Attia and El-Morshedy *Boundary Value Problems* (2023) 2023:80
https://doi.org/10.1186/s13661-023-01743-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

New product-type oscillation criteria for first-order linear differential equations with several nonmonotone arguments

Emad R. Attia[1,2] and Hassan A. El-Morshedy[2*]

*Correspondence:
elmorshedy@yahoo.com
[2]Department of Mathematics, Faculty of Science, Damietta University, New Damietta, 34517, Egypt
Full list of author information is available at the end of the article

Abstract

We use an improved technique to establish new sufficient criteria of product type for the oscillation of the delay differential equation

$$x'(t) + \sum_{l=1}^{m} b_l(t)x\big(\sigma_l(t)\big) = 0, \quad t \geq t_0,$$

with $b_l, \sigma_l \in C([t_0,\infty),[0,\infty))$ such that $\sigma_l(t) \leq t$ and $\lim_{t\to\infty} \sigma_l(t) = \infty$, $l = 1,2,\dots,m$. The obtained results are applicable for the nonmonotone delay case. Their strength is supported by a detailed practical example.

MSC: 34K11; 34K06

Keywords: Oscillation; Delay differential equations; Several nonmonotone delays

1 Introduction

Consider the first-order differential equation with several delays of the form

$$x'(t) + \sum_{l=1}^{m} b_l(t)x\big(\sigma_l(t)\big) = 0, \quad t \geq t_0, \tag{1.1}$$

with $b_l, \sigma_l \in C([t_0,\infty),[0,\infty))$ such that $\sigma_l(t) \leq t$ and $\lim_{t\to\infty} \sigma_l(t) = \infty$, $l = 1,2,\dots,m$.

Let t_{-1} be a real number defined by $t_{-1} = \min_{1\leq l\leq m}\{\inf_{t\geq t_0} \sigma_l(t)\}$. A function $x(t)$ is called a solution of Eq. (1.1) if $x \in C([t_{-1},\infty),\mathbb{R})$ is continuously differentiable on $[t_0,\infty)$ and satisfies Eq. (1.1) for all $t \geq t_0$. If $x(t)$ has arbitrary large zeros, then it is said to be oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory; otherwise, it is nonoscillatory.

Oscillation and delay phenomena appear in various models from real-world applications; see, e.g., [30, 31] for models from mathematical biology, where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. In particular, the oscillation of first-order delay differential equations has numerous applications in the analysis of higher-order differential equations with deviating arguments (e.g., we can investigate the oscillation and asymptotic behavior of higher-order differential equations with deviating arguments by relating the oscillation of these equations to that of associated first-order delay differential equations); see, e.g., [16, 22, 28, 32] for more detail. Indeed, the oscillation of first-order delay differential equations has attracted the attention of many mathematicians; see [1–15, 17–21, 23–27, 29, 33–42] and the references therein.

Note that most known criteria require the delays to be nondecreasing, although in many situations, relaxation of the monotonicity of the delay is required for some equations to be more realistic; see [13]. Indeed, the nonmonotonicity of the delay adds difficulties to the problem. As a result, some known criteria for the monotonic case fail to extend to the nonmonotone one; see Braverman and Karpuz [9]. This motivates us to investigate the oscillation of Eq. (1.1) without restricting the monotonic behavior of the delays. Our focus will be only on the lim sup-type conditions in the product form. Next, we give a brief summary of these criteria. First, we introduce the following important notation:

$$\zeta_{r,l} = \liminf_{t\to\infty} \int_{\sigma_r(t)}^{t} b_l(u)\,du, \quad \zeta_{r,l} \le \frac{1}{\mathrm{e}},$$

$$\zeta_r = \liminf_{t\to\infty} \int_{\sigma_r(t)}^{t} \sum_{l=1}^{m} b_l(u)\,du, \quad \zeta_r \le \frac{1}{\mathrm{e}},$$

$$\zeta = \liminf_{t\to\infty} \int_{\sigma_{\max}(t)}^{t} \sum_{l=1}^{m} b_l(u)\,du, \quad \zeta \le \frac{1}{\mathrm{e}},$$

$$\eta_l = \liminf_{t\to\infty} \int_{\varphi_l(t)}^{t} b_l(u)\,du, \quad \eta_l \le \frac{1}{\mathrm{e}},$$

and

$$\eta = \liminf_{t\to\infty} \int_{\varphi(t)}^{t} \sum_{l=1}^{m} b_l(u)\,du, \quad \eta \le \frac{1}{\mathrm{e}},$$

where $r, l = 1, 2, \dots, m$, and $\varphi_l(t)$ and $\varphi(t)$ are nondecreasing continuous functions such that

$$\sigma_l(t) \le \varphi_l(t), \quad \text{and} \quad \varphi_l(t) \le \varphi(t), \quad t \ge t_1, t_1 \ge t_0, l = 1, 2, \dots, m,$$

and

$$\sigma_{\max}(t) = \max_{1\le l\le m} \sigma_l(t) \quad \text{and} \quad \theta_l(t) = \sup_{t_0\le u\le t} \sigma_l(u), \quad l = 1, 2, \dots, m, t \ge t_0. \tag{1.2}$$

Furthermore, the number $\lambda(\alpha)$ is defined as the smaller real root of the equation $\mathrm{e}^{\alpha z} = z$, and the number $Q(\alpha)$ is defined by

$$Q(\alpha) = \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \quad 0 \le \alpha \le \frac{1}{\mathrm{e}}.$$

The first work in our summary of oscillation criteria is due to Infante et al. [24]. They obtained the following two criteria:

$$\limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} b_{r_1}(u)\mathrm{e}^{\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}\sum_{l=1}^{m} b_l(u_1)\mathrm{e}^{\int_{\sigma_l(u_1)}^{u_1}\sum_{l_1=1}^{m} b_{l_1}(u_2)\,du_2}\,du_1}\,du\right]^{\frac{1}{m}} > \frac{1}{m^m} \tag{1.3}$$

and

$$\limsup_{\epsilon\to 0^+}\left[\limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} b_{r_1}(u)\mathrm{e}^{\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}\sum_{l=1}^{m}(\lambda(\zeta_{l,l})-\epsilon)b_l(u_1)\,du_1}\,du\right]^{\frac{1}{m}}\right] > \frac{1}{m^m}, \tag{1.4}$$

where $\zeta_{l,l} > 0$, $l = 1, 2, \dots, m$.

Koplatadze [25] established the following three conditions:

$$\bar{d} > \frac{1}{\mathrm{e}} \quad \text{and} \quad \limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} b_{r_1}(u)\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}\left(\prod_{r_2=1}^{m} b_{r_2}(u_1)\right)^{\frac{1}{m}} du_1\,du\right]^{\frac{1}{m}} > 0, \tag{1.5}$$

where $\bar{d} = \liminf_{t\to\infty}\sum_{l=1}^{m}\int_{\sigma_l(t)}^{t}(\prod_{r=1}^{m} b_r(u))^{\frac{1}{m}}\,du$,

$$\begin{aligned}&\limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} b_{r_1}(u)\mathrm{e}^{m(\lambda(\bar{d})-\epsilon)\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}(\prod_{r_2=1}^{m} b_{r_2}(u_1))^{\frac{1}{m}}\,du_1}\,du\right]^{\frac{1}{m}}\\&\quad > \frac{1}{m^m} - \frac{\prod_{r=1}^{m} Q(\eta_r)}{m^m},\end{aligned} \tag{1.6}$$

where $0 < \bar{d} \le \frac{1}{\mathrm{e}}$, $\epsilon \in (0, \lambda(\bar{d}))$, and finally

$$\begin{aligned}&\limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} b_{r_1}(u)\mathrm{e}^{m\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}(\prod_{r_2=1}^{m} b_{r_2}(u_1))^{\frac{1}{m}}\Upsilon_i(u_1)\,du_1}\,du\right]^{\frac{1}{m}}\\&\quad > \frac{1}{m^m} - \frac{\prod_{r=1}^{m} Q(\eta_r)}{m^m},\end{aligned} \tag{1.7}$$

where $\Upsilon_1(t) = 0$ and $\Upsilon_i(t) = \mathrm{e}^{\sum_{l=1}^{m}\int_{\sigma_l(t)}^{t}(\prod_{r=1}^{m} b_r(u))^{\frac{1}{m}}\Upsilon_{i-1}(u)\,du}$, $i = 2, 3, \dots$.

