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Multiplicity of solutions for the Dirichlet boundary value problem to a fractional quasilinear differential model with impulses

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Abstract

This paper aims to consider the multiplicity of solutions for a kind of boundary value problem to a fractional quasilinear differential model with impulsive effects. By establishing a new variational structure and overcoming the difficulties brought by the influence of impulsive effects, some new results are acquired via the symmetry mountain-pass theorem, which extend and enrich some previous results.

MSC: 26A33; 34G20; 34B15

Keywords: Fractional differential equation; Boundary value problem; Multiplicity; Impulsive effect

1 Introduction

In this paper, we are concerned with the following fractional quasilinear differential model with impulsive effects.

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) + b(t)u(t) + 2u(t)|{}_0D_t^\alpha u(t)|^2 + 2{}_tD_T^\alpha(|u(t)|^2{}_0D_t^\alpha u(t)) \\ \quad = f(t,u(t)), \quad \text{a.e. } t \in J, \\ \Delta({}_tI_T^{1-\alpha}({}_0D_t^\alpha u(t_j))) = I_{1j}(u(t_j)), \quad j = 1,2,\ldots,m, \\ \Delta({}_tI_T^{1-\alpha}(|u(t_j)|^2{}_0D_t^\alpha u(t_j))) = I_{2j}(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where D_t^α and ${}_tD_b^\alpha$ are the left and right Riemann–Liouville fractional derivatives, respectively, ${}_tI_b^\alpha$ is the right Riemann–Liouville fractional integral, $f(t,u) = g(t,u) + \zeta h(t)|u(t)|^{\nu-2}u(t)$, $g \in C([0,T] \times \mathbb{R},\mathbb{R})$, $\alpha \in (\frac{1}{2},1]$, $b,h \in C([0,T],\mathbb{R})$, $t_0 = 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $J = [0,T] \setminus \{t_1,t_2,\ldots,t_m\}$, $m \in \mathbb{N}$, $I_j \in C(\mathbb{R},\mathbb{R})$, $\zeta \in \mathbb{R}$, $\nu \in [1,2)$,

$$\Delta\left({}_tI_T^{1-\alpha}\left({}_0D_t^\alpha u(t_j)\right)\right) = {}_tI_T{}^{1-\alpha}\left({}_0D_t^\alpha u\left(t_j^+\right)\right) - {}_tI_T{}^{1-\alpha}\left({}_0D_t^\alpha u\left(t_j^-\right)\right),$$

$${}_tI_T{}^{1-\alpha}\left({}_0D_t^\alpha u\left(t_j^+\right)\right) = \lim_{t\to t_j^+} {}_tI_T^{1-\alpha}\left({}_0D_t^\alpha u(t_j)\right),$$

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$$
\begin{aligned}
&{}_tI_T{}^{1-\alpha}\left({}_0D_t^\alpha u\left(t_j^-\right)\right)=\lim_{t\to t_j^-}{}_tI_T^{1-\alpha}\left({}_0D_t^\alpha u(t_j)\right),\\
&\Delta\left({}_tI_T^{1-\alpha}\left(\left|u(t_j)\right|^2{}_0D_t^\alpha u(t_j)\right)\right)={}_tI_T^{1-\alpha}\left(\left|u\left(t_j^+\right)\right|^2{}_0D_t^\alpha u\left(t_j^+\right)\right)-{}_tI_T^{1-\alpha}\left(\left|u\left(t_j^-\right)\right|^2{}_0D_t^\alpha u\left(t_j^-\right)\right),\\
&{}_tI_T{}^{1-\alpha}\left(\left|u\left(t_j^+\right)\right|_0^2D_t^\alpha u\left(t_j^+\right)\right)=\lim_{t\to t_j^+}{}_tI_T^{1-\alpha}\left(\left|u(t_j)\right|_0^2D_t^\alpha u(t_j)\right),\\
&{}_tI_T{}^{1-\alpha}\left(\left|u\left(t_j^-\right)\right|_0^2D_t^\alpha u\left(t_j^-\right)\right)=\lim_{t\to t_j^-}{}_tI_T^{1-\alpha}\left(\left|u(t_j)\right|_0^2D_t^\alpha u(t_j)\right).
\end{aligned}
$$

In fact, the idea of a fractional quasilinear differential model comes from the standing-wave solutions ($\varphi(t,x)=e^{-iwt}u(x)$, $w\in\mathbb{R}$) of the following integer quasilinear Schrödinger equation.

$$
i\partial_t\varphi=-\partial_{xx}\varphi+V(x)\varphi-\partial_{xx}\left(|\varphi|^2\right)\varphi-|\varphi|^{\nu-1}\varphi,\quad x\in\mathbb{R},\nu>1,\tag{1.2}
$$

which plays an important role in some research fields of physics (see [1, 2] and the references therein). An interesting question as to whether the existence or multiplicity of solutions to this fractional quasilinear differential model with suitable boundary conditions generated by impulsive effects can be obtained naturally comes to mind. It is well known that the impulsive differential models describe the discontinuous process and originate from some important research fields. In recent years, critical-point theory has been successfully applied to deal with the existence and multiplicity of solutions of boundary value problems (BVPs for short) to differential equations with impulsive effects. Based on some critical-point theorems, Nieto and O'Regan [3] considered the impulsive Dirichlet BVP

$$
\begin{cases}
-u''(t)+\lambda u(t)=f(t,u(t)), & \text{a.e. } t\in J,\\
\Delta(u'(t_j))=I_j(u(t_j)), & j=1,2,\dots,m,\\
u(0)=u(T)=0
\end{cases}\tag{1.3}
$$

and obtained some existence results. Subsequently, more and more scholars have paid attention to this problem, such as Sun and Chen [4], Zhou and Li [5], Zhang and Yuan [6], etc. Moreover, for the case of impulsive BVPs with p-Laplacian operator, one can refer to [7, 8] and references therein.

