# Multiplicity of positive periodic solutions to third-order variable coefficients singular dynamical systems 

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#### Abstract

In this paper, by applying a nonlinear alternative principle of Leray-Schauder and Guo-Krasnosel'skii fixed point theorem on compression and expansion of cones, together with truncation technique, we study the existence of multiplicity noncollision periodic solutions to third-order singular dynamical systems. By combining the analysis of the sign of Green's function for a linear equation, we consider the systems where the potential has a repulsive singularity at origin. The so-called strong force condition is not needed, and the nonlinearity may have sign changing behavior. Recent results in the literature, even in the scalar case, are generalized and improved.


MSC: 34C25
Keywords: Singular; Dynamical systems; Leray-Schauder alternative principle; Guo-Krasnosel'skii fixed point theorem

## 1 Introduction

The purpose of this work is to study the existence of noncollision periodic solutions to the third-order singular dynamical systems

$$
\begin{equation*}
x^{\prime \prime \prime}+a(t) x=f(t, x)+e(t) \tag{1.1}
\end{equation*}
$$

where $a \in C(\mathbb{R} / T \mathbb{Z}, \mathbb{R}), e=\left(e_{1}, \ldots, e_{n}\right)^{\mathrm{T}} \in C\left((\mathbb{R} / T \mathbb{Z}), \mathbb{R}^{n}\right)$, the nonlinearity $f=\left(f_{1}, \ldots, f_{n}\right)^{\mathrm{T}} \in$ $C\left((\mathbb{R} / T \mathbb{Z}) \times \mathbb{R}^{n} \backslash\{0\}, \mathbb{R}^{n}\right)$ is a continuous vector-valued function with repulsive singularity at $x=0$.

Let $\mathbb{R}_{+}=[0, \infty)$ and $\mathbb{R}_{+}^{n}=\prod_{i=1}^{n} \mathbb{R}_{+}$. For $x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right) \in \mathbb{R}^{n}$, the usual scalar product is denoted by $\langle x, y\rangle=\sum_{i=1}^{n} x_{i} y_{i}$. We say that (1.1) has a repulsive singularity at the origin if there exists a fixed vector $v \in \mathbb{R}_{+}^{n}$ such that

$$
\lim _{x \rightarrow 0, x \in \mathbb{R}_{+}^{n}}\langle v, f(t, x)\rangle=+\infty \quad \text { uniformly in } t .
$$

As usual, by a noncollision nontrivial periodic solution we mean a function $x=\left(x_{1}, \ldots\right.$, $\left.x_{n}\right)^{\mathrm{T}} \in C^{3}\left((\mathbb{R} / T \mathbb{Z}), \mathbb{R}^{n}\right)$ solving (1.1) such that $x(t) \neq 0$ for all $t$ and satisfying the periodic
boundary conditions

$$
\begin{equation*}
x(0)=x(T), \quad x^{\prime}(0)=x^{\prime}(T), \quad x^{\prime \prime}(0)=x^{\prime \prime}(T) . \tag{1.2}
\end{equation*}
$$

In the pioneering paper [15], Lazer and Solimini investigated the singular equation

$$
\begin{equation*}
x^{\prime \prime}=\frac{1}{x^{\lambda}}+h(t), \tag{1.3}
\end{equation*}
$$

where $\lambda \geq 1$, and $h$ is periodic function with period $T$; by using the method of upper and lower solutions they proved that a sufficient and necessary condition for the existence of a positive $T$-periodic solution is $\int_{0}^{T} h(t) d t<0$. We say that $0<\lambda<1$ is the weak force condition for equation (1.3) and $\lambda \geq 1$ is the strong force condition for ir (the strong force condition was first introduced by Gordon [9]). During the last few decades, the question of existence of noncollision periodic solutions for singular scalar equations and dynamical systems has attracted much attention [1, 5, 15, 21, 22, 25, 27, 28]. For example, in 2019, Jiang [13] investigated a kind of second-order nonautonomous dynamical systems

$$
\begin{equation*}
x^{\prime \prime}+a(t) x=f(t, x) . \tag{1.4}
\end{equation*}
$$

By a nonlinear alternative principle of Leray-Schauder and the fixed point theorem in cones the author showed that the singular system (1.4) has at least two positive solutions when the Green's function is nonnegative.
Singular differential equations and singular dynamical systems have a wide range of applications in biology, physics, and mechanics, such as the nonlinear elasticity [6] and Brillouin focusing system [7]. Usually, the proof is based on either variational approach [23, 29] or topological methods. In particular, degree theory [16, 17], Schauder's fixed point theorem [14], some fixed point theorems in cones for completely continuous operators [3, 8, 18, 26], and a nonlinear alternative principle of Leray-Schauder type [13, 20] are the most relevant tools.
To avoid collision of the solution with singularity, the strong force condition plays an important role and is standard in the related works. Compared with the strong singularity case, the case of weak singularity was less studied by topological methods [5, 11, 12].
At the same time, some authors began to consider third-order singular differential equations and singular dynamical systems [2, 4, 14, 24], for example, the third-order differential equation with constant coefficient

$$
\begin{equation*}
x^{\prime \prime \prime}+K x=f(t, x), \quad 0 \leq t \leq 2 \pi \tag{1.5}
\end{equation*}
$$

with periodic boundary conditions (1.2). Here $K$ is a positive constant, and the nonlinearity $f(t, x)$ is singular at $x=0$. In [24], using the Green's function and fixed point index theory, the existence of multiple positive solutions is obtained. In [14], by using Schauder's fixed point theorem, together with perturbation technique, the existence of at least one positive solution is established. The main result in [14] is the following.