Attia et al. [4] introduced the condition

$$\begin{aligned}&\limsup_{t\to\infty}\left(\prod_{r=1}^{m}\left(\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t} W_{r_1}(u)\,du\right)^{\frac{1}{m}} + \frac{\prod_{r=1}^{m} Q(\eta_r)}{m^m}\mathrm{e}^{\sum_{r=1}^{m}\int_{\varphi_r(t)}^{t}\sum_{r_1=1}^{m} b_{r_1}(u)\,du}\right)\\&\quad > \frac{1}{m^m},\end{aligned} \tag{1.8}$$

where

$$W_{r_1}(u) = \mathrm{e}^{\int_{\varphi_{r_1}(u)}^{u}\sum_{r_2=1}^{m} b_{r_2}(u_1)\,du_1}\sum_{r_2=1}^{m} b_{r_2}(u)\int_{\sigma_{r_2}(u)}^{u} b_{r_1}(u_1)\mathrm{e}^{(\lambda(\eta)-\epsilon)\int_{\sigma_{r_2}(u_1)}^{\varphi_{r_1}(u)}\sum_{r_3=1}^{m} b_{r_3}(u_2)\,du_2}\,du_1,$$

with $\eta > 0$, $\epsilon \in (0, \lambda(\eta))$, and $r_1 = 1, 2, \dots, m$.

Bereketoglu et al. [7] defined the sequence $\{\Phi_\ell(t)\}_{\ell\geq 0}$ by

$$\Phi_0(t)=m\left(\prod_{r_1=1}^{m}b_{r_1}(t)\right)^{\frac{1}{m}},$$

$$\Phi_\ell(t)=\sum_{l_1=1}^{m}b_{l_1}(t)\left[1+m\left(\prod_{r=1}^{m}\int_{\varphi_{l_1}(t)}^{t}b_r(u)e^{\int_{\sigma_r(u)}^{t}\Phi_{\ell-1}(u_1)\,du_1}\,du\right)^{\frac{1}{m}}\right],\quad \ell=1,2,\ldots,$$

and obtained the condition

$$\limsup_{t\to\infty}\prod_{l_1=1}^{m}\left[\prod_{l_2=1}^{m}\int_{\varphi_{l_1}(t)}^{t}b_{l_2}(u)e^{\int_{\sigma_{l_2}(u)}^{\varphi_{l_2}(t)}\Phi_\ell(u_1)\,du_1}\,du\right]^{\frac{1}{m}}>\frac{1}{m^m}\left(1-\prod_{r=1}^{m}Q(\eta_r)\right). \tag{1.9}$$

Moremedi et al. [33] established the criterion

$$\limsup_{t\to\infty}\prod_{r=1}^{m}\left[\prod_{r_1=1}^{m}\int_{\varphi_r(t)}^{t}b_{r_1}(u)e^{\int_{\sigma_{r_1}(u)}^{\varphi_{r_1}(t)}\Lambda_\ell(u_1)\,du_1}\,du\right]^{\frac{1}{m}}>\frac{1}{m^m}\left(1-\prod_{r=1}^{m}Q(\eta_r)\right), \tag{1.10}$$

where $\Lambda_0(t)=\sum_{l=1}^{m}b_l(t)$ and

$$\Lambda_\ell(t)=\sum_{l=1}^{m}b_l(t)\left[1+\int_{\varphi_l(t)}^{t}\sum_{l_1=1}^{m}b_{l_1}(u)e^{\int_{\sigma_{l_1}(u)}^{t}\sum_{l_2=1}^{m}b_{l_2}(u_1)e^{\int_{\sigma_{l_2}(u_1)}^{u_1}\Lambda_{\ell-1}(u_2)\,du_2}\,du_1}\,du\right],\quad \ell\in\mathbb{N}.$$

Attia and El-Morshedy [5] improved (1.3) and (1.7) with $i=3$ and obtained the criterion

$$\begin{aligned}&\limsup_{t\to\infty}\left(m\left(\prod_{r=1}^{m}Q(\eta_r)\right)^{1-\frac{1}{m}}\sum_{l=1}^{m}Z_l(t)+\sum_{l=2}^{m}m^l\left(\prod_{l_1=1}^{m}Q(\eta_{l_1})\right)^{1-\frac{l}{m}}\prod_{r=1}^{l}Z_r(t)\right)\\&\quad>1-\prod_{r=1}^{m}Q(\eta_r),\end{aligned} \tag{1.11}$$

where

$$Z_l(t)=\left(\prod_{r=1}^{m}\int_{\varphi_l(t)}^{t}b_r(u)e^{\int_{\sigma_r(u)}^{\varphi_r(t)}\sum_{l_1=1}^{m}b_{l_1}(u_1)e^{(\lambda(\eta)-\epsilon)\int_{\sigma_{l_1}(u_1)}^{u_1}\sum_{l_2=1}^{m}b_{l_2}(u_2)\,du_2}\,du_1}\,du\right)^{\frac{1}{m}}$$

for $l=1,2,\ldots,m$, $\eta>0$, and $\epsilon\in(0,\lambda(\eta))$.

In the next section, we obtain several new oscillation criteria for Eq. (1.1). Moreover, we give a practical example to show that our results can be used to test the oscillation of a certain equation, whereas the criteria listed above fail.

2 Main results

We state some important results for Eq. (1.1) when it possesses a positive solution $x(t)$. In this case, $x(t)$ is eventually nonincreasing and eventually satisfies the inequalities

$$x'(t)+x(\sigma_l(t))b_l(t)\leq 0,\quad l=1,2,\ldots,m,$$

and

$$x'(t) + x\big(\sigma_{\max}(t)\big) \sum_{l=1}^{m} b_l(t) \leq 0.$$

Therefore [42], [24, Lemma 3.1], [19, Lemma 2.1.2], and the nonincreasing nature of $x(t)$ imply respectively, for $l = 1, 2, \dots, m$, that

$$\liminf_{t\to\infty} \frac{x(t)}{x(\varphi_l(t))} \geq Q(\eta_l), \tag{2.1}$$

$$\liminf_{t\to\infty} \frac{x(\sigma_l(t))}{x(t)} \geq \lambda(\zeta_{l,l}), \quad l = 1, 2, \dots, m, \tag{2.2}$$

and

$$\liminf_{t\to\infty} \frac{x(\sigma_l(t))}{x(t)} \geq \liminf_{t\to\infty} \frac{x(\sigma_{\max}(t))}{x(t)} \geq \lambda(\zeta), \tag{2.3}$$

where $\zeta, \zeta_{l,l} > 0$.

If nothing else is stated, all inequalities are assumed to hold eventually.