On the other hand, recently, Jiao and Zhou [9] proved that under the Dirichlet boundary condition $u(0)=u(T)=0$, the operator ${}_t^cD_T^\alpha{}_0D_t^\alpha$ has a variational structure. Also, by the mountain-pass theorem, the existence of solutions to the following systems was obtained under the Ambrosetti–Rabinowitz condition:

$$
\begin{cases}
{}_tD_T^\alpha({}_0D_t^\alpha u(t))=\nabla F(t,u(t)), & \text{a.e. } t\in[0,T],\\
u(0)=u(T)=0,
\end{cases}\tag{1.4}
$$

where $\alpha\in(\frac{1}{2},1]$. After that, Bonanno, Rodríguez-López and Tersian [10] discussed the existence of three solutions to the following problem with impulsive effects and parameters:

$$
\begin{cases}
{}_tD_T^\alpha({}_0^cD_t^\alpha u(t))+a(t)u(t)=\lambda f(t,u(t)), & \text{a.e. } t\in J\\
\Delta({}_tI_T^{1-\alpha}({}_0^cD_t^\alpha u(t_j)))=\mu I_j(u(t_j)), & j=1,2,\dots,m,\\
u(0)=u(T)=0,
\end{cases}\tag{1.5}
$$

where $\alpha \in (\frac{1}{2}, 1]$. Nyamoradi and Rodríguez-López [11] extended the scalar model of (1.5) to the case of Hamiltonian systems and obtained the multiplicity of solutions by the variant Fountain theorems. Moreover, by the gene property and the mountain-pass theorem, Ledesma and Nyamoradi [12] investigated the eigenvalue problem ${}_tD_T^\alpha \phi_p({}_0D_t^\alpha u) = \lambda \phi_p(u)$ with the Dirichlet boundary conditions $u(0) = u(T) = 0$ and obtained the existence of solutions to the Dirichlet boundary problem of a fractional p-Laplacian equation with impulsive effects. Liu, Wang and Shen [13] extended the results of [12] to the case of combined nonlinearity. Furthermore, for the Dirichlet BVPs and other BVPs of fractional differential equations with or without impulsive effects, one can refer to [14–21] and references therein.

Motivated by the works mentioned above, we are concerned with the multiplicity of solutions to the fractional quasilinear differential model with impulsive effects (1.1). Let us present our paper's contribution: To begin with, the variational structure of (1.1) is established, which makes the critical-point theory applicable to discuss the existence and multiplicity of solutions to this problem. Moreover, the impulsive effects produced by the quasilinear term $u|{}_0D_t^\alpha u|^2 + {}_tD_T^\alpha(|u|^2 {}_0D_t^\alpha u)$ are more complex than the case of ${}_t^cD_T^\alpha({}_0D_t^\alpha u)$, which make this problem challenging. Furthermore, there are few papers considering this problem.

In order to describe our main conclusion, the following assumptions are presented:

(I1) For any $t \in \mathbb{R}$, $I_{1j}(t)$ and $I_{2j}(t)$ are odd on t and

$$\int_0^t \bigl(I_{1j}(s) + I_{2j}(s)\bigr)\, ds \geq 0.$$

(I2) There exist constants $a_{1j}, a_{2j}, d_{1j}, d_{2j} > 0$ such that

$$\begin{aligned} \bigl|I_{1j}(t)\bigr| &\leq a_{1j} + d_{1j}|t|^{\gamma_{1j}} \quad \text{for any } t \in \mathbb{R}, \gamma_{1j} \in [0,1), \\ \bigl|I_{2j}(t)\bigr| &\leq a_{2j} + d_{2j}|t|^{\gamma_{2j}} \quad \text{for any } t \in \mathbb{R}, \gamma_{2j} \in [2,3). \end{aligned}$$

(I3) For any $t \in \mathbb{R}$, $I_{1j}(t)$ and $I_{2j}(t)$ satisfy

$$\theta \int_0^t \bigl(I_{1j}(s) + I_{2j}(s)\bigr)\, ds - \bigl(I_{1j}(t) + I_{2j}(t)\bigr)t \geq 0,$$

where $\theta \geq 4$ is a constant.

(G1) $\lim_{|u|\to+\infty} \frac{G(t,u)}{|u|^4} = +\infty$ uniformly for $t \in [0,T]$.

(G2) There exist constants $M_1 > 0$, $L_1 > 0$ such that for $t \in [0,T]$, $|u| \geq L_1$,

$$ug(t,u) - \theta G(t,u) \geq -M_1|u|^2.$$

(G3) There exist constants $M_2 > 0$, $L_2 > 0$, $\mu > \theta$ such that for $t \in [0,T]$, $|u| \geq L_2$,

$$G(t,u) \leq M_2|u|^\mu.$$

(G4) $g(t,u) = o(|u|)$ as $|u| \to 0$ uniformly for $t \in [0,T]$.

(G5) $g(t,u)$ is odd on u.

Now, we state our main results.

Theorem 1.1 *Assuming that the conditions* (I1)–(I3) *and* (G1)–(G5) *are satisfied, there exists a constant* $\zeta_* > 0$ *such that the problem* (1.1) *has infinitely many nontrivial weak solutions, provided that* $\zeta \in [0, \zeta_*)$.