## Theorem 1.1 Let the following three assumptions hold:

$\left(A_{1}\right) f(t, u)$ is a nonnegative function on $[0,2 \pi] \times(0,+\infty)$, and $f(t, u)$ is integrable on $[0,2 \pi]$ for each fixed $u \in(0,+\infty)$;
$\left(A_{2}\right) f(t, u)$ is nonincreasing in $u>0$ for almost all $t \in[0,2 \pi]$, and

$$
\lim _{u \rightarrow 0^{+}} f(t, u)=+\infty, \quad \lim _{u \rightarrow+\infty} f(t, u)=0
$$

uniformly for $t \in[0,2 \pi]$;
$\left(A_{3}\right) \int_{0}^{2 \pi} f(s, \tau) d s<+\infty$ for all $\tau>0$.
Then equation (1.5) has at least one positive solution if $K \in\left(0, \frac{1}{3 \sqrt{3}}\right)$.
This paper is mainly motivated by the recent papers [13, 14], but we do not require that all components of the nonlinearity $f(t, x)$ have a singularity. The new results cover both strong and weak singularities. The structure of the paper is as follows. In Sect. 2, we present a survey on some known results concerning the sign of the Green's function of the linear equation

$$
\begin{equation*}
x^{\prime \prime \prime}+K x=0, \tag{1.6}
\end{equation*}
$$

associated with periodic boundary conditions (1.2). In Sects. 3 and 4, by employing a nonlinear alternative principle of Leray-Schauder and Guo-Krasnosel'skii's fixed point theorem, we prove the main existence results for (1.1) under the positiveness of the Green's function associated with (1.6)-(1.2).

In this paper, we use the following notations. The usual Euclidean norm is denoted by $|x|$. More generally, for a fixed vector $v=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{R}_{+}^{n}$, we have the well-defined norm

$$
|x|_{v}=\sum_{i=1}^{n} v_{i}\left|x_{i}\right| .
$$

In particular, we get the $l_{1}$-norm $|x|_{v}=|x|_{1}=\sum_{i=1}^{n}\left|x_{i}\right|$ if $v=(1, \ldots, 1)$. Let $\|\cdot\|$ denote the supremum norm of $C_{T}=\{x: x \in C(\mathbb{R} / T \mathbb{Z}), \mathbb{R}\}$ and take $X=C_{T} \times \cdots \times C_{T}$ ( $n$ copies). Then for $x=\left(x_{1}, \ldots, x_{n}\right) \in X$, the natural norm becomes

$$
\|x\|=\sum_{i=1}^{n} v_{i}\left\|x_{i}\right\|=\sum_{i=1}^{n} v_{i} \cdot \max _{t}\left|x_{i}(t)\right| .
$$

Obviously, $X$ is a Banach space.

## 2 Sign of Green's function and its properties

As we know, it is very complicated to calculate the Green's function of the third-order scalar linear differential equation

$$
\begin{equation*}
x^{\prime \prime \prime}+a(t) x=h(t), \tag{2.1}
\end{equation*}
$$

with with variable coefficients and periodic boundary conditions (1.2), where $h \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$ is a $T$-periodic function. In this section, we first discuss the Green's function of the thirdorder scalar linear differential equation with constant coefficients

$$
\left\{\begin{array}{ll}
x^{\prime \prime \prime}+A x=h(t), & 0 \leq t \leq T  \tag{2.2}\\
x(0)=x(T), & x^{\prime}(0)=x^{\prime}(T),
\end{array} x^{\prime \prime}(0)=x^{\prime \prime}(T), ~ \$\right.
$$

where $A:=\max _{t \in[0, T]} a(t)$. We will use it to investigate the existence of a positive periodic solution for (1.1). In the following, we introduce the Green's functions of (2.2) and some properties, which can be found in [24]. Let $A=\rho^{3}$. Then (2.2) is transformed into

$$
\left\{\begin{array}{l}
y^{\prime}(t)+\rho x=h(t),  \tag{2.3}\\
y(0)=y(T),
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t)-\rho x^{\prime}(t)+\rho^{2} x(t)=h(t)  \tag{2.4}\\
x(0)=x(T), \quad x^{\prime}(0)=x^{\prime}(T)
\end{array}\right.
$$

Moreover, the solutions of (2.3) can be written as

$$
y(t)=\int_{0}^{T} G_{1}(t, s) h(s) d s
$$

where

$$
G_{1}(t, s)= \begin{cases}\frac{e^{-\rho(t-s)}}{1-e^{-\rho T}}, & 0 \leq s \leq t \leq T \\ \frac{e^{-\rho(T+t-s)}}{1-e^{-\rho T}}, & 0 \leq t \leq s \leq T\end{cases}
$$