Lemma 2.1 *If $x(t)$ is an eventually positive solution of Eq.* (1.1), *then*

$$\liminf_{t\to\infty} \frac{x(\sigma_r(t))}{x(t)} \geq \mathrm{e}^{\max\{\sum_{l=1}^{m} \zeta_{r,l}\lambda_l^*, \lambda^*(\zeta)\zeta_r\}}, \quad r = 1, 2, \dots, m, \tag{2.4}$$

where $\lambda_l^* = \max\{\lambda^*(\zeta_{l,l}), \lambda^*(\zeta)\}$, *and*

$$\lambda^*(z) = \begin{cases} 1, & z = 0, \\ \lambda(z), & z > 0. \end{cases}$$

Proof Dividing Eq. (1.1) by $x(t)$ and integrating from u to t, $u \leq t$, we obtain

$$-\ln\left(\frac{x(t)}{x(u)}\right) = \sum_{l=1}^{m} \int_u^t b_l(u_1) \frac{x(\sigma_l(u_1))}{x(u_1)}\, du_1,$$

which is equivalent to

$$x(u) = x(t)\mathrm{e}^{\int_u^t \sum_{l=1}^{m} b_l(u_1) \frac{x(\sigma_l(u_1))}{x(u_1)}\, du_1}. \tag{2.5}$$

Therefore

$$\frac{x(\sigma_r(t))}{x(t)} = \mathrm{e}^{\int_{\sigma_r(t)}^t \sum_{l=1}^{m} b_l(u_1) \frac{x(\sigma_l(u_1))}{x(u_1)}\, du_1}. \tag{2.6}$$

Equation (2.6) leads to the following two inequalities, using (2.2) and (2.3), for all sufficiently small $\epsilon > 0$:

$$\frac{x(\sigma_r(t))}{x(t)} \geq \mathrm{e}^{\sum_{l=1}^{m} (\zeta_{r,l}-\epsilon)(\lambda_l^*-\epsilon)}$$

and

$$\frac{x(\sigma_r(t))}{x(t)} \geq \mathrm{e}^{(\lambda^*(\zeta)-\epsilon)(\zeta_r-\epsilon)}.$$

Now taking the lower limits as $t \to \infty$ and then letting $\epsilon \to 0$, we get

$$\liminf_{t\to\infty} \frac{x(\sigma_r(t))}{x(t)} \geq \mathrm{e}^{\sum_{l=1}^{m} \zeta_{r,l}\lambda_l^*}$$

and

$$\liminf_{t\to\infty} \frac{x(\sigma_r(t))}{x(t)} \geq \mathrm{e}^{\lambda^*(\zeta)\zeta_r}.$$

The last two inequalities are equivalent to (2.4). □

For an easy reference, the sequences $\{\Omega_r^{(n)}(t)\}_{n\geq 0}$, $r = 1,2,\dots,m$, are defined as follows:

$$\Omega_r^{(0)}(t) = \begin{cases} 1 & \zeta = \zeta_{r,l} = 0 \text{ for all } l = 1,2,\dots,m, \\ \mathrm{e}^{\max\{\sum_{l=1}^{m} \zeta_{r,l}\lambda_l^*,\lambda^*(\zeta)\zeta_r\}} - \epsilon_r & \text{otherwise,} \end{cases}$$

$$\Omega_r^{(n)}(t) = \frac{\mathrm{e}^{\int_{\varphi_r(t)}^{t} \sum_{k=1, k\neq r}^{m} b_k(u)\Omega_k^{(n-1)}(u)\,du}}{1 - G_{r,r}^{(n-1)}(t)}, \quad n = 1,2,\dots,$$

where $\epsilon_r \in (0, \mathrm{e}^{\max\{\sum_{l=1}^{m} \zeta_{r,l}\bar{\lambda}_l,\lambda(\zeta)\zeta_r\}})$, and

$$G_{i,k}^{(n)}(t) = \int_{\varphi_i(t)}^{t} b_k(u)\mathrm{e}^{\int_{\sigma_k(u)}^{\varphi_k(t)} \sum_{l=1}^{m} b_l(u_1)\Omega_l^{(n)}(u_1)\,du_1}\,du, \quad i,k = 1,2,\dots,m.$$

Lemma 2.2 *Assume that $x(t)$ is an eventually positive solution of Eq.* (1.1), $n \in \mathbb{N}_0$, *and* $j \in \{1,2,\dots,m\}$. *Then the inequalities $G_{j,j}^{(n)}(t) < 1$ and*

$$\prod_{r=1}^{m}\left(\frac{1}{1-G_{r,r}^{(n)}(t)}\right)\left(\prod_{r=1}^{m}\left(\frac{x(t)}{x(\varphi_r(t))}\right) + (m-1)^m \prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m} G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right) \leq 1 \tag{2.7}$$

are satisfied.

Proof Since $x(t)$ is an eventually positive solution of Eq. (1.1), for any sufficiently small $\epsilon_r > 0$, inequality (2.4) yields

$$\frac{x(\sigma_r(t))}{x(t)} > \mathrm{e}^{\max\{\sum_{l=1}^{m} \zeta_{r,l}\lambda_l^*,\lambda^*(\zeta)\zeta_r\}} - \epsilon_r, \quad \zeta > 0 \text{ or } \zeta_{r,l} > 0 \text{ for some } l = 1,2,\dots,m.$$

Combining this inequality with the fact that $\frac{x(\sigma_r(t))}{x(t)} \geq 1$, we obtain

$$\frac{x(\sigma_r(t))}{x(t)} \geq \Omega_r^{(0)}(t). \tag{2.8}$$

Integrating Eq. (1.1) from $\varphi_i(t)$ to t, $i = 1, 2, \ldots, m$, we get

$$x(t) - x\big(\varphi_i(t)\big) + \int_{\varphi_i(t)}^{t} b_i(u)x\big(\sigma_i(u)\big)\,du + \sum_{\substack{l=1\\ l\neq i}}^{m} \int_{\varphi_i(t)}^{t} b_l(u)x\big(\sigma_l(u)\big)\,du = 0. \tag{2.9}$$

On the other hand, proceeding as in the proof of Lemma 2.1, we obtain (2.5), which yields

$$x\big(\sigma_i(u)\big) = x\big(\varphi_i(t)\big)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}, \quad \varphi_i(t) \le u \le t,$$

and

$$x\big(\sigma_l(u)\big) = \frac{x(\sigma_l(u))}{x(u)}x(u) = x(t)\frac{x(\sigma_l(u))}{x(u)}\mathrm{e}^{\int_u^t \sum_{l_1=1}^{m} b_{l_1}(u_1)\frac{x(\sigma_{l_1}(u_1))}{x(u_1)}\,du_1}, \quad \varphi_i(t) \le u \le t.$$

Substituting into (2.9), we get

$$\begin{aligned} x\big(\varphi_i(t)\big) = {} & x(t) + x\big(\varphi_i(t)\big)\int_{\varphi_i(t)}^{t} b_i(u)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}\,du \\ & + x(t)\sum_{\substack{l=1\\ l\neq i}}^{m} \int_{\varphi_i(t)}^{t} b_l(u)\frac{x(\sigma_l(u))}{x(u)}\mathrm{e}^{\int_u^t \sum_{l_1=1}^{m} b_{l_1}(u_1)\frac{x(\sigma_{l_1}(u_1))}{x(u_1)}\,du_1}\,du. \end{aligned}$$