Remark 1.2 It should be pointed out that if $\zeta = 0$, the condition (G3) can be removed. Moreover, the conditions (G1) and (G2) are weaker than the following classical Ambrosetti–Rabinowitz condition:

$$0 < \theta G(t,u) \leq ug(t,u), \quad \theta > 4, u \in \mathbb{R} \setminus \{0\}.$$

Remark 1.3 If $\alpha = 1$, the quasilinear term $2u|{}_0D_t^\alpha u|^2 + 2{}_tD_T^\alpha(|u|^2{}_0D_t^\alpha u)$ is equal to $-(|u|^2)''u$. Moreover, $I_{2j}(u) = |u|^2 I_{1j}(u)$.

Corollary 1.4 *If the assumptions* (I2) *and* (G1) *in Theorem* 1.1 *are replaced by*

(I2)$_*$ *there exist constants* $a_{1j}, a_{2j}, d_{1j}, d_{2j} > 0$ *such that*

$$\begin{aligned} \left|I_{1j}(t)\right| &\leq a_{1j} + d_{1j}|t|^{\gamma_{1j}} \quad \textit{for any } t \in \mathbb{R}, \gamma_{1j} \in [0, \theta - 3), \\ \left|I_{2j}(t)\right| &\leq a_{2j} + d_{2j}|t|^{\gamma_{2j}} \quad \textit{for any } t \in \mathbb{R}, \gamma_{2j} \in [2, \theta - 1). \end{aligned}$$

(G1)$_*$ $\lim_{|u|\to+\infty} \frac{F(t,u)}{|u|^\theta} = +\infty$ *uniformly for* $t \in [0, T]$.

Then, the conclusion of Theorem 1.1 *is also true.*

Remark 1.5 It should be pointed out that the impulsive nonlinearity I_{1j} could be superlinear growth when $\theta > 4$.

2 Preliminaries

Set $C := C([0,T],\mathbb{R})$ with norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ and $L^p := L^p([0,T],\mathbb{R})$ with norm $\|u\|_{L^p} = (\int_0^T |u(t)|^p\,dt)^{\frac{1}{p}}$. For the definitions of fractional integrals and derivatives relating to the well-known left and right Riemann–Liouville and Caputo, one can refer to references [22, 23]. Next, some of the necessary results and properties will be presented. Define the Sobolev space

$$E_0^\alpha = \left\{u : [0,T] \to \mathbb{R} \mid u, {}_0D_t^\alpha u \in L^2, u(0) = u(T) = 0\right\}$$

by $\overline{C_0^\infty([0,T],\mathbb{R})}^{\|\cdot\|_\alpha}$, where $\|\cdot\|_\alpha = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$,

$$\langle u,v\rangle = \int_0^T \left(\left(u(t)v(t) + {}_0D_t^\alpha u(t){}_0D_t^\alpha v(t)\right) dt.$$

Let

$$P(u) = \frac{1}{2}\|u\|_\alpha^2 - \frac{1}{2}\int_0^T \left(1 - b(t)\right)u^2(t)\,dt.$$

By the method of [24], the space E_0^α can be decomposed as follows. In fact, based on the Riesz representation theorem, we can find a linear self-adjoint operator $Q : E_0^\alpha \to E_0^\alpha$ such that

$$\langle Qu, v\rangle = \int_0^T \bigl(1 - b(t)\bigr)u(t)v(t)\,dt \quad \text{for } u, v \in E_0^\alpha,$$

which implies that

$$P(u) = \frac{1}{2}\bigl\langle (I - Q)u, u\bigr\rangle.$$

Noting that the embedding $E_0^\alpha \hookrightarrow C$ is compact (see [9]), it implies that Q is compact. In view of the well-known compact operator's spectral theory, for the operator $I - Q$, we can decompose the Sobolev space E_0^α into the orthogonal sum of invariant subspaces as

$$E_0^\alpha = E^- \oplus E^0 \oplus E^+,$$

where E^- and E^+ are negative and positive spectral subspaces corresponding to the operator $I - Q$, $E^0 = N(I - Q)$. Moreover, letting $\Pi = \{1, 2, \dots, \iota\}$ with $\iota \in \mathbb{N}$, Q possesses only finitely many eigenvalues $\{\lambda_i\}_{i\in\Pi}$ satisfying $\lambda_i > 1$ because Q is compact on E_0^α, which implies that the dimension of subspace E^- is finite. By the classical self-adjoint operator theory, for $I - Q$ that can be viewed as a compact perturbation relating to the self-adjoint operator I, it is clear that 0 is excluded in the essential spectrum of $I - Q$. Thus, the dimension of subspace E^0 is also finite. Furthermore, there exists a positive constant κ such that

$$\pm P(u) \geq \kappa \|u\|_\alpha^2, \quad u \in E^\pm. \tag{2.1}$$

Lemma 2.1 ([22, 23]) *Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If u is a function defined on $[a, b]$ for which the Caputo fractional derivatives ${}_a^cD_t^\alpha u(t)$ and ${}_t^cD_b^\alpha u(t)$ of order α exist together with the Riemann–Liouville fractional derivatives ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, then*

$${}_a^cD_t^\alpha u(t) = {}_aD_t^\alpha u(t) - \sum_{j=0}^{n-1} \frac{u^j(a)}{\Gamma(j - \alpha + 1)}(t - a)^{j-\alpha}, \quad t \in [a, b], \tag{2.2}$$

$${}_t^cD_b^\alpha u(t) = {}_tD_b^\alpha u(t) - \sum_{j=0}^{n-1} \frac{u^j(b)}{\Gamma(j - \alpha + 1)}(b - t)^{j-\alpha}, \quad t \in [a, b]. \tag{2.3}$$

Remark 2.2 From (2.2) and (2.3), one has ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, ${}_t^cD_T^\alpha u(t) = {}_tD_T^\alpha u(t)$, $t \in [0, T]$ by $u(0) = u(T) = 0$.