Lemma 2.1 ([24]) The boundary value problem (2.4) is equivalent to the integral equation

$$
x(t)=\int_{0}^{T} G_{2}(t, s) y(s) d s
$$

where

$$
G_{2}(t, s)= \begin{cases}\frac{2 e^{\frac{\rho(t-s)}{2}}\left[\sin \frac{\sqrt{3} \rho(T-t+s)}{2}+e^{-\frac{\rho T}{2}} \sin \frac{\sqrt{3} \rho(t-s)}{2}\right]}{\sqrt{3} \rho\left(e^{\frac{\rho T}{2}}+e^{-\frac{\rho T}{3}}-2 \cos \frac{\sqrt{3} \rho T}{\rho}\right)}, & 0 \leq s \leq t \leq T, \\ \frac{2 e^{\frac{\rho(T t-s)}{2}}\left[\sin \frac{\sqrt{3} \rho(s-t)}{2}+e^{-\frac{\rho T}{2}} \sin \frac{\sqrt{3} \rho(T+t-s)}{2}\right]}{\sqrt{3} \rho\left(e^{\frac{\rho T}{2}}+e^{-\frac{\rho T}{3}}-2 \cos \frac{\sqrt{3} \rho T}{2}\right)}, & 0 \leq t \leq s \leq T .\end{cases}
$$

Moreover, for $G_{2}(t, s)$, if $\rho \in\left(0, \frac{2 \sqrt{3} \pi}{3 T}\right)$, then we have the estimates

$$
0<\frac{2 \sin \left(\frac{\sqrt{3} \rho T}{2}\right)}{\sqrt{3} \rho\left(e^{\frac{\rho T}{2}}+1\right)^{2}} \leq G_{2}(t, s) \leq \frac{2}{\sqrt{3} \sin \left(\frac{\sqrt{3} \rho T}{2}\right)} .
$$

The solution of (2.2) can be written as

$$
\begin{aligned}
x(t) & =\int_{0}^{T} G_{2}(t, \tau) \int_{0}^{T} G_{1}(\tau, s) h(s) d s d \tau \\
& =\int_{0}^{T} \int_{0}^{T} G_{2}(t, \tau) G_{1}(\tau, s) h(s) d s d \tau \\
& =\int_{0}^{T}\left[\int_{0}^{T} G_{2}(t, s) G_{1}(s, \tau) d s\right] h(\tau) d \tau
\end{aligned}
$$

$$
=\int_{0}^{T}\left[\int_{0}^{T} G_{2}(t, \tau) G_{1}(\tau, s) d \tau\right] h(s) d s
$$

Thus letting

$$
G(t, s)=\int_{0}^{T} G_{2}(t, \tau) G_{1}(\tau, s) d \tau
$$

we can get

$$
x(t)=\int_{0}^{T} G(t, s) h(s) d s
$$

Let $A:=\max _{t \in[0, T]} a(t)$. Since $G_{1}(t, s)>0$ and $G_{2}(t, s) \geq 0$, we easily get the following.
Lemma 2.2 Assume that $0<A<\frac{8 \sqrt{3} \pi^{3}}{9 T^{3}}$. Then the Green's function $G(t, s)$ associated with the boundary value problem (2.2) is positive for all $(t, s) \in[0, T] \times[0, T]$.

We denote

$$
\begin{equation*}
m=\min _{0 \leq t, s \leq T} G(t, s), \quad M=\max _{0 \leq t, s \leq T} G(t, s), \quad \sigma=m / M \tag{2.5}
\end{equation*}
$$

and thus $M>m>0$ and $0<\sigma<1$.

## 3 Existence result (I)

In this section, we state and prove the first existence result for (1.1). The proof is based on the following nonlinear alternative of Leray-Schauder, which can be found in [19] and has been used in $[13,18]$.

Lemma 3.1 Let $C$ be a convex subset of a normed linear space $E$, and let $U$ be an open subset of $C$ with $0 \in U$. Then every compact continuous map $F: \bar{U} \rightarrow C$ has at least one of the following properties,
(i) $F$ has a fixed point in $\bar{U}$; or
(ii) There are $u \in \partial U$ and $0<\lambda<1$ such that $x=\lambda F x$.

Define two functions $\omega$ and $\gamma$ by

$$
\omega(t)=\int_{0}^{T} G(t, s) d s
$$

and

$$
\gamma(t)=\int_{0}^{T} G(t, s) e(s) d s
$$

which is the unique $T$-periodic solution of the linear system

$$
x^{\prime \prime \prime}+a(t) x=e(t) .
$$

Observe that $u(t)=x(t)+\gamma(t)$ is a $T$-periodic solution of (1.1) if the system

$$
x^{\prime \prime \prime}+a(t) x=f(t, x(t)+\gamma(t))
$$

has a $T$-periodic solution $x(t)$, since

$$
u^{\prime \prime \prime}+a(t) u=x^{\prime \prime \prime}+\gamma^{\prime \prime \prime}+a(t) x+a(t) \gamma=f(t, x+\gamma)+e(t)=f(t, u)+e(t)
$$

Theorem 3.2 Assume that $0<A<\frac{8 \sqrt{3} \pi^{3}}{9 T^{3}}$. In addition, suppose that there exists a positive constant $r>0$ satisfying the following conditions.
$\left(\mathrm{H}_{1}\right)$ For each constant $L>0$, there exists a continuous function $\phi_{L} \succ 0$ (which means that $\phi(t) \geq 0$ for all $t \in[0, T]$ and it is positive for $t$ in a subset of positive measure) such that

$$
\langle v, f(t, x)\rangle \geq \phi_{L}(t) \quad \text { for }(t, x) \in[0, T] \times \mathbb{R}_{+}^{n} \text { with } 0<|x|_{v} \leq L .
$$

$\left(\mathrm{H}_{2}\right)$ There exist continuous nonnegative functions $g$ and $h$ on $(0, \infty)$ such that

$$
\langle v, f(t, x)\rangle \leq g\left(|x|_{v}\right)+h\left(|x|_{v}\right)
$$

for all $t$ and $x \in \mathbb{R}_{+}^{n}$ with $0<|x|_{v} \leq r+\gamma^{*}$, where $g>0$ is nonincreasing, and $h / g$ is nondecreasing.
$\left(\mathrm{H}_{3}\right)$ We have the following inequality:

$$
\frac{r}{g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h\left(r+\gamma^{*}\right)}{g\left(r+\gamma^{*}\right)}\right\}}>\|\omega\|,
$$

where

$$
\gamma_{*}=\min _{t}\langle\nu, \gamma(t)\rangle, \quad \gamma *=\max _{t}|\gamma(t)|_{v} .
$$

Then system (1.1) has at least one positive T-periodic solution.

