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Decay rate of the solutions to the Cauchy problem of the Bresse system in thermoelasticity of type III with distributed delay

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Abstract

The decay rate of solutions to a Bresse system in thermoelasticity of type III with respect to the distributed delay term is the subject of this study. We demonstrate our major finding utilising the energy approach in the Fourier space.

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1 Introduction and preliminaries

Fourier law provides the fundamental principle governing classical heat conduction:

$$q(x, t) = -\kappa \nabla v(x, t), \quad (1.1)$$

where t represent the time, x is the Lagrangian coordinates material point, v is the temperature, measured with respect to a reference temperature, ∇ is the gradient operator, q is the heat flux and κ is the thermal conductivity of the material which is a thermodynamic state property. According to equation (1.1), the heat flux is caused by the temperature gradient at the same material point x and at the same time t . Equation (1.1) and the conservation law together (assuming for simplicity that no heat sources are present)

$$J v_t + \varrho \operatorname{div} q = 0, \quad (1.2)$$

produces the classical heat transport equation (of parabolic type)

$$J v_t - \varrho \kappa \Delta v = 0, \quad (1.3)$$

Green & Naghdi [6, 7] created a thermoelasticity model that incorporates the temperature gradient and thermal displacement gradient among the constitutive variables, and

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presented a heat conduction law as

$$q(x, t) = -\kappa \nabla v - \kappa^* \nabla r, \quad (1.4)$$

where $r_t = v$ and r is the thermal displacement gradient and the constants κ and κ^* are both positive. The energy balance law (1.2) and equation (1.4) result in the equation

$$J v_{tt} - \kappa \varrho \Delta v_t - \kappa^* \varrho \Delta v = 0, \quad (1.5)$$

this allows thermal waves to travel at a finite speed.

Several authors have discussed the interaction between Fourier law of heat conduction and various systems, and there are numerous outcomes. Examples include the Timoshenko system in [9, 13], the Bresse system (Bresse–Fourier) in [5, 10, 15–17], the Bresse system combined with the Cattaneo law of heat conduction in [14] and the MGT problem in [1]. We recommend the following papers [2–4, 8] to the reader for more information.

We would like to demonstrate the general decay result in the Fourier space to the Cauchy issue of the Bresse system in type III thermoelasticity using all of the papers cited above, particularly [15]. This is one of the earliest papers that we are aware of that look at this issue in Fourier space.

Therefore, the primary objective of this paper is to investigate the rate at which the following system's solutions decay:

$$\begin{cases} \varsigma_{tt} - (\varsigma_x - \hbar - l\mathfrak{J})_x - k_0^2 l(\mathfrak{J}_x - l\varsigma) = 0, \\ \hbar_{tt} - a^2 \hbar_{xx} - (\varsigma_x - \hbar - l\mathfrak{J}) + mv_x = 0, \\ \mathfrak{J}_{tt} - k_0^2 (\mathfrak{J}_x - l\varsigma)_x - l(\varsigma_x - \hbar - l\mathfrak{J}) + \aleph_1 \mathfrak{J}_t + \int_{\wp_1}^{\wp_2} \aleph_2(s) \mathfrak{J}_t(x, t-s) ds = 0, \\ v_{tt} - k_1 v_{xx} + \beta \hbar_{tx} - k_2 v_{tx} = 0, \end{cases} \quad (1.6)$$

where

$$(x, s, t) \in \mathbb{R} \times (\wp_1, \wp_2) \times \mathbb{R}_+,$$

with the initial and boundary conditions

$$\begin{aligned} (\varsigma, \varsigma_t, \hbar, \hbar_t, \mathfrak{J}, \mathfrak{J}_t, v, v_t)(x, 0) &= (\varsigma_0, \varsigma_1, \hbar_0, \hbar_1, \mathfrak{J}_0, \mathfrak{J}_1, v_0, v_1), \quad x \in \mathbb{R}, \\ \mathfrak{J}_t(x, -t) &= f_0(x, t), \quad (x, t) \in (0, 1) \times (0, \wp_2), \end{aligned} \quad (1.7)$$

where the functions ς , \mathfrak{J} and \hbar denote the vertical displacements of the beam, longitudinal displacements and the rotation angle of the linear filaments material, respectively; $a, l, m, k_0, k_1, k_2, \aleph_1$ and β are positive constants and the function v is the temperature difference; the integral represent the distributed delay terms with $\wp_1, \wp_2 > 0$ being a time delay, \aleph_2 is an L^∞ function satisfying:

(H1) $\aleph_2 : [\wp_1, \wp_2] \rightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\wp_1}^{\wp_2} |\aleph_2(s)| ds < \aleph_1. \quad (1.8)$$

The sections of this paper are as follows: In this section, we apply our assumptions and preliminary findings to the major decay result. We build the Lyapunov functional and determine the estimate for the Fourier image in the following section by employing the energy approach in Fourier space. The conclusion is covered in the final section.

As in [12], we begin by introducing the new variable

$$\mathcal{Y}(x, J, s, t) = \mathfrak{J}_t(x, t - s_J),$$

then, we get

$$\begin{cases} s\mathcal{Y}_t(x, J, s, t) + \mathcal{Y}_J(x, J, s, t) = 0, \\ \mathcal{Y}(x, 0, s, t) = \mathfrak{J}_t(x, t), \end{cases}$$

and utilize the transformation [18]

$$\bar{v} := \int_0^t v(x, s) ds + \chi(x), \quad (1.9)$$

with a function $\chi := \chi(x)$ satisfying

$$k_1 \chi_{xx} = v_1 - k_2 v_{0xx} + \beta \bar{h}_{1x}. \quad (1.10)$$

We can also write the proposed problem in the form (by writing, v instead of \bar{v})

$$\begin{cases} \zeta_{tt} - (\zeta_x - \bar{h} - l\mathfrak{J})_x - k_0^2 l(\mathfrak{J}_x - l\zeta) = 0, \\ \bar{h}_{tt} - a^2 \bar{h}_{xx} - (\zeta_x - \bar{h} - l\mathfrak{J}) + mv_{tx} = 0, \\ \mathfrak{J}_{tt} - k_0^2 (\mathfrak{J}_x - l\zeta)_x - l(\zeta_x - \bar{h} - l\mathfrak{J}) + \mathfrak{J}_1 \mathfrak{J}_t + \int_{\wp_1}^{\wp_2} \mathfrak{J}_2(s) \mathcal{Y}(x, 1, s, t) ds = 0, \\ v_{tt} - k_1 v_{xx} + \beta \bar{h}_{tx} - k_2 v_{txx} = 0, \\ s\mathcal{Y}_t(x, J, s, t) + \mathcal{Y}_J(x, J, s, t) = 0, \end{cases} \quad (1.11)$$

where

$$(x, J, s, t) \in \mathbb{R} \times (0, 1) \times (\wp_1, \wp_2) \times \mathbb{R}_+,$$

with initial conditions

$$\begin{cases} (\zeta, \zeta_t, \bar{h}, \bar{h}_t, \mathfrak{J}, \mathfrak{J}_t, v, v_t)(x, 0) = (\zeta_0, \zeta_1, \bar{h}_0, \bar{h}_1, \mathfrak{J}_0, \mathfrak{J}_1, \bar{v}(x, 0), \bar{v}_t(x, 0)), \\ \mathcal{Y}(x, J, s, 0) = f_0(x, s_J), \quad (x, J, s) \in \mathbb{R} \times (0, 1) \times (0, \wp_2), \end{cases} \quad (1.12)$$

In order to get the main result, we require the Hausdorff–Young inequality in the following lemma.