Therefore

$$\begin{aligned} x\big(\varphi_i(t)\big) \ge {} & x(t) + x\big(\varphi_i(t)\big)\int_{\varphi_i(t)}^{t} b_i(u)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}\,du \\ & + x(t)\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\frac{x(\sigma_l(u))}{x(u)}\mathrm{e}^{\int_u^t \sum_{\substack{j=1\\ j\neq i}}^{m} b_j(u_1)\frac{x(\sigma_j(u_1))}{x(u_1)}\,du_1}\,du, \end{aligned}$$

that is,

$$\begin{aligned} x\big(\varphi_i(t)\big) \ge {} & x\big(\varphi_i(t)\big)\int_{\varphi_i(t)}^{t} b_i(u)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}\,du \\ & + x(t)\mathrm{e}^{\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\frac{x(\sigma_l(u))}{x(u)}\,du}. \end{aligned}$$

Hence

$$\frac{x(\varphi_i(t))}{x(t)} \ge \frac{\mathrm{e}^{\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\frac{x(\sigma_l(u))}{x(u)}\,du}}{1 - \int_{\varphi_i(t)}^{t} b_i(u)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}\,du}.$$

Now by (2.8) it follows that

$$\frac{x(\varphi_i(t))}{x(t)} \ge \frac{\mathrm{e}^{\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\Omega_l^{(0)}(u)\,du}}{1 - \int_{\varphi_i(t)}^{t} b_i(u)\mathrm{e}^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\Omega_l^{(0)}(u_1)\,du_1}\,du}$$

$$= \frac{e^{\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\Omega_l^{(0)}(u)\,du}}{1-G_{i,i}^{(0)}(t)} = \Omega_i^{(1)}(t).$$

Continuing in this way, we can prove that

$$\frac{x(\varphi_i(t))}{x(t)} \geq \frac{e^{\int_{\varphi_i(t)}^{t} \sum_{\substack{l=1\\ l\neq i}}^{m} b_l(u)\Omega_l^{(n-1)}(u)\,du}}{1-G_{i,i}^{(n-1)}(t)} = \Omega_i^{(n)}(t), \quad n \geq 1. \tag{2.10}$$

Returning to (2.5), we obtain

$$x\bigl(\sigma_i(u)\bigr) = x\bigl(\varphi_i(t)\bigr)e^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}, \quad \varphi_i(t) \leq u \leq t. \tag{2.11}$$

Therefore (2.9) implies that

$$\begin{aligned} x\bigl(\varphi_i(t)\bigr) &= x(t) + x\bigl(\varphi_i(t)\bigr)\int_{\varphi_i(t)}^{t} b_i(u)e^{\int_{\sigma_i(u)}^{\varphi_i(t)} \sum_{l=1}^{m} b_l(u_1)\frac{x(\sigma_l(u_1))}{x(u_1)}\,du_1}\,du \\ &\quad + \sum_{\substack{l=1\\ l\neq i}}^{m} x\bigl(\varphi_l(t)\bigr)\int_{\varphi_i(t)}^{t} b_l(u)e^{\int_{\sigma_l(u)}^{\varphi_l(t)} \sum_{l_1=1}^{m} b_{l_1}(u_1)\frac{x(\sigma_{l_1}(u_1))}{x(u_1)}\,du_1}\,du. \end{aligned}$$

However, (2.10) leads to

$$\frac{x(\sigma_l(u_1))}{x(u_1)} \geq \frac{x(\varphi_i(t))}{x(t)} \geq \Omega_l^{(n)}(t).$$

Consequently, the previous equation leads to the inequality

$$x\bigl(\varphi_i(t)\bigr)\bigl(1-G_{i,i}^{(n)}(t)\bigr) \geq x(t) + \sum_{\substack{l=1\\ l\neq i}}^{m} x\bigl(\varphi_l(t)\bigr)G_{i,l}^{(n)}(t) > 0.$$

This proves that $G_{i,i}^{(n)}(t) < 1$ and

$$\frac{x(\varphi_i(t))}{x(t)} \geq \frac{1+\sum_{\substack{l=1\\ l\neq i}}^{m} \frac{x(\varphi_l(t))}{x(t)}G_{i,l}^{(n)}(t)}{1-G_{i,i}^{(n)}(t)}.$$

Then the arithmetic–geometric mean leads to

$$\frac{x(\varphi_i(t))}{x(t)} \geq \frac{1+(m-1)(\prod_{\substack{r=1\\ r\neq i}}^{m} \frac{x(\varphi_r(t))}{x(t)})^{\frac{1}{m-1}}(\prod_{\substack{r=1\\ r\neq i}}^{m} G_{i,r}^{(n)}(t))^{\frac{1}{m-1}}}{1-G_{i,i}^{(n)}(t)}.$$

Taking the product of both sides, we get

$$\begin{aligned} &\prod_{r=1}^{m}\left(\frac{x(\varphi_r(t))}{x(t)}\right) \\ &\quad \geq A^{(n)}(t)\left(1+(m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}\frac{x(\varphi_{r_1}(t))}{x(t)}\right)^{\frac{1}{m-1}}\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right), \end{aligned}$$

where $A^{(n)}(t)=\prod_{r=1}^{m}(\frac{1}{1-G_{r,r}^{(n)}(t)})$. Therefore

$$\prod_{r=1}^{m}\left(\frac{x(\varphi_r(t))}{x(t)}\right) \geq A^{(n)}(t)\left(1+(m-1)^m\left(\left(\prod_{r=1}^{m}\frac{x(\varphi_r(t))}{x(t)}\right)^{m-1}\right)^{\frac{1}{m-1}}\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right).$$

Thus

$$\prod_{r=1}^{m}\left(\frac{x(\varphi_r(t))}{x(t)}\right)\geq A^{(n)}(t)\left(1+(m-1)^m\prod_{r=1}^{m}\left(\frac{x(\varphi_r(t))}{x(t)}\right)\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right).$$

Then

$$A^{(n)}(t)\left(\prod_{r=1}^{m}\left(\frac{x(t)}{x(\varphi_r(t))}\right)+(m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right)\leq 1.$$

□

Theorem 2.1 *Assume that* $i\in\{1,2,\dots,m\}$ *and either one of the following conditions is satisfied for some* $n\in\mathbb{N}_0$:

(i) *there exists a sequence* $\{c_k\}_{k\geq 0}$ *such that* $\lim_{k\to\infty}c_k=\infty$ *and*

$$G_{i,i}^{(n)}(c_k)\geq 1 \quad \text{for all } k\in\mathbb{N}_0, \tag{2.12}$$

(ii)

$$\limsup_{t\to\infty}\left(\prod_{r=1}^{m}\frac{1}{1-G_{r,r}^{(n)}(t)}\left(\prod_{r=1}^{m}Q(\eta_r)+(m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right)\right) > 1. \tag{2.13}$$