Proof Step 1. We first consider a family of systems.
Since $\left(\mathrm{H}_{3}\right)$ holds, we can choose $n_{0} \in\{1,2, \ldots\}$ such that $\frac{1}{n_{0}}<\sigma r+\gamma_{*}$ and

$$
\|\omega\| g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h(r+\gamma *)}{g\left(r+\gamma^{*}\right)}\right\}+\frac{1}{n_{0}}<r .
$$

Let $N_{0}=\left\{n_{0}, n_{0}+1, \ldots\right\}$ and fix $n \in N_{0}$. Consider the family of systems

$$
\begin{equation*}
x^{\prime \prime \prime}(t)+a(t) x(t)=\mu f_{n}(t, x(t)+\gamma(t))+\frac{a(t) \vartheta}{n} \tag{3.1}
\end{equation*}
$$

where $\mu \in[0,1], \vartheta \in \mathbb{R}_{+}^{n}$ is chosen such that $(v, \vartheta)=1$, with truncation functions

$$
f_{n}(t, x)= \begin{cases}f(t, x) & \text { if }|x|_{v} \geq \frac{1}{n}  \tag{3.2}\\ \tilde{f}(t, x) & \text { if }|x|_{\nu}<\frac{1}{n}\end{cases}
$$

where $\tilde{f}$ is chosen such that $f_{n}$ are continuous on $[0, T] \times \mathbb{R}^{n}$.

So, (3.1) is equivalent to the fixed point problem

$$
\begin{equation*}
x(t)=\mu\left(\mathcal{T}^{n} x\right)(t)+\ell \tag{3.3}
\end{equation*}
$$

where $\ell=\vartheta / n, \mathcal{T}^{n}$ is defined by

$$
\left(\mathcal{T}^{n} x\right)(t)=\int_{0}^{T} G(t, s) f^{n}(s, x(s)+\gamma(s)) d s
$$

and we have used the fact that

$$
\int_{0}^{T} a(s) G(t, s) d s=1
$$

We claim that any fixed point $x$ of (3.3) for all $\mu \in[0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that $x$ is a fixed point of (3.3) for some $\mu \in[0,1]$ such that $\|x\|=r$. From Lemma 2.2 we have

$$
\begin{aligned}
\langle v, x(t)\rangle-\langle v, \ell\rangle & =\mu \int_{0}^{T}\left\langle v, G(t, s) f_{n}(s, x(s)+\gamma(s))\right\rangle d s \\
& \geq \mu m \int_{0}^{T}\left\langle v, f_{n}(s, x(s)+\gamma(s))\right\rangle d s \\
& =\sigma M \mu \int_{0}^{T}\left\langle v, f_{n}(s, x(s)+\gamma(s))\right\rangle d s \\
& \geq \sigma\left\langle v, \max _{t}\left\{\mu \int_{0}^{T} G(t, s) f_{n}(s, x(s)+\gamma(s)) d s\right\}\right\rangle \\
& =\sigma\|x(t)-\ell\| .
\end{aligned}
$$

Therefore, for all $t$, we have

$$
\begin{aligned}
\langle v, x(t)\rangle & \geq \sigma\|x(t)-\ell\|+\langle v, \ell\rangle \\
& \geq \sigma(\|x(t)\|-\langle v, \ell\rangle)+\langle v, \ell\rangle \\
& \geq \sigma r .
\end{aligned}
$$

So we have

$$
\|x(t)+\gamma(t)\| \geq\langle v, x(t)+\gamma(t)\rangle \geq \sigma r+\gamma_{*}>\frac{1}{n}
$$

since $\frac{1}{n} \leq \frac{1}{n_{0}}<\sigma r+\gamma_{*}$, which implies that

$$
f_{n}(t, x(t)+\gamma(t))=f(t, x(t)+\gamma(t)) .
$$

Thus from $\left(\mathrm{H}_{2}\right)$ we have

$$
\langle v, x(t)\rangle=\lambda \int_{0}^{T}\langle v, G(t, s) f(s, x(s)+\gamma(s))\rangle d s+\langle v, \ell\rangle
$$

$$
\begin{aligned}
& \leq \int_{0}^{T}\langle v, G(t, s) f(s, x(s)+\gamma(s))\rangle d s+\langle v, \ell\rangle \\
& =\int_{0}^{T} G(t, s)\langle v, f(s, x(s)+\gamma(s))\rangle d s+\langle v, \ell\rangle \\
& \leq \int_{0}^{T} G(t, s) g\left(|x(s)+\gamma(s)|_{v}\right)\left\{1+\frac{h\left(|x(s)+\gamma(s)|_{v}\right)}{g\left(|x(s)+\gamma(s)|_{v}\right)}\right\} d s+\langle v, \ell\rangle \\
& \leq\|\omega\| g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h\left(r+\gamma^{*}\right)}{g\left(r+\gamma^{*}\right)}\right\}+\frac{1}{n_{0}} .
\end{aligned}
$$