Proposition 2.3 ([23]) *The following property of fractional integration*

$$\int_a^b \bigl[{}_aI_t^\alpha f(t)\bigr]g(t)\,dt = \int_a^b \bigl[{}_tI_b^\alpha g(t)\bigr]f(t)\,dt, \quad \alpha > 0$$

holds, provided that $u \in L^p([a, b], \mathbb{R}^N)$, $g \in L^q([a, b], \mathbb{R}^N)$ and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \alpha$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \alpha$, where ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ are the left and right Riemann–Liouville fractional integrals, respectively.

Lemma 2.4 ([9]) *Let $0 < \alpha \leq 1$. If $u \in E_0^\alpha$, one has*

$$\|u\|_{L^2} \leq S_\alpha \left\|{}_0D_t^\alpha u\right\|_{L^2}, \tag{2.4}$$

where $S_\alpha = \frac{T^\alpha}{\Gamma(\alpha+1)}$. Moreover, if $\alpha > \frac{1}{2}$, then

$$\|u\|_\infty \leq S_\infty \|{}_0D_t^\alpha u\|_{L^2}, \tag{2.5}$$

where $S_\infty = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}}$. Based on (2.4), clearly, the norm of E_0^α is equivalent to $\|{}_0D_t^\alpha u\|_{L^2}$.

Proposition 2.5 ([9]) *Let $0 < \alpha \leq 1$. Assume that $\alpha > \frac{1}{2}$ and the sequence u_n converges weakly to u in E_0^α, i.e., $u_n \rightharpoonup u$. Then, $u_n \to u$ in C, i.e., $\|u_n - u\|_\infty \to 0$, $n \to +\infty$.*

If $u \in E_0^\alpha$ is a solution of the problem (1.1), for $v \in E_0^\alpha$, based on Lemma 2.1 and Proposition 2.3, it implies that

$$\begin{aligned}
&\int_0^T {}_tD_T^\alpha\left(\left|u(t)\right|^2 {}_0D_t^\alpha u(t)\right)v(t)\,dt \\
&\quad = -\sum_{j=0}^m \int_{t_j}^{t_{j+1}} v(t)\,d\left[{}_tI_T^{1-\alpha}\left(\left|u(t)\right|^2 {}_0D_t^\alpha u(t)\right)\right] \\
&\qquad - \sum_{j=0}^m {}_tI_T^{1-\alpha}\left(\left|u(t)\right|^2 {}_0D_t^\alpha u(t)\right)v(t)\Big|_{t_j^+}^{t_{j+1}^-} + \sum_{j=0}^m \int_{t_j}^{t_{j+1}} \left|u(t)\right|^2 {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt \\
&\quad = \sum_{j=1}^m \left({}_tI_T^{1-\alpha}\left(\left|u\left(t_j^+\right)\right|^2 {}_0D_t^\alpha u\left(t_j^+\right)\right)v(t_j) - {}_tI_T^{1-\alpha}\left(\left|u\left(t_j^-\right)\right|^2 {}_0D_t^\alpha u\left(t_j^-\right)\right)v(t_j)\right) \\
&\qquad + \int_0^T \left|u(t)\right|^2 {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt \\
&\quad = \sum_{j=1}^m I_{2j}\left(u(t_j)\right)v(t_j) + \int_0^T \left|u(t)\right|^2 {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt.
\end{aligned}$$

Similarly, one has

$$\int_0^T {}_tD_T^\alpha\left({}_0D_t^\alpha u(t)\right)v(t)\,dt = \sum_{j=1}^m I_{1j}\left(u(t_j)\right)v(t_j) + \int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt.$$

As a conclusion, the definition of a weak solution is shown as follows.

Definition 2.6 A function $u \in E_0^\alpha$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&\int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt + \int_0^T b(t)u(t)v(t)\,dt + \int_0^T 2\left|{}_0D_t^\alpha u(t)\right|^2 u(t)v(t)\,dt \\
&\qquad + \int_0^T 2\left|u(t)\right|^2 {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t)\,dt + \sum_{j=1}^m \left(I_{1j}\left(u(t_j)\right) + I_{2j}\left(u(t_j)\right)\right)v(t_j) \\
&\quad = \int_0^T f\left(t, u(t)\right)v(t)\,dt
\end{aligned}$$

holds for any $v \in E_0^\alpha$.

Define the functional $\Phi : E_0^\alpha \to \mathbb{R}$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_0^T \left|{}_0D_t^\alpha u(t)\right|^2 dt + \frac{1}{2}\int_0^T b(t)\left|u(t)\right|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} \left(I_{1j}(t) + I_{2j}(t)\right) dt \\ &\quad + \int_0^T \left|{}_0D_t^\alpha u(t)\right|^2 \left|u(t)\right|^2 dt - \int_0^T G\left(t, u(t)\right) dt - \frac{\zeta}{\nu}\int_0^T h(t)\left|u(t)\right|^\nu dt,\end{aligned}$$

where $G(t,u) = \int_0^u g(t,s)\,ds$. Since g, I_{1j} and I_{2j} are continuous, by the standard arguments, one can obtain that $\Phi(u) \in C^1(E_0^\alpha, \mathbb{R})$. Moreover, it is clear that the critical points of $\Phi(u)$ are weak solutions of the problem (1.1).