Then Eq. (1.1) *is oscillatory.*

Proof We assume for contradiction that Eq. (1.1) has a nonoscillatory solution $x(t)$. Because of the linearity of Eq. (1.1), there is no loss of generality to assume the existence of a sufficiently large $T\geq t_0$ such that $x(t)>0$ for all $t\geq T$. Then Lemma 2.2 leads to $G_{i,i}^{(n)}(t)<1$ for all $i=1,2,\dots,m$ and $n\in\mathbb{N}_0$. This contradicts (2.12) and hence proves (i). For the proof of (ii), we see from (2.1) and (2.7) that

$$\limsup_{t\to\infty}\left(\prod_{r=1}^{m}\frac{1}{1-G_{r,r}^{(n)}(t)}\left(\prod_{r=1}^{m}Q(\eta_r)+(m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m}G_{r,r_1}^{(n)}(t)\right)^{\frac{1}{m-1}}\right)\right)\leq 1,$$

which is impossible due to (2.13). □

Next, we define the functions $C_r^{(n)}(t)$ and $D_r^{(n)}(t)$ for some $n \in \mathbb{N}_0$ as follows:

$$\begin{aligned} C_r^{(n)}(t) = {} & \int_t^{\varphi_r^{-1}(t)} b_r(u)\,du \\ & + \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^t \sum_{\substack{l=1 \\ l\neq r}}^m b_l(u_1) e^{\int_{\sigma_l(u_1)}^t \sum_{l_1=1}^m b_{l_1}(u_2)\Omega_{l_1}^{(n)}(u_2)\,du_2}\,du_1\,du \\ & + \int_t^{\varphi_r^{-1}(t)} \sum_{\substack{l=1 \\ l\neq r}}^m b_l(u) e^{\int_{\sigma_l(u)}^t \sum_{l_1=1}^m b_{l_1}(u_1)\Omega_{l_1}^{(n)}(u_1)\,du_1}\,du, \end{aligned}$$

and

$$D_r^{(n)}(t) = \frac{\int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^t b_r(u_1)\,du_1\,du}{1 - C_r^{(n)}(t)},$$

where $\varphi_r(t)$ are strictly increasing functions for all $r = 1, 2, \dots, m$.

Theorem 2.2 *Assume that the function $\varphi_r(t)$ is strictly increasing for each $r = 1, 2, \dots, m$. Suppose that for some $n \in \mathbb{N}_0$,*

(i) *there exists a sequence $\{d_k\}_{k\geq 0}$ such that $\lim_{k\to\infty} d_k = \infty$,*

$$C_r^{(n)}(d_k) \geq 1 \quad \textit{for some } r \in \{1, 2, \dots, m\} \textit{ and all } k \in \mathbb{N}_0, \tag{2.14}$$

or

(ii)

$$\limsup_{t\to\infty} \left(\prod_{r=1}^m \left(\frac{1}{1 - G_{r,r}^{(n)}(t)} \right) \left(\prod_{r=1}^m D_r^{(n)}(t) + (m-1)^m \prod_{r=1}^m \left(\prod_{\substack{r_1=1 \\ r_1\neq r}}^m G_{r,r_1}^{(n)}(t) \right)^{\frac{1}{m-1}} \right) \right) > 1. \tag{2.15}$$

Then Eq. (1.1) *is oscillatory.*

Proof As in the proof of the previous theorem, we assume that Eq. (1.1) has an eventually positive solution $x(t)$. Integrating Eq. (1.1) from t to $\varphi_r^{-1}(t)$, we have

$$x\big(\varphi_r^{-1}(t)\big) - x(t) + \int_t^{\varphi_r^{-1}(t)} \sum_{l=1}^m b_l(u) x\big(\sigma_l(u)\big)\,du = 0,$$

that is,

$$x\big(\varphi_r^{-1}(t)\big) - x(t) + \int_t^{\varphi_r^{-1}(t)} b_r(u) x\big(\sigma_r(u)\big)\,du + \int_t^{\varphi_r^{-1}(t)} \sum_{\substack{l=1 \\ l\neq r}}^m b_l(u) x\big(\sigma_l(u)\big)\,du = 0. \tag{2.16}$$

Again, integrating Eq. (1.1) from $\sigma_r(u)$ to $t \le u \le \varphi_r^{-1}(t)$, we obtain

$$x(\sigma_r(u)) = x(t) + \int_{\sigma_r(u)}^{t} \sum_{l=1}^{m} b_l(u_1) x(\sigma_l(u_1))\, du_1.$$

Substituting into (2.16), we get

$$\begin{aligned} x(t) = {} & x(\varphi_r^{-1}(t)) + x(t)\int_t^{\varphi_r^{-1}(t)} b_r(u)\, du + \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} b_r(u_1) x(\sigma_r(u_1))\, du_1\, du \\ & + \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u_1) x(\sigma_l(u_1))\, du_1\, du + \int_t^{\varphi_r^{-1}(t)} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u) x(\sigma_l(u))\, du. \end{aligned}$$

Recalling that (2.5) holds and $x(t)$ is nonincreasing, it follows that

$$\begin{aligned} x(t) \ge {} & x(\varphi_r^{-1}(t)) + x(t)\int_t^{\varphi_r^{-1}(t)} b_r(u)\, du + x(\varphi_r(t)) \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} b_r(u_1)\, du_1\, du \\ & + x(t) \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u_1) \mathrm{e}^{\int_{\sigma_l(u_1)}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_2) \frac{x(\sigma_{l_1}(u_2))}{x(u_2)}\, du_2}\, du_1\, du \\ & + x(t) \int_t^{\varphi_r^{-1}(t)} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u) \mathrm{e}^{\int_{\sigma_l(u)}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_1) \frac{x(\sigma_{l_1}(u_1))}{x(u_1)}\, du_1}\, du. \end{aligned}$$

Since $\frac{x(\sigma_{l_1}(u_2))}{x(u_2)} \ge \Omega_{l_1}^{(n)}(u_2)$ (from (2.11)), we have

$$\begin{aligned} x(t) \ge {} & x(\varphi_r^{-1}(t)) + x(t)\int_t^{\varphi_r^{-1}(t)} b_r(u)\, du + x(\varphi_r(t)) \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} b_r(u_1)\, du_1\, du \\ & + x(t) \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u_1) \mathrm{e}^{\int_{\sigma_l(u_1)}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_2) \Omega_{l_1}^{(n)}(u_2)\, du_2}\, du_1\, du \\ & + x(t) \int_t^{\varphi_r^{-1}(t)} \sum_{\substack{l=1 \\ l\neq r}}^{m} b_l(u) \mathrm{e}^{\int_{\sigma_l(u)}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_1) \Omega_{l_1}^{(n)}(u_1)\, du_1}\, du. \end{aligned}$$

Therefore

$$x(t)\bigl(1 - C_r^{(n)}(t)\bigr) \ge x(\varphi_r^{-1}(t)) + x(\varphi_r(t)) \int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} b_r(u_1)\, du_1\, du > 0, \tag{2.17}$$

which leads to $C_r^{(n)}(t) < 1$. This contradicts (2.14) and completes the proof of (i).

To prove (ii), we notice from (2.17) that

$$\frac{x(t)}{x(\varphi_r(t))} > \frac{\int_t^{\varphi_r^{-1}(t)} b_r(u) \int_{\sigma_r(u)}^{t} b_r(u_1)\, du_1\, du}{1 - C_r^{(n)}(t)} = D_r^{(n)}(t).$$

Substituting into (2.7) and then taking the upper limit of both sides, we get a contradiction with (2.15). The proof of the theorem is complete. □

Corollary 2.1 *Let $q_k, \mu_k > 0$ be such that $\sigma_k(t) \le t - \mu_k$, $b_k(t) \ge q_k$ on $(a_j, a_j + 3\mu^*)$, $k \in \{1, 2, \dots, m\}$ and $j \in \mathbb{N}_0$, $\mu^* = \max \mu_{k_{1\le k\le m}}$, and $\lim_{j\to\infty} a_j = \infty$. If*

$$\prod_{r=1}^{m} \frac{1}{1 - D_{r,r}} \left(\prod_{r=1}^{m} Q(\eta_r) + (m-1)^m \prod_{r=1}^{m} \left(\prod_{\substack{r_1=1 \\ r_1 \ne r}}^{m} D_{r,r_1} \right)^{\frac{1}{m-1}} \right) > 1, \tag{2.18}$$

where $D_{i,k} = \frac{q_k}{B}(\mathrm{e}^{\mu_i B} - 1)$ and $B = \sum_{l=1}^{m} \frac{q_l}{1-\mu_l q_l}$, $i = 1, 2, \dots, m$, then Eq. (1.1) is oscillatory.