Therefore we have

$$
r=\|x\|_{\nu} \leq\|\omega\| g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h\left(r+\gamma^{*}\right)}{g\left(r+\gamma^{*}\right)}\right\}+\frac{1}{n_{0}} .
$$

This is a contradiction to the choice of $n_{0}$, and thus the claim is proved.
By this claim Lemma 3.1 guarantees that

$$
x=\mathcal{T}^{n} x
$$

has a fixed point, denoted by $x_{n}$, i.e., the system

$$
\begin{equation*}
x^{\prime \prime \prime}(t)+a(t) x(t)=f_{n}(t, x(t)+\gamma(t))+\frac{a(t) \vartheta}{n} \tag{3.4}
\end{equation*}
$$

has a periodic solution $x_{n}$ with $\left\|x_{n}\right\|<r$. Since $\left\langle v, x_{n}(t)\right\rangle \geq\langle v, \ell\rangle>0$ for all $t \in[0, T], x_{n}$ is in fact a positive $T$-periodic solution of (3.4).
Now we show that $\left\langle v, x_{n}(t)+\gamma(t)\right\rangle$ have an uniform positive lower bound, i.e., there exists a constant $\delta>0$, independent of $n \in N_{0}$, such that

$$
\begin{equation*}
\left\langle v, x_{n}(t)+\gamma(t)\right\rangle \geq \delta \tag{3.5}
\end{equation*}
$$

for all $n \in N_{0}$. To see this, we know that by $\left(\mathrm{H}_{1}\right)$ there exists a continuous function $\phi_{L}(t) \succ 0$ such that

$$
\langle v, f(t, x)\rangle \geq \phi_{L}(t)
$$

for all $t$ and $0<\|x\| \leq L$. Then we have

$$
\begin{aligned}
\left\langle v, x_{n}(t)+\gamma(t)\right\rangle & =\int_{0}^{T} G(t, s)\left\langle v, f_{n}\left(s, x_{n}(s)+\gamma(s)\right)\right\rangle d s+\frac{1}{n} \\
& \geq m \int_{0}^{T}\left\langle v, f\left(s, x_{n}(s)+\gamma(s)\right)\right\rangle d s+\frac{1}{n} \\
& \geq m \int_{0}^{T} \phi_{L}(s) d s:=\delta .
\end{aligned}
$$

So we have $\left\langle v, x_{n}(t)+\gamma(t)\right\rangle \geq \delta$ for all $n$.
Step 2. To pass the solutions $x_{n}$ of the truncation systems (3.4) to that of the original system (1.1), we need to show that $\left\{x_{n}\right\}_{n \in N_{0}}$ is compact.

Firstm we claim that

$$
\begin{equation*}
\left\|x_{n}^{\prime}\right\| \leq H \tag{3.6}
\end{equation*}
$$

for some constant $H>0$ and all $n \geq n_{0}$.
Here we denote by $x_{n i}$ and $\vartheta_{i}$ the $i$ th components of $x_{n}$ and $\vartheta$. Since $x_{n i}$ are $T$-periodic solutions of (3.4), we have

$$
\begin{equation*}
x_{n i}^{\prime \prime \prime}(t)+a(t) x_{n i}(t)=f_{n}\left(t, x_{n i}(t)+\gamma(t)\right)+\frac{a(t) \vartheta_{i}}{n} \tag{3.7}
\end{equation*}
$$

for each $i=1,2, \ldots, n$.
Multiplying both sides of (3.7) by $x_{n i}^{\prime}(t)$ and integrating from 0 to $T$, we have

$$
\begin{align*}
\int_{0}^{T} x_{n i}^{\prime \prime \prime}(t) x_{n i}^{\prime}(t) d t+\int_{0}^{T} a(t) x_{n i}(t) x_{n i}^{\prime}(t) d t= & \int_{0}^{T} f_{n}\left(t, x_{n i}(t)+\gamma(t)\right) x_{n i}^{\prime}(t) d t  \tag{3.8}\\
& +\int_{0}^{T} \frac{a(t) \vartheta_{i}}{n} x_{n i}^{\prime}(t) d t
\end{align*}
$$

Substituting $\int_{0}^{T} x_{n i}^{\prime \prime \prime}(t) x_{n i}^{\prime}(t) d t=-\int_{0}^{T}\left|x_{n i}^{\prime \prime}(t)\right|^{2} d t$ into (3.8), we have