Lemma 2.7 ([25]) *Let E be a Banach space and $\Phi \in C^1(E,\mathbb{R})$ be even with $\Phi(0) = 0$. Assume that $E = V \oplus X$, where V is finite-dimensional. Moreover, Φ satisfies the (PS)-condition and the following conditions.*

- (i) *There exist constants $\rho, \sigma > 0$ such that $\Phi\,|_{\partial B_\rho \cap X} \geq \sigma$.*
- (ii) *For each finite-dimensional subspace $\widetilde{X} \subset E$, there exists an $l = l(\widetilde{X})$ such that $\Phi \leq 0$ on $\widetilde{X} \setminus B_l$.*

Then, Φ has an unbounded sequence of critical values.

3 Main results

In order to prove our main conclusions, we need the following lemmas. First, in E_0^α, let $V = E^- \oplus E^0$ and $X = E^+$, then the dimension of subspace V is finite and $E_0^\alpha = V \oplus X$.

Lemma 3.1 *Assuming that the conditions* (I1), (G3), *and* (G4) *are satisfied, we can find constants $\rho, \sigma, \zeta^* > 0$ such that $\Phi|_{\partial B_\rho \cap X} \geq \sigma$, provided that $\zeta \in [0, \zeta^*)$.*

Proof Based on (G3) and (G4), for any $\varepsilon > 0$, we can find a constant c_ε such that for $t \in [0,T]$,

$$G(t,u) \leq \varepsilon |u|^2 + c_\varepsilon |u|^\mu, \tag{3.1}$$

which shows that

$$\begin{aligned}\int_0^T G(t,u)\,dt &\leq \varepsilon \int_0^T |u|^2\,dt + c_\varepsilon \int_0^T |u|^\mu\,dt \\ &\leq \varepsilon T S_\infty^2 \|u\|_\alpha^2 + c_\varepsilon T S_\infty^\mu \|u\|_\alpha^\mu.\end{aligned}$$

Hence, for $u \in E_0^\alpha$, by (I1), one has

$$\begin{aligned}\Phi(u) &\geq \kappa\|u\|_\alpha^2 + \sum_{j=1}^m \int_0^{u(t_j)} \left(I_{1j}(t) + I_{2j}(t)\right) dt + \int_0^T \left|{}_0D_t^\alpha u(t)\right|^2 \left|u(t)\right|^2 dt \\ &\quad - \int_0^T G\left(t,u(t)\right) dt - \frac{\zeta}{\nu}\int_0^T h(t)\left|u(t)\right|^\nu dt \\ &\geq \kappa\|u\|_\alpha^2 - \int_0^T G\left(t,u(t)\right) dt - \frac{\zeta}{\nu}\int_0^T h(t)\left|u(t)\right|^\nu dt\end{aligned}$$

$$\geq \|u\|_\alpha^\nu \left(\left(\kappa - \varepsilon T S_\infty^2 \right) \|u\|_\alpha^{2-\nu} - c_\varepsilon T S_\infty^\mu \|u\|_\alpha^{\mu-\nu} - \frac{\zeta}{\nu} T S_\infty^\nu \|h\|_{L^1} \right).$$

Letting $\varepsilon = \frac{\zeta}{2TS_\infty^2}$, leads to

$$\Phi(u) \geq \|u\|_\alpha^\nu \left(\frac{\kappa}{2} \|u\|_\alpha^{2-\nu} - c_\varepsilon T S_\infty^\mu \|u\|_\alpha^{\mu-\nu} - \frac{\zeta}{\nu} T S_\infty^\nu \|h\|_{L^1} \right).$$

Set

$$y(t) = \frac{\kappa}{2} t^{2-\nu} - c_\varepsilon T S_\infty^\mu t^{\mu-\nu}, \quad t \geq 0.$$

Clearly, there exists a $\rho = [\frac{\kappa(2-\nu)}{2c_\varepsilon T S_\infty^\mu (\mu-\nu)}]^{\frac{1}{\mu-2}}$ such that

$$y(\rho) = \max_{t\geq 0} y(t) = \frac{\kappa(\mu-2)}{2(\mu-\nu)} \left[\frac{\kappa(2-\nu)}{2c_\varepsilon T S_\infty^\mu (\mu-\nu)} \right]^{\frac{2-\nu}{\mu-2}} > 0.$$

Therefore, we can find

$$\zeta^* = \frac{\nu\kappa(\mu-2)}{T S_\infty^\nu (\mu-\nu) \|h\|_{L^1}} \left[\frac{\kappa(2-\nu)}{2c_\varepsilon T S_\infty^\mu (\mu-\nu)} \right]^{\frac{2-\nu}{\mu-2}}.$$

If $\zeta \in [0, \zeta^*)$, there exists a constant $\sigma > 0$ such that $\Phi|_{X\cap\partial B_\rho} \geq \sigma$. □

Lemma 3.2 *If the conditions* (I2) *and* (G1) *are satisfied, there exists a constant* $l > 0$ *such that for each finite-dimensional subspace* $\widetilde{X} \subset E_0^\alpha$, $\Phi(u) \leq 0$, $\forall u \in \widetilde{X} \backslash B_l$, *provided that* $\zeta \in [0, +\infty)$.

