}(\mathfrak{s})\bigl(\cos\bar{g}(\mathfrak{s}) - \cos g(\mathfrak{s})\bigr) - \mathscr{B}(\mathfrak{s})\left(\sin\bar{g}\left(\frac{1}{2}\mathfrak{s}\right) - \sin g\left(\frac{1}{2}\mathfrak{s}\right)\right) \\
&\qquad - \frac{2}{3\pi}\bigl(\mathscr{B}(\mathfrak{s}) + \varepsilon\bigr)\left(\bar{g}\left(\frac{1}{3}\mathfrak{s}\right) - g\left(\frac{1}{3}\mathfrak{s}\right)\right) \\
&\quad \geq -\mathscr{A}(\mathfrak{s})\bigl(\bar{g}(\mathfrak{s}) - g(\mathfrak{s})\bigr) - \mathscr{B}(\mathfrak{s})\sin\left(\bar{g}\left(\frac{1}{2}\mathfrak{s}\right) - g\left(\frac{1}{2}\mathfrak{s}\right)\right) \\
&\qquad - \frac{2}{3\pi}\bigl(\mathscr{B}(\mathfrak{s}) + \varepsilon\bigr)\left(\bar{g}\left(\frac{1}{3}\mathfrak{s}\right) - g\left(\frac{1}{3}\mathfrak{s}\right)\right) \\
&\quad = -\mathscr{L}(\mathfrak{s})\bigl(\bar{g}(\mathfrak{s}) - g(\mathfrak{s})\bigr) - \bigl(\Gamma(\bar{g} - g)\bigr)(\mathfrak{s}),
\end{aligned}$$

where $\phi(\mathfrak{s}) \leq g(\mathfrak{s}) \leq \bar{g}(\mathfrak{s}) \leq \psi(\mathfrak{s})$ on $\mathfrak{s} \in J$, $\mathscr{L}(\mathfrak{s}) = \mathscr{A}(\mathfrak{s})$, and

$$\int_0^1 \left(\int_s^1 \bigl[\mathscr{L}(\mathfrak{s}) + (\Gamma\mathbf{1})(\mathfrak{s})\bigr]\,d\mathfrak{s}\right) ds < 1.$$

In view of Theorem 3.2, we obtain the existence of monotone sequences that approximate the extremal solutions to (8) in $[\phi, \psi]$.

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