$$
\begin{aligned}
\int_{0}^{T}\left|x_{n i}^{\prime \prime}(t)\right|^{2} d t= & \int_{0}^{T} a(t) x_{n i}(t) x_{n i}^{\prime}(t) d t-\int_{0}^{T} f_{n}\left(t, x_{n i}(t)+\gamma(t)\right) x_{n i}^{\prime}(t) d t \\
& -\int_{0}^{T} \frac{a(t) \vartheta_{i}}{n} x_{n i}^{\prime}(t) d t \\
\leq & A \int_{0}^{T}\left|x_{n i}(t)\right|\left|x_{n i}^{\prime}(t)\right| d t+\int_{0}^{T}\left|f_{n}\left(t, x_{n i}(t)+\gamma(t)\right)\right|\left|x_{n i}^{\prime}(t)\right| d t \\
& +\frac{A \vartheta_{i}}{n} \int_{0}^{T}\left|x_{n i}^{\prime}(t)\right| d t \\
\leq & A r \sqrt{T}\left(\int_{0}^{T}\left|x_{n i}^{\prime}(t)\right|^{2} d t\right)^{\frac{1}{2}} \\
& +\left(\int_{0}^{T}\left|f_{n}\left(t, x_{n i}(t)+\gamma(t)\right)\right|^{2} d t\right)^{\frac{1}{2}}\left(\int_{0}^{T}\left|x_{n i}^{\prime}(t)\right|^{2} d t\right)^{\frac{1}{2}} \\
& +\frac{A \vartheta_{i}}{n_{0}} \sqrt{T}\left(\int_{0}^{T}\left|x_{n i}^{\prime}(t)\right|^{2} d t\right)^{\frac{1}{2}} \\
= & C\left(\int_{0}^{T}\left|x_{n i}^{\prime}(t)\right|^{2} d t\right)^{\frac{1}{2}},
\end{aligned}
$$

where

$$
C=A r \sqrt{T}+\max _{\delta \leq x_{n i} \leq r}\left(\int_{0}^{T}\left|f_{n}\left(t, x_{n i}(t)+\gamma(t)\right)\right|^{2} d t\right)^{\frac{1}{2}}+\frac{A \vartheta_{i}}{n_{0}} \sqrt{T} .
$$

Using the Writinger inequality, we have

$$
\int_{0}^{T}\left|x_{n i}^{\prime \prime}(t)\right|^{2} d t \leq \frac{C T}{2 \pi}\left(\int_{0}^{T}\left|x_{n i}^{\prime \prime}(t)\right|^{2} d t\right)^{\frac{1}{2}}
$$

It is easy to see that there is constant $D>0$ such that

$$
\int_{0}^{T}\left|x_{n i}^{\prime \prime}(t)\right|^{2} d t \leq D
$$

For each $i=1, \ldots, n$, by the periodic boundary conditions $x_{n i}(0)=x_{n i}(T)$ we know that there exists a point $t_{0} \in[0, T]$ such that $x_{n i}^{\prime}\left(t_{0}\right)=0$. Therefore we have

$$
\begin{aligned}
\left\|x_{n i}^{\prime}\right\| & =\max _{t}\left|\int_{t_{i}}^{t} x_{n i}^{\prime \prime}(s) d s\right| \\
& \leq \int_{0}^{T}\left|x_{n i}^{\prime \prime}(s)\right| d s \\
& \leq T^{\frac{1}{2}}\left(\int_{0}^{T}\left|x_{n i}^{\prime \prime}(s)\right|^{2} d s\right)^{\frac{1}{2}} \leq \sqrt{T D} .
\end{aligned}
$$

Therefore

$$
\left\|x_{n}^{\prime}\right\|=\left\langle v,\left\|x_{n i}^{\prime}\right\|\right\rangle \leq \sum_{i=1}^{n} v_{i} \sqrt{T D}:=H .
$$

Step 3. The facts (3.6) and (3.5) show that $\left\{x_{n}\right\}_{n \in N_{0}}$ is a bounded and equicontinuous family.
Now the Arzelà-Ascoli theorem guarantees that $\left\{x_{n}\right\}_{n \in N_{0}}$ has a subsequence $\left\{x_{n_{k}}\right\}_{k \in \mathbb{N}}$ converging uniformly on $[0, T]$ to a function $x \in X$. Moreover, we have

$$
\delta \leq\langle v, x(t)+\gamma(t)\rangle \leq r+\gamma^{*} \quad \text { for all } t .
$$

Furthermore, $x_{n_{k}}$ satisfies the integral equation

$$
x_{n_{k}}(t)=\int_{0}^{T} G(t, s) f\left(s, x_{n_{k}}(s)+\gamma(s)\right) d s+\frac{\ell}{n_{k}} .
$$

Letting $k \rightarrow \infty$, we arrive at

$$
x(t)=\int_{0}^{T} G(t, s) f(s, x(s)+\gamma(t)) d s
$$

Therefore $x$ is a positive $T$-periodic solution of (1.1) and satisfies $0<\|x\| \leq r$.
Corollary 3.3 Assume that $0<A<\frac{8 \sqrt{3} \pi^{3}}{9 T^{3}}, b, c \in C[0, T]$ are positive functions, $e_{1}, e_{2} \in$ $C(\mathbb{R} / T \mathbb{Z}, \mathbb{R}), \alpha, \beta>0$, and $\mu \in \mathbb{R}$ is a given positive parameter. Consider the following twodimensional third-order nonlinear systems:

$$
\left\{\begin{array}{l}
x_{1}^{\prime \prime \prime}+a(t) x_{1}=b(t)\left(x_{1}+x_{2}\right)^{-\alpha}+e_{1}(t),  \tag{3.9}\\
x_{2}^{\prime \prime \prime}+a(t) x_{2}=\lambda c(t)\left(x_{1}+x_{2}\right)^{\beta}+e_{2}(t) .
\end{array}\right.
$$

(i) if $\beta<1$, then (3.9) has at least one positive periodic solution for each $\lambda>0$;
(ii) if $\beta \geq 1$, then (3.9) has at least one positive periodic solution for each $0<\lambda<\lambda_{1}$, where $\lambda_{1}$ is some positive constant;