Proof Actually, for $\zeta \in [0, +\infty)$, the key point is to prove that $\Phi(u)$ is anticoercive, i.e.,

$$\Phi(u) \to -\infty \quad \text{as } \|u\|_\alpha \to +\infty \text{ for } u \in \widetilde{X}. \tag{3.2}$$

If not, let the sequence $\{u_n\} \subset \widetilde{X}$ and $\tau \in \mathbb{R}$ such that

$$\Phi(u_n) \geq \tau \quad \text{when } \|u_n\|_\alpha \to +\infty \text{ as } n \to +\infty. \tag{3.3}$$

Setting $\omega_n = \frac{u_n}{\|u_n\|_\alpha}$, then $\|\omega_n\|_\alpha = 1$. Since $\dim \widetilde{X} < \infty$, we can find a subsequence of $\{\omega_n\}$ (named again $\{\omega_n\}$) such that $\omega_n \to \omega$ in E_0^α, which implies $\|\omega\|_\alpha = 1$. From $\omega \neq 0$, one has $|u_n(t)| \to +\infty$ as $n \to +\infty$. Define

$$W(t,u) = G(t,u) + \frac{\zeta}{\nu} h(t)|u|^\nu - \frac{1}{2} b(t)|u|^2.$$

In view of (G1), it follows that for any $t \in [0, T]$,

$$\lim_{|u|\to+\infty} \frac{W(t,u)}{|u|^4} = +\infty. \tag{3.4}$$

Moreover, by a standard measure estimation on a finite-dimensional space (see [4]), it follows that there exists a positive constant $\epsilon > 0$ such that

$$\operatorname{meas}\bigl\{t \in [0,T] : \bigl|u(t)\bigr| \geq \epsilon \|u\|_\alpha\bigr\} \geq \epsilon \quad \text{for } u \in \widetilde{X} \setminus \{0\}. \tag{3.5}$$

Let $\Pi = \{t \in [0,T] : |u(t)| \geq \epsilon \|u\|_\alpha\}$. Based on (3.4), it means that for $\frac{2S_\infty^2}{\epsilon^4} > 0$, there exists $\eta > 0$ such that

$$W(t,u) \geq \frac{2S_\infty^2}{\epsilon^4}|u|^4 \quad \text{for } |u| \geq \eta. \tag{3.6}$$

Hence, for $u \in \widetilde{X}$ with $\|u\|_\alpha \geq \frac{\eta}{\epsilon}$, we can obtain that

$$W(t,u) \geq 2S_\infty^2 \|u\|_\alpha^4 \quad \text{for } t \in \Pi. \tag{3.7}$$

Let $\|u_n\|_\alpha \geq \frac{\eta}{\epsilon}$ for n large enough. From (I2), one has

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_0^T \bigl|{}_0D_t^\alpha u_n(t)\bigr|^2\,dt + \sum_{j=1}^m \int_0^{u_n(t_j)} \bigl(I_{1j}(t) + I_{2j}(t)\bigr)\,dt \\
&\quad + \int_0^T \bigl|{}_0D_t^\alpha u_n(t)\bigr|^2 \bigl|u_n(t)\bigr|^2\,dt \\
&\quad - \int_0^T W\bigl(t,u_n(t)\bigr)\,dt \\
&\leq \frac{1}{2}\|u_n\|_\alpha^2 + \sum_{j=1}^m a_{1j}S_\infty \|u_n\|_\alpha + \sum_{j=1}^m a_{2j}S_\infty \|u_n\|_\alpha + \sum_{j=1}^m d_{1j}S_\infty^{\gamma_{1j}+1} \|u_n\|_\alpha^{\gamma_{1j}+1} \\
&\quad + \sum_{j=1}^m d_{2j}S_\infty^{\gamma_{2j}+1} \|u_n\|_\alpha^{\gamma_{2j}+1} + S_\infty^2 \|u_n\|_\alpha^4 - \int_0^T W\bigl(t,u(t)\bigr)\,dt \\
&= \|u_n\|_\alpha^4 \Biggl(\frac{1}{2\|u_n\|_\alpha^2} + \sum_{j=1}^m a_{1j}S_\infty \frac{1}{\|u_n\|_\alpha^3} + \sum_{j=1}^m a_{2j}S_\infty \frac{1}{\|u_n\|_\alpha^3} \\
&\quad + \sum_{j=1}^m d_{1j}S_\infty^{\gamma_{1j}+1} \frac{1}{\|u_n\|_\alpha^{3-\gamma_{1j}}} \\
&\quad + \sum_{j=1}^m d_{2j}S_\infty^{\gamma_{2j}+1} \frac{1}{\|u_n\|_\alpha^{3-\gamma_j}} + S_\infty^2 - \int_0^T \frac{W(t,u_n)}{\|u_n\|_\alpha^4}\,dt \Biggr) \\
&\leq \|u_n\|_\alpha^4 \Biggl(\frac{1}{2\|u_n\|_\alpha^2} + \sum_{j=1}^m a_{1j}S_\infty \frac{1}{\|u_n\|_\alpha^3} + \sum_{j=1}^m a_{2j}S_\infty \frac{1}{\|u_n\|_\alpha^3} \\
&\quad + \sum_{j=1}^m d_{1j}S_\infty^{\gamma_{1j}+1} \frac{1}{\|u_n\|_\alpha^{3-\gamma_{1j}}} \\
&\quad + \sum_{j=1}^m d_{2j}S_\infty^{\gamma_{2j}+1} \frac{1}{\|u_n\|_\alpha^{3-\gamma_j}} + S_\infty^2 - \int_\Pi \frac{W(t,u_n)}{\|u_n\|_\alpha^4}\,dt \Biggr) \\
&\to -\infty \quad \text{if } \|u_n\|_\alpha \to +\infty \text{ as } n \to +\infty,
\end{aligned}$$

which is in contradiction to (3.3). Hence, $\Phi(u)$ is anticoercive. Therefore, there exists a constant $l > 0$ such that $\Phi(u) \leq 0$, $\forall u \in \widetilde{X} \backslash B_l$ for $\zeta \in [0, +\infty)$. $\square$

Lemma 3.3 *If the assumptions* (I2), (I3), (G1), (G2), *and* (G4) *are satisfied,* $\Phi(u)$ *meets the* (*PS*)*-condition, provided that* $\zeta \in [0, +\infty)$.