Proof Let $\varphi_k(t) = t - \mu_k$, $k = 1, 2, \dots, m$. Then

$$G_{k,k}^{(0)}(t) = \int_{\varphi_k(t)}^{t} b_k(u) \mathrm{e}^{\int_{\sigma_k(u)}^{\varphi_k(t)} \sum_{l=1}^{m} b_l(u_1)\Omega_l^{(0)}(u_1)\,du_1}\,du \ge \int_{t-\mu_k}^{t} b_k(u)\,du.$$

This leads to

$$\Omega_k^{(1)}(t) = \frac{\mathrm{e}^{\int_{\varphi_k(t)}^{t} \sum_{\substack{l=1 \\ l\ne k}}^{m} b_l(u)\Omega_l^{(0)}(u)\,du}}{1 - G_{k,k}^{(0)}(t)} \ge \frac{1}{1 - \int_{t-\mu_k}^{t} b_k(u)\,du}. \tag{2.19}$$

Also,

$$\begin{aligned} G_{i,k}^{(1)}(t) &= \int_{\varphi_i(t)}^{t} b_k(u) \mathrm{e}^{\int_{\sigma_k(u)}^{\varphi_k(t)} \sum_{l=1}^{m} b_l(u_1)\Omega_l^{(1)}(u_1)\,du_1}\,du \\ &\ge \int_{t-\mu_i}^{t} b_k(u) \mathrm{e}^{\int_{u-\mu_k}^{t-\mu_k} \sum_{l=1}^{m} b_l(u_1)\Omega_l^{(1)}(u_1)\,du_1}\,du \\ &\ge \int_{t-\mu_i}^{t} b_k(u) \mathrm{e}^{\int_{u-\mu_k}^{t-\mu_k} \sum_{l=1}^{m} \frac{b_l(u_1)}{1-\int_{u_1-\mu_l}^{u_1} b_l(u_2)\,du_2}\,du_1}\,du, \quad i = 1, 2, \dots, m. \end{aligned}$$

Therefore

$$\begin{aligned} G_{i,k}^{(1)}(a_j + 3\mu_k) &\ge \int_{a_j+3\mu_k-\mu_i}^{a_j+3\mu_k} q_k \mathrm{e}^{\int_{u-\mu_k}^{a_j+3\mu_k-\mu_k} \sum_{l=1}^{m} \frac{q_l}{1-\mu_l q_l}\,du_1}\,du \\ &= \frac{q_k}{B}\left(\mathrm{e}^{\mu_i B} - 1\right) = D_{i,k}. \end{aligned} \tag{2.20}$$

Now let

$$I(t) = \prod_{r=1}^{m} \frac{1}{1 - G_{r,r}^{(1)}(t)} \left(\prod_{r=1}^{m} Q(\eta_r) + (m-1)^m \prod_{r=1}^{m} \left(\prod_{\substack{r_1=1 \\ r_1 \ne r}}^{m} G_{r,r_1}^{(n)}(t) \right)^{\frac{1}{m-1}} \right).$$

Then (2.20) leads to

$$I(a_j + 3\mu_k) \ge \prod_{r=1}^{m} \frac{1}{1 - D_{r,r}} \left(\prod_{r=1}^{m} Q(\eta_r) + (m-1)^m \prod_{r=1}^{m} \left(\prod_{\substack{r_1=1 \\ r_1 \ne r}}^{m} D_{r,r_1} \right)^{\frac{1}{m-1}} \right) > 1.$$

It follows that (2.13) with $n = 1$ is satisfied, and hence (ii) of Theorem 2.1 guarantees the oscillation of Eq. (1.1). □

Corollary 2.2 *Let $q_k, \mu_k > 0$ be such that $\sigma_k(t) \le t - \mu_k$, $b_k(t) \ge q_k$ on $(a_j, a_j + 4\mu^*)$, $k \in \{1, 2, \dots, m\}$, and $j \in \mathbb{N}$, $\mu^* = \max \mu_{k_{1\le k\le m}}$, and $\lim_{j\to\infty} a_j = \infty$. If*

$$\prod_{r=1}^{m} \frac{1}{1 - D_{r,r}} \left(\prod_{r=1}^{m} \frac{q_r^2 \mu_r^2}{2(1 - H_r)} + (m-1)^m \prod_{r=1}^{m} \left(\prod_{\substack{r_1=1 \\ r_1 \ne r}}^{m} D_{r,r_1} \right)^{\frac{1}{m-1}} \right) > 1, \tag{2.21}$$

where B, $D_{i,k}$ are defined as in Corollary 2.1, and

$$H_k = q_k \mu_k + q_k \left(1 - \mathrm{e}^{-B\mu_k}\right) \sum_{\substack{l=1 \\ l \ne k}}^{m} q_l \frac{\mathrm{e}^{B\mu_l}}{B} + \left(\mathrm{e}^{B\mu_k} - B\mu_k - 1\right) \sum_{\substack{l=1 \\ l \ne k}}^{m} q_l \frac{\mathrm{e}^{B\mu_l}}{B^2},$$

then Eq. (1.1) is oscillatory.

Proof Let $\varphi_k(t) = t - \mu_k$, $k = 1, 2, \dots, m$. Then

$$\begin{aligned} C_k^{(1)}(t) \ge{}& \int_t^{t+\mu_k} b_k(u)\, du \\ &+ \int_t^{t+\mu_k} b_k(u) \int_{u-\mu_k}^{t} \sum_{\substack{l=1 \\ l \ne k}}^{m} b_l(u_1) \mathrm{e}^{\int_{u_1-\mu_l}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_2)\Omega_{l_1}^{(1)}(u_2)\, du_2}\, du_1\, du \\ &+ \int_t^{t+\mu_k} \sum_{\substack{l=1 \\ l \ne k}}^{m} b_l(u) \mathrm{e}^{\int_{u-\mu_l}^{t} \sum_{l_1=1}^{m} b_{l_1}(u_1)\Omega_{l_1}^{(1)}(u_1)\, du_1}\, du. \end{aligned}$$

In view of (2.19), we have

$$\begin{aligned} C_k^{(1)}(t) \ge{}& \int_t^{t+\mu_k} b_k(u)\, du \\ &+ \int_t^{t+\mu_k} b_k(u) \int_{u-\mu_k}^{t} \sum_{\substack{l=1 \\ l \ne k}}^{m} b_l(u_1) \mathrm{e}^{\int_{u_1-\mu_l}^{t} \sum_{l_1=1}^{m} \frac{b_{l_1}(u_2)}{1 - \int_{u_2-\mu_{l_1}}^{u_2} b_{l_1}(u_3)\, du_3}\, du_2}\, du_1\, du \\ &+ \int_t^{t+\mu_k} \sum_{\substack{l=1 \\ l \ne k}}^{m} b_l(u) \mathrm{e}^{\int_{u-\mu_l}^{t} \sum_{l_1=1}^{m} \frac{b_{l_1}(u_1)}{1 - \int_{u_1-\mu_{l_1}}^{u_1} b_{l_1}(u_2)\, du_2}\, du_1}\, du. \end{aligned}$$