Proof We will apply Theorem 3.2. For a fixed vector $v=(1,1),\left(\mathrm{H}_{1}\right)$ is fulfilled by $\phi_{L}=$ $b(t) L^{-\alpha}$. For $s \in \mathbb{R}, s>0$, to verify $\left(\mathrm{H}_{2}\right)$, we may take

$$
g(s)=b^{*} s^{-\alpha}, \quad h(s)=\mu c^{*} s^{\beta},
$$

where

$$
b^{*}=\max _{t} b(t), \quad c^{*}=\max _{t} c(t) .
$$

Condition $\left(\mathrm{H}_{3}\right)$ becomes

$$
\lambda<\frac{r\left(\sigma r+\gamma_{*}\right)^{\alpha}-b^{*} M T}{c^{*}\left(r+\gamma^{*}\right)^{\alpha+\beta} M T}
$$

for some $r>0$. So (3.9) has at least one positive periodic solution for

$$
0<\lambda<\lambda_{1}:=\sup _{r>0} \frac{r\left(\sigma r+\gamma_{*}\right)^{\alpha}-b^{*} M T}{c^{*}\left(r+\gamma^{*}\right)^{\alpha+\beta} M T} .
$$

Note that $\lambda_{1}=\infty$ if $\beta<1$ and $\lambda_{1}<\infty$ if $\beta \geq 1$. So we have (i) and (ii).

## 4 Existence result (II)

In this section, by using Guo-Krasnosel'skii's fixed point theorem on compression and expansion of cones, we establish the second existence results for (1.1).

Lemma 4.1 ([10]) Let $X$ be a Banach space, and let $K(\subset X)$ be a cone. Assume that $\Omega_{1}$, $\Omega_{2}$ are open subsets of $X$ with $0 \in \Omega_{1}, \bar{\Omega}_{1} \subset \Omega_{2}$, and let

$$
\mathcal{A}: K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \rightarrow K
$$

be a completely continuous operator such that either
(i) $\|\mathcal{A} u\| \geq\|u\|, u \in K \cap \partial \Omega_{1}$, and $\|\mathcal{A} u\| \leq\|u\|$, $u \in K \cap \partial \Omega_{2}$; or
(ii) $\|\mathcal{A} u\| \leq\|u\|, u \in K \cap \partial \Omega_{1}$, and $\|\mathcal{A} u\| \geq\|u\|, u \in K \cap \partial \Omega_{2}$.

Then $\mathcal{A}$ has a fixed point in $K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

Let $X=C_{T} \times \cdots \times C_{T}(n$ copies $)$ and define

$$
\begin{equation*}
K=\left\{x \in X: \min _{t}\langle v, x(t)\rangle \geq \sigma\|x\|\right\}, \tag{4.1}
\end{equation*}
$$

where $\sigma$ is as in (2.5).
We can readily verify that $K$ is a cone in the Banach space $X$. Define the operator

$$
(\Phi x)(t)=\int_{0}^{T} G(t, s) f(s, x(s)) d s
$$

for $x \in X$ and $t \in[0, T]$. Then $\Phi$ is well defined and maps $X$ into $K$.
Indeed, for $t \in \mathbb{R}$ and $x \in X$, we have

$$
\|\Phi x\|=\max _{t}\langle v,(\Phi x)(t)\rangle
$$

$$
\begin{aligned}
& =\max _{t} \int_{0}^{T} G(t, s)\langle v, f(s, x(s))\rangle d s \\
& \leq M \int_{0}^{T}\langle v, f(s, x(s))\rangle d s
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle v,(\Phi x)(t)\rangle & =\int_{0}^{T} G(t, s)\langle v, f(s, x(s))\rangle d s \\
& \geq m \int_{0}^{T}\langle v, f(s, x(s))\rangle d s
\end{aligned}
$$

Thus

$$
\min _{t}\langle v,(\Phi x)(t)\rangle \geq m \int_{0}^{T}\langle v, f(s, x(s))\rangle d s \geq \sigma\|\Phi x\| .
$$

This implies that $\Phi(X) \subset K$. It is easy to prove $\Phi: X \rightarrow K$ is completely continuous.
Theorem 4.2 Assume that $0<A<\frac{8 \sqrt{3} \pi^{3}}{9 T^{3}}$ and $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{3}\right)$ hold. In addition, we assume that the following two conditions are satisfied:
$\left(\mathrm{H}_{4}\right)$ There exist continuous nonnegative functions $g$ and $h_{1}$ such that

$$
\langle v, f(t, x)\rangle \geq g_{1}\left(|x|_{v}\right)+h_{1}\left(|x|_{v}\right) \quad \text { for all }(t, x) \in[0, T] \times \mathbb{R}_{+}^{n} \backslash\{0\},
$$

where $g_{1}>0$ is nonincreasing, and $h_{1} / g_{1}$ is nondecreasing.
$\left(\mathrm{H}_{5}\right)$ There exists $R>r$ such that

$$
\|\omega\| g_{1}\left(R+\gamma^{*}\right)\left\{1+\frac{h_{1}\left(\sigma R+\gamma_{*}\right)}{g_{1}\left(\sigma R+\gamma_{*}\right)}\right\} \geq R .
$$

Then, besides the solution $x$ constructed in Theorem 3.2, problem (1.1)-(1.2) has another positive T-periodic solution $\tilde{x}$ with $r<\|\tilde{x}-\gamma\| \leq R$.