Proof Let $\{u_n\} \subset E_0^\alpha$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ as $n \to +\infty$, which implies that there exists a constant $\beta > 0$ such that

$$\left|\Phi(u_n)\right| \leq \beta, \qquad \left\|\Phi'(u_n)\right\|_{(E_0^\alpha)^*} \leq \beta.$$

We claim that the sequence $\{u_n\}$ is bounded. If not, let $\|u_n\| \to +\infty$ as $n \to +\infty$. Setting $\omega_n = \frac{u_n}{\|u_n\|_\alpha}$, it follows that ω_n is bounded in E_0^α. Noting that E_0^α is a reflexive Banach space, it implies that $\{\omega_n\}$ has a convergent subsequence (named again $\{\omega_n\}$) such that $\omega_n \rightharpoonup \omega$ in E_0^α and $\omega_n \to \omega$ uniformly in C.

In view of (I2), one has

$$\begin{aligned}
\int_0^T W\big(t, u_n(t)\big)\, dt &= \frac{1}{2} \int_0^T \left|{}_0D_t^\alpha u_n(t)\right|^2 dt + \sum_{j=1}^m \int_0^{u_n(t_j)} \big(I_{1j}(t) + I_{2j}(t)\big)\, dt \\
&\quad + \int_0^T \left|{}_0D_t^\alpha u_n(t)\right|^2 \left|u_n(t)\right|^2 dt - \Phi(u_n) \\
&\leq \frac{1}{2} \|u_n\|_\alpha^2 + \sum_{j=1}^m a_{1j} S_\infty \|u_n\|_\alpha + \sum_{j=1}^m a_{2j} S_\infty \|u_n\|_\alpha \\
&\quad + \sum_{j=1}^m d_{1j} S_\infty^{\gamma_{1j}+1} \|u_n\|_\alpha^{\gamma_{1j}+1} \\
&\quad + \sum_{j=1}^m d_{2j} S_\infty^{\gamma_{2j}+1} \|u_n\|_\alpha^{\gamma_{2j}+1} + S_\infty^2 \|u_n\|_\alpha^4 + \beta,
\end{aligned}$$

which shows that for n large enough,

$$\int_0^T \frac{W(t, u_n)}{\|u_n\|_\alpha^4}\, dt \leq S_\infty^2 + o(1). \tag{3.8}$$

Based on the continuity of g, we can find a constant $\vartheta_1 > 0$ such that

$$\left|ug(t,u) - \theta G(t,u)\right| \leq \vartheta_1 \quad \text{for } |u| \leq L_1, t \in [0,T],$$

which together with (G2) yields

$$ug(t,u) - \theta G(t,u) \geq -M_1 |u|^2 - \vartheta_1 \quad \text{for } |u| \in \mathbb{R}, t \in [0,T]. \tag{3.9}$$

In view of (I3) and (3.9), we have

$$\theta\beta + \beta \|u_n\|_\alpha \geq \theta\Phi(u_n) - \left\langle \Phi'(u_n), u_n \right\rangle$$

$$
\begin{aligned}
&= \left(\frac{\theta}{2}-1\right)\|u_n\|_\alpha^2 + (\theta-4)\int_0^T \left|{}_0D_t^\alpha u_n(t)\right|^2 \left|u_n(t)\right|^2 dt \\
&\quad + \left(\frac{\theta}{2}-1\right)\int_0^T b(t)u_n^2(t)\,dt \\
&\quad + \theta\sum_{j=1}^m \int_0^{u_n(t_j)} \left(I_{1j}(t)+I_{2j}(t)\right)dt - \sum_{j=1}^m \left(I_{1j}\left(u_n(t_j)\right)+I_{1j}\left(u_n(t_j)\right)\right)u_n(t_j) \\
&\quad + \int_0^T \left(u_n(t)g\left(t,u_n(t)\right)-\theta G\left(t,u_n(t)\right)\right)dt - \zeta\frac{\theta-\nu}{\nu}\int_0^T h(t)\left|u_n(t)\right|^\nu dt \\
&\geq \left(\frac{\theta}{2}-1\right)\|u_n\|_\alpha^2 + \int_0^T \left(u_n(t)g\left(t,u_n(t)\right)-\theta G\left(t,u_n(t)\right)\right)dt \\
&\quad + \left(\frac{\theta}{2}-1\right)\int_0^T b(t)u_n^2(t)\,dt - \zeta\frac{\theta-\nu}{\nu}\int_0^T h(t)\left|u_n(t)\right|^\nu dt \\
&\geq \left(\frac{\theta}{2}-1\right)\|u_n\|_\alpha^2 - \left(M_1T + \left(\frac{\theta}{2}-1\right)\|b\|_{L^1}\right)\|u_n\|_\infty^2 \\
&\quad - \zeta\frac{\theta-\nu}{\nu}S_\infty^\nu\|h\|_{L^1}\|u_n\|_\alpha^\nu - \vartheta_1 T,
\end{aligned}
$$

which means that there exists a positive constant ϑ_2 such that

$$\lim_{n\to+\infty}\|\omega_n\|_\infty = \lim_{n\to+\infty}\frac{\|u_n\|_\infty}{\|u_n\|_\alpha} \geq \vartheta_2 > 0.$$

Therefore, we can obtain $\omega \not\equiv 0$. Define

$$\Xi_1 = \left\{t\in[0,T]:\omega\not\equiv 0\right\},\qquad \Xi_2 = [0,T]\setminus\Xi_1.$$

In view of (G1), there exists a constant $\vartheta_3 > 0$ such that $G(t,u)\geq 0$, for$t\in[0,T]$, $|u|\geq\vartheta_3$, which together with (G4) yields that there exist constants $\vartheta_4, \vartheta_5 > 0$ such that

$$G(t,u)\geq -\vartheta_4u^2-\vartheta_5 \quad \text{for } t\in[0,T], u\in\mathbb{R}.$$