Thus

$$\begin{aligned} C_k^{(1)}(a_j + 3\mu_k) \ge{}& \mu_k q_k + q_k \sum_{\substack{l=1 \\ l \ne k}}^{m} q_l \int_{a_j+3\mu_k}^{a_j+4\mu_k} \int_{u-\mu_k}^{a_j+3\mu_k} \mathrm{e}^{\int_{u_1-\mu_l}^{a_j+3\mu_k} B\, du_2}\, du_1\, du \\ &+ \sum_{\substack{l=1 \\ l \ne k}}^{m} q_l \int_{a_j+3\mu_k}^{a_j+4\mu_k} \mathrm{e}^{\int_{u-\mu_l}^{a_j+3\mu_k} B\, du_1}\, du, \end{aligned}$$

that is,

$$C_k^{(1)}(a_j+3\mu_k) \ge q_k\mu_k + q_k\left(1-\mathrm{e}^{-B\mu_k}\right)\sum_{\substack{l=1\\ l\neq k}}^{m} q_l\frac{\mathrm{e}^{B\mu_l}}{B} + \left(\mathrm{e}^{B\mu_k}-B\mu_k-1\right)\sum_{\substack{l=1\\ l\neq k}}^{m} q_l\frac{\mathrm{e}^{B\mu_l}}{B^2}$$
$$= H_k. \tag{2.22}$$

Also,

$$\int_{a_j+3\mu_k}^{\varphi_k^{-1}(a_j+3\mu_k)} b_k(u)\int_{\sigma_k(u)}^{a_j+3\mu_k} b_k(u_1)\,du_1\,du \ge q_k^2\int_{a_j+3\mu_k}^{a_j+4\mu_k}\int_{u-\mu_k}^{a_j+3\mu_k} du_1\,du = \frac{1}{2}q_k^2\mu_k^2.$$

This inequality and (2.22) lead to

$$D_k^{(1)}(a_j+3\mu_k) = \frac{\int_{a_j+3\mu_k}^{\varphi_k^{-1}(a_j+3\mu_k)} b_k(u)\int_{\sigma_k(u)}^{a_j+3\mu_k} b_k(u_1)\,du_1\,du}{1-C_k^{(1)}(a_j+3\mu_k)} \ge \frac{q_k^2\mu_k^2}{2(1-H_k)}. \tag{2.23}$$

Let

$$I_1(t) = \prod_{r=1}^{m}\left(\frac{1}{1-G_{r,r}^{(1)}(t)}\right)\left(\prod_{r=1}^{m} D_r^{(1)}(t) + (m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m} G_{r,r_1}^{(1)}(t)\right)^{\frac{1}{m-1}}\right).$$

Then (2.20), (2.21), and (2.23) imply that

$$I_1(a_j+3\mu_k) \ge \prod_{r=1}^{m}\frac{1}{1-D_{r,r}}\left(\prod_{r=1}^{m}\frac{q_r^2\mu_r^2}{2(1-H_r)} + (m-1)^m\prod_{r=1}^{m}\left(\prod_{\substack{r_1=1\\ r_1\neq r}}^{m} D_{r,r_1}\right)^{\frac{1}{m-1}}\right) > 1.$$

Therefore condition (2.15) with $n=1$ is satisfied, so Eq. (1.1) is oscillatory. The proof is complete. □

The following illustrative example highlights the significance of some of our results. All calculations are done using a Maple code.

Example 2.1 Consider the equation

$$x'(t) + \sum_{l=1}^{2} b_l(t)x\big(\sigma_l(t)\big) = 0, \quad t \ge 2, \tag{2.24}$$

where $\sigma_2(t) = t - 1 - 0.0001\sin^2(20000\pi t)$, and

$$\sigma_1(t) = \begin{cases} t-0.1, & t\in[4k,4k+3],\\ \frac{1}{\delta}[(0.1-\delta)(4k+3-t)]+4k+2.9, & t\in[4k+3,4k+3+\delta],\\ \frac{1.1-\delta}{1-\delta}(t-4k-3-\delta)+4k+2.8+\delta, & t\in[4k+3+\delta,4k+4], \end{cases} \quad k\in\mathbb{N},$$

where $0 < \delta < 0.1$. Also,

$$b_1(t) = \begin{cases} 0, & t \in [b_k, c_k], \\ \frac{1}{2e\delta}(t - c_k), & t \in [c_k, c_k + \delta], \\ \frac{1}{2e} & t \in [c_k + \delta, c_k + 3.0001 + \delta], \\ \frac{1}{2e}(1 - \frac{t - c_k - 3.0001 - \delta}{b_{k+1} - c_k - 3.0001 - \delta}), & t \in [c_k + 3.0001 + \delta, b_{k+1}], \end{cases} \quad k \in \mathbb{N}_0,$$

and

$$b_2(t) = \begin{cases} 0, & t \in [b_k, c_k], \\ \frac{\beta}{\delta}(t - c_k), & t \in [c_k, c_k + \delta], \\ \beta & t \in [c_k + \delta, c_k + 3.0001 + \delta], \\ \beta(1 - \frac{t - c_k - 3.0001 - \delta}{b_{k+1} - e_k - 3.0001 - \delta}), & t \in [c_k + 3.0001 + \delta, b_{k+1}], \end{cases} \quad k \in \mathbb{N}_0,$$

where $\beta \geq 0$, and $\{b_k\}_{k\geq 0}$, $\{c_k\}_{k\geq 0}$ are sequences of positive integers such that $c_k > b_k + 1$, $b_{k+1} > c_k + 3.0001 + \delta$, and $\lim_{k\to\infty} b_k = \infty$. Let us assume that $\varphi_i(t) = \theta_i(t)$, $i = 1, 2$ (see (1.2) for definition). It is not difficult to see that $0 \leq b_1(t) \leq \frac{1}{2e}$, $0 \leq b_2(t) \leq \beta$,

$$t - 0.2 \leq \sigma_1(t) \leq \varphi_1(t) \leq t - 0.1, \quad \text{and} \quad t - 1.0001 \leq \sigma_2(t) \leq \varphi_2(t) \leq t - 1. \tag{2.25}$$

Since

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} \int_{\sigma_2(t)}^{t} \sum_{l=1}^{2} b_l(u)\,du \leq \lim_{k\to\infty} \int_{\sigma_2(c_k)}^{c_k} \sum_{l=1}^{2} b_l(u)\,du \\ &= \lim_{k\to\infty} \int_{c_k - 1}^{c_k} \sum_{l=1}^{2} b_l(u)\,du = 0, \end{aligned}$$

we conclude that

$$\liminf_{t\to\infty} \int_{\sigma_2(t)}^{t} \sum_{l=1}^{2} b_l(u)\,du = 0.$$

On the other hand,

$$\liminf_{t\to\infty} \int_{\sigma_1(t)}^{t} \sum_{l=1}^{2} b_l(u)\,du \leq \liminf_{t\to\infty} \int_{\sigma_2(t)}^{t} \sum_{k=1}^{2} b_k(u)\,du$$

and

$$\liminf_{t\to\infty} \sum_{l=1}^{2} \int_{\sigma_l(t)}^{t} \left(\prod_{r=1}^{2} b_r(u) \right)^{\frac{1}{2}} du \leq 2 \liminf_{t\to\infty} \int_{\sigma_2(t)}^{t} \sum_{l=1}^{2} b_l(u)\,du.$$

It follows that $\bar{d} = \liminf_{t\to\infty} \sum_{l=1}^{2} \int_{\sigma_l(t)}^{t} (\prod_{l_1=1}^{2} b_{l_1}(u))^{\frac{1}{2}}\,du = 0$ and $\zeta_{i,l} = \zeta = \eta_l = \eta = Q(\zeta_{i,l}) = Q(\eta_l) = 0$ for $l, i = 1, 2$. Consequently, conditions (1.4), (1.5), (1.6), (1.8), and (1.11) cannot be applied.