Proof Let $K$ be a cone in $X$ defined by (4.1). Define

$$
\Omega_{1}=\{x \in X:\|x\|<r\}, \quad \Omega_{2}=\{x \in X:\|x\|<R\} .
$$

First, we claim that $\|\Phi x\| \leq\|x\|$ for $x \in K \cap \partial \Omega_{1}$. Indeed, if $x \in K \cap \partial \Omega_{1}$, then $\|x\|=r$, and we have

$$
\sigma r \leq|x(t)|_{v} \leq r .
$$

Thus

$$
\begin{aligned}
\|\Phi x\| & =\max _{t}|\Phi x|_{v} \\
& =\max _{t}\left\langle v, \int_{0}^{T} G(t, s) f(s, x(s)+\gamma(s)) d s\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& =\max _{t} \int_{0}^{T} G(t, s)\langle v, f(s, x(s)+\gamma(s)\rangle d s \\
& \leq \max _{t} \int_{0}^{T} G(t, s) g\left(|x(t)++\gamma(s)|_{v}\right)\left\{1+\frac{h\left(|x(t)+\gamma(s)|_{v}\right)}{g\left(|x(t)+\gamma(s)|_{v}\right)}\right\} d s \\
& \leq g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h\left(r+\gamma^{*}\right)}{g\left(r+\gamma^{*}\right)}\right\} \max _{t} \int_{0}^{T} G(t, s) d s \\
& \leq\|\omega\| g\left(\sigma r+\gamma_{*}\right)\left\{1+\frac{h\left(r+\gamma^{*}\right)}{g\left(r+\gamma^{*}\right)}\right\} \\
& \leq r=\|x\| .
\end{aligned}
$$

Next, we prove that $\|\Phi x\| \geq\|x\|$ for $x \in K \cap \partial \Omega_{2}$. Indeed, if $x \in K \cap \partial \Omega_{2}$, then $\|x\|=R$, and we have

$$
\sigma R+\gamma_{*} \leq|x(t)+\gamma(t)|_{v} \leq R+\gamma^{*} .
$$

Thus

$$
\begin{aligned}
\|\Phi x\| & =\max _{t} \int_{0}^{T} G(t, s)\langle v, f(s, x(s)+\gamma(s)) d s\rangle \\
& \geq \max _{t} \int_{0}^{T} G(t, s) g_{1}\left(|x(t)+\gamma(s)|_{v}\right)\left\{1+\frac{h_{1}\left(|x(t)+\gamma(s)|_{v}\right)}{g_{1}\left(|x(t)+\gamma(s)|_{v}\right)}\right\} d s \\
& \geq g_{1}\left(R+\gamma^{*}\right)\left\{1+\frac{h_{1}\left(\sigma R+\gamma_{*}\right)}{g_{1}\left(\sigma R+\gamma_{*}\right)}\right\} \max _{t} \int_{0}^{T} G(t, s) d s \\
& \geq\|\omega\| g_{1}\left(R+\gamma^{*}\right)\left\{1+\frac{h_{1}\left(\sigma R+\gamma_{*}\right)}{g_{1}\left(\sigma R+\gamma_{*}\right)}\right\} \\
& \geq R=\|x\| .
\end{aligned}
$$

Now Lemma 4.1 guarantees that $\Phi$ has at least one fixed point $\tilde{x} \in K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$ with $r \leq\|\tilde{x}\| \leq R$.

Let us consider again example (3.9) in Corollary 3.3.
Corollary 4.3 Assume in (3.9) that $0<A<\frac{8 \sqrt{3} \pi^{3}}{9 T^{3}}, b(t)>0$ and $c(t)>0$ for all $t \in[0, T]$, and $\beta>1$. Then, for each $\mu$ with $0<\lambda<\lambda_{1}$, where $\lambda_{1}$ is given as in Corollary 3.3, problem (3.9) has at least two different positive solutions.

To verify $\left(\mathrm{H}_{4}\right)$, for $s \in \mathbb{R}, s>0$, we may take

$$
v=(1,1), \quad g_{1}(s)=b_{*} s^{-\alpha}, \quad h_{1}(s)=\mu c_{*} s^{\beta},
$$

where

$$
c_{*}=\min _{t} c(t), \quad d_{*}=\min _{t} d(t)
$$

If $\beta>1$, then condition $\left(\mathrm{H}_{5}\right)$ becomes

$$
\begin{equation*}
\lambda \geq \frac{R\left(R+\gamma^{*}\right)^{\alpha}-M T b_{*}}{c_{*}\left(\sigma R+\gamma_{*}\right)^{\alpha+\beta}} . \tag{4.2}
\end{equation*}
$$

Since $\beta>1$, the right-hand side goes to 0 as $R \rightarrow+\infty$. Thus, for any given $0<\lambda<\lambda_{1}$, it is always possible to find $R \gg r$ such that (4.2) is satisfied. Thus (3.9) has an additional positive periodic solution $\tilde{x}$.

## Acknowledgements

We would like to express our great thanks to the referees for their valuable suggestions. Shengjun Li was supported by Hainan Provincial Natural Science Foundation of China (Grant No. 120RC450), the National Natural Science Foundation of China (Grant No. 11861028), Key Laboratory of Engineering Modeling and Statistical Computation of Hainan Province. Fang Zhang was supported by the Natural Science Foundation of Jiangsu Province (Grant No. BK20201447), and Science and Technology Innovation Talent Support Project of Jiangsu Advanced Catalysis and Green Manufacturing Collaborative Innovation Center (Grant No. ACGM2022-10-02).

Availability of data and materials
All data analyzed in this study are included in this paper.

## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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Received: 6 April 2023 Accepted: 3 June 2023 Published online: 29 June 2023

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