Based on Fatou's lemma, it follows that

$$\liminf_{n\to+\infty}\int_{\Xi_2}\frac{G(t,u_n)}{\|u_n\|_\alpha^4}\,dt > -\infty.$$

By (G1), for $t\in[0,T]$, we can obtain that

$$
\begin{aligned}
&\liminf_{n\to+\infty}\int_0^T \frac{G(t,u_n)}{\|u_n\|_\alpha^4}\,dt \\
&\quad = \liminf_{n\to+\infty}\left(\int_{\Xi_1}\frac{G(t,u_n)}{|u_n|^4}|\omega_n|^4\,dt + \int_{\Xi_2}\frac{G(t,u_n)}{|u_n|^4}|\omega_n|^4\,dt\right) \to +\infty,
\end{aligned} \tag{3.10}
$$

which is in contradiction to (3.8). Thus, $\{u_n\}$ is bounded, which implies that $\{u_n\}$ possesses a convergent subsequence (named again $\{u_n\}$) such that $u_n = u_n^+ + u_n^- + u_n^0 \rightharpoonup u = u^+ + u^- + u^0$ and $u_n^+ \rightharpoonup u^+$ in E_0^α. Moreover, $u_n\to u$ and $u_n^+\to u^+$ uniformly in C. It should be mentioned that the dimensions of subspaces E^- and E^0 are finite. Hence, $u_n^- \to u^-$ and $u_n^0 \to u^0$ in E_0^α. Furthermore, if $n \to +\infty$, one has

$$\begin{aligned}
&\langle \Phi'(u_n) - \Phi'(u), u_n^+ - u^+ \rangle \to 0, \\
&\int_0^T b(t)\bigl(u_n(t) - u(t)\bigr)\bigl(u_n^+(t) - u^+(t)\bigr)\,dt \to 0, \\
&\sum_{j=1}^m \bigl(I_{1j}\bigl(u_n(t_j)\bigr) - I_{1j}\bigl(u(t_j)\bigr)\bigr)\bigl(u_n^+(t_j) - u^+(t_j)\bigr) \to 0, \\
&\sum_{j=1}^m \bigl(I_{2j}\bigl(u_n(t_j)\bigr) - I_{2j}\bigl(u(t_j)\bigr)\bigr)\bigl(u_n^+(t_j) - u^+(t_j)\bigr) \to 0, \\
&\int_0^T \bigl(f\bigl(t, u_n(t)\bigr) - f\bigl(t, u(t)\bigr)\bigr)\bigl(u_n^+(t) - u^+(t)\bigr)\,dt \to 0, \\
&\int_0^T \bigl(\bigl|{}_0D_t^\alpha u_n(t)\bigr|^2 u_n(t) - \bigl|{}_0D_t^\alpha u(t)\bigr|^2 u(t)\bigr)\bigl(u_n^+(t) - u^+(t)\bigr)\,dt \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \bigl(\bigl|u_n(t)\bigr|^2 {}_0D_t^\alpha u_n(t) - \bigl|u(t)\bigr|^2 {}_0D_t^\alpha u(t)\bigr)\bigl({}_0D_t^\alpha u_n^+(t) - {}_0D_t^\alpha u^+(t)\bigr)\,dt \\
&\quad = \int_0^T \bigl(\bigl(\bigl|u_n(t)\bigr|^2 - \bigl|u(t)\bigr|^2\bigr){}_0D_t^\alpha u_n(t) - \bigl|u(t)\bigr|^2\bigl({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\bigr)\bigr) \\
&\qquad \times \bigl({}_0D_t^\alpha u_n^+(t) - {}_0D_t^\alpha u^+(t)\bigr)\,dt \\
&\quad = \int_0^T \bigl|u(t)\bigr|^2\bigl|{}_0D_t^\alpha u_n^+(t) - {}_0D_t^\alpha u^+(t)\bigr)\bigr|^2\,dt + o(1),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\langle \Phi'(u_n) - \Phi'(u), u_n^+ - u^+ \rangle \to 0 \\
&\quad = \int_0^T \bigl|{}_0D_t^\alpha u_n^+(t) - {}_0D_t^\alpha u^+(t)\bigr|^2\,dt + \int_0^T \bigl|u(t)\bigr|^2\bigl|{}_0D_t^\alpha u_n^+(t) - {}_0D_t^\alpha u^+(t)\bigr|^2\,dt + o(1).
\end{aligned}$$

Since the norm of E_0^α is equivalent to $\|{}_0D_t^\alpha u\|_{L^2}$, it is clear that $u_n^+ \to u^+$ in E_0^α. Thus, $u_n \to u$ in E_0^α. Therefore, $\Phi(u)$ satisfies the (PS)-condition. □

Proof of Theorem 1.1 From Lemma 3.1, Lemma 3.2, and Lemma 3.3, Theorem 1.1 can be proven immediately by Lemma 2.7. □

Proof of Corollary 1.4 The proof is similar to Theorem 1.1. Therefore, we omit the detail. □

4 Conclusions

By establishing a new variational structure and overcoming the difficulties brought by the influence of impulsive effects, the multiplicity of solutions for a kind of boundary value problem to a fractional quasilinear differential model with impulsive effects is obtained, which extend and enrich some previous results. Moreover, the impulsive effects produced by the quasilinear term $u|_0D_t^\alpha u|^2 + {}_tD_T^\alpha(|u|^2{}_0D_t^\alpha u)$ are more complex than the case of ${}_t^cD_T^\alpha({}_0D_t^\alpha u)$, which makes this problem challenging. Furthermore, there are few papers that consider this problem.

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