Also, since

$$\Lambda_0(t) = \sum_{l=1}^{2} b_l(t) \le \frac{1}{2\mathrm{e}} + \beta$$

and

$$\begin{aligned}\Lambda_1(t) &= \sum_{l=1}^{2} b_l(t)\left[1 + \int_{\varphi_l(t)}^{t} \sum_{l_1=1}^{2} b_{l_1}(u) \mathrm{e}^{\int_{\sigma_{l_1}(u)}^{t} \sum_{l_2=1}^{2} b_{l_2}(u_1) \mathrm{e}^{\int_{\sigma_{l_2}(u_1)}^{u_1} \Lambda_0(u_2)\,du_2}\,du_1}\,du\right] \\ &\le \frac{1}{2\mathrm{e}}\left[1 + \int_{t-0.2}^{t} \frac{1}{2\mathrm{e}} \mathrm{e}^{(t-u+0.2)A_1}\,ds + \int_{t-0.2}^{t} \beta \mathrm{e}^{(t-u+1.0001)A_1}\,ds\right] \\ &\quad + \beta\left[1 + \int_{t-1.0001}^{t} \frac{1}{2\mathrm{e}} \mathrm{e}^{(t-u+0.2)A_1}\,ds + \int_{t-1.0001}^{t} \beta \mathrm{e}^{(t-u+1.0001)A_1}\,ds\right] < \frac{8.373}{\mathrm{e}}\end{aligned}$$

for all $\beta \in [0, \frac{1.43}{\mathrm{e}}]$, where $A_1 = (\frac{1}{2\mathrm{e}}\mathrm{e}^{0.2(\frac{1}{2\mathrm{e}}+\beta)} + \beta \mathrm{e}^{1.0001(\frac{1}{2\mathrm{e}}+\beta)})$, we have

$$\limsup_{t\to\infty} \prod_{l=1}^{2}\left[\prod_{l_1=1}^{2} \int_{\varphi_l(t)}^{t} b_{l_1}(u) \mathrm{e}^{\int_{\sigma_{l_1}(u)}^{\varphi_{l_1}(t)} \Lambda_1(u_1)\,du_1}\,du\right]^{\frac{1}{m}} < 0.246 < \frac{1}{4}\left(1 - \prod_{l=1}^{2} Q(\eta_l)\right).$$

Consequently, condition (1.10) with $\ell = 1$ fails for all $\beta \in [0, \frac{1.43}{\mathrm{e}}]$.

Moreover, we have

$$\Phi_0(t) = 2\left(\prod_{l=1}^{2} b_l(t)\right)^{\frac{1}{2}} \le 2\sqrt{\frac{\beta}{2\mathrm{e}}}$$

and

$$\begin{aligned}\Phi_1(t) &= \sum_{l=1}^{2} b_l(t)\left[1 + 2\left(\prod_{r=1}^{2} \int_{\varphi_l(t)}^{t} b_r(u) \mathrm{e}^{\int_{\sigma_r(u)}^{t} \Phi_0(u_1)\,du_1}\,du\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{2\mathrm{e}}\left[1 + 2\left(\int_{t-0.2}^{t} \frac{1}{2\mathrm{e}} \mathrm{e}^{\int_{u-0.2}^{t} \Phi_0(u_1)\,du_1}\,du \int_{t-0.2}^{t} \beta \mathrm{e}^{\int_{u-1.0001}^{t} \Phi_0(u_1)\,du_1}\,du\right)^{\frac{1}{2}}\right] \\ &\quad + \beta\left[1 + 2\left(\int_{t-1.0001}^{t} \frac{1}{2\mathrm{e}} \mathrm{e}^{\int_{u-0.2}^{t} \Phi_0(u_1)\,du_1}\,du \int_{t-1.0001}^{t} \beta \mathrm{e}^{\int_{u-1.0001}^{t} \Phi_0(u_1)\,du_1}\,du\right)^{\frac{1}{2}}\right] \\ &< \frac{7.1}{\mathrm{e}}\end{aligned}$$

for all $\beta \in [0, \frac{2.23}{\mathrm{e}}]$. Consequently,

$$\begin{aligned}&\limsup_{t\to\infty} \prod_{l=1}^{2}\left[\prod_{l_1=1}^{2} \int_{\varphi_l(t)}^{t} b_{l_1}(u) \mathrm{e}^{\int_{\sigma_{l_1}(u)}^{\varphi_{l_1}(t)} \Phi_1(u_1)\,du_1}\,du\right]^{\frac{1}{2}} \\ &\quad < \limsup_{t\to\infty} \prod_{l=1}^{2}\left[\prod_{l_1=1}^{2} \int_{\varphi_l(t)}^{t} b_{l_1}(u) \mathrm{e}^{\int_{\sigma_{l_1}(u)}^{\varphi_{l_1}(t)} \frac{7.1}{\mathrm{e}}\,du_1}\,du\right]^{\frac{1}{2}}\end{aligned}$$

$$< 0.249 < \frac{1}{4}\left(1 - \prod_{r=1}^{2} Q(\eta_r)\right)$$

for all $\beta \in [0, \frac{2.23}{e}]$. This means that condition (1.9) with $\ell = 1$ and $\beta \in [0, \frac{2.23}{e}]$ is not satisfied. Similarly, condition (1.3) is not satisfied for all $\beta \in [0, \frac{2.294}{e}]$, and condition (1.7) with $i = 4$ is not satisfied for all $\beta \in [0, \frac{3}{e}]$.

Next, we show that Eq. (2.24) is oscillatory for all $\beta \in [\frac{1.3735}{e}, \frac{1.384}{e}]$. Indeed,

$$b_1(t) = \frac{1}{2e} \quad \text{and} \quad b_2(t) = \beta \quad \text{for } t \in [c_k + \delta, c_k + 3.0001 + \delta] \text{ and all } k \in \mathbb{N}.$$

From this and (2.25) the parameters of Corollary 2.1 can be chosen as follows:

$$q_1 = \frac{1}{2e}, \qquad q_2 = \beta, \qquad \mu_1 = 0.1, \qquad \mu_2 = \mu^* = 1.$$

Let

$$I_2(\beta) = \frac{D_{1,2}D_{2,1}}{(1 - D_{1,1})(1 - D_{2,2})},$$

where $D_{l,k}$, $l, k = 1, 2$, are defined as in Corollary 2.1. Then

$$I_2(\beta) = \frac{\beta(e^{0.1B} - 1)(e^B - 1)}{(1 + 2Be - e^{0.1B})(1 + B - \beta e^B)} > 1.09 \quad \text{for all } \beta \in \left[\frac{1.3735}{e}, \frac{1.384}{e}\right],$$

where $B = \frac{1}{2e-0.1} + \frac{\beta}{1-\beta}$. Hence condition (2.18) is satisfied, and Corollary 2.1 implies that Eq. (2.24) is oscillatory for all $\beta \in [\frac{1.3735}{e}, \frac{1.384}{e}]$.

Acknowledgements
The authors express their sincere gratitude to the anonymous reviewers for their constructive comments.

Funding
This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
EA made the major analysis and the original draft preparation. HE revised the calculations, made corrections and provide several improvements. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Sciences and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj, 11942, Saudi Arabia. [2]Department of Mathematics, Faculty of Science, Damietta University, New Damietta, 34517, Egypt.

Received: 24 March 2023 Accepted: 3 May 2023 Published online: 16 August 2023

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