Lemma 1.1 ([11]) *For any $k, \alpha \geq 0, c > 0$, a constant $C > 0$ exist in such a way that $\forall t \geq 0$ the following estimate hold:*

$$\int_{|l| \leq 1} |l|^k e^{-c|l|^\alpha t} dl \leq C(1+t)^{-(k+n)/\alpha}, \quad l \in \mathbb{R}^n. \quad (1.13)$$

2 Energy method and decay estimates

We will obtain a decay estimate of the Fourier image of the solution for problem (1.11)–(1.12) in this section. This approach enables us to provide the decay rate of the solution in the energy space by utilising Plancherel's theorem along with some integral estimates, such as Lemma (1.1). Using the energy approach in Fourier space, we create the proper Lyapunov functionals for this problem. Lastly, we prove our major finding.

2.1 The energy method in the Fourier space

Now, we introduce the new variables to construct the Lyapunov functional in the Fourier space

$$\begin{aligned} r &= (\zeta_x - \hbar - l\mathfrak{J}), & g &= \zeta_t, & v &= a\hbar_x, & w &= \hbar_t \\ \phi &= k_0(\mathfrak{J}_x - l\zeta), & \varpi &= \mathfrak{J}_t, & \vartheta &= v_t, & \sigma &= v_x. \end{aligned} \quad (2.1)$$

Then, the system (1.11) takes the following form

$$\left\{ \begin{array}{l} r_t - g_x + w + l\varpi = 0, \\ g_t - r_x - k_0 l\phi = 0, \\ v_t - ay_x = 0, \\ w_t - az_x - r + m\vartheta_x = 0, \\ \phi_t - k_0 \varpi_x + k_0 l u = 0, \\ \varpi_t - k_0 \phi_x - lv + \aleph_1 \varpi + \int_{\mathcal{P}_1}^{\mathcal{P}_2} \aleph_2(s) \mathcal{Y}(x, 1, s, t) ds = 0, \\ \vartheta_t - k_1 \sigma_x + \beta w_x - k_2 \vartheta_{xx} = 0, \\ \sigma_t - \vartheta_x = 0, \\ s\mathcal{Y}_t + \mathcal{Y}_j = 0, \end{array} \right. \quad (2.2)$$

with initial conditions

$$(r, g, v, w, \phi, \varpi, \vartheta, \sigma, \mathcal{Y})(x, 0) = (r_0, g_0, v_0, w_0, \phi_0, \varpi_0, \vartheta_0, \sigma_0, f_0), \quad x \in \mathbb{R}, \quad (2.3)$$

where

$$\begin{aligned} r_0 &= (\zeta_{0,x} - \hbar_0 - l\mathfrak{J}_0), & g_0 &= \zeta_1, & v_0 &= a\hbar_{0,x}, & w_0 &= \hbar_1, \\ \phi_0 &= k_0(\mathfrak{J}_{0,x} - l\zeta_0), & \varpi_0 &= \mathfrak{J}_1, & \vartheta_0 &= v_1, & \sigma_0 &= v_{0,x}. \end{aligned}$$

Hence, the problem (2.2)–(2.3) is written as

$$\left\{ \begin{array}{l} Z_t + \mathcal{A}Z_x + \mathcal{L}Z = \mathcal{B}Z_{xx}, \\ Z(x, 0) = Z_0(x), \end{array} \right. \quad (2.4)$$

with $Z = (r, g, v, w, \phi, \varpi, \vartheta, \sigma, \mathcal{Y})^T$, $Z_0 = (r_0, g_0, v_0, w_0, \phi_0, \varpi_0, \vartheta_0, \sigma_0, f_0)$ and

$$\begin{aligned} \mathcal{A}Z &= \begin{pmatrix} -g \\ -r \\ -ay \\ -az + m\vartheta \\ -k_0\varpi \\ -k_0\phi \\ -k_1\sigma + \beta w \\ -\vartheta \\ 0 \end{pmatrix}, \quad \mathcal{L}Z = \begin{pmatrix} w + l\varpi \\ -k_0l\phi \\ 0 \\ r \\ k_0lu \\ -lv + \aleph_1\varpi + \int_{\varphi_1}^{\varphi_2} \aleph_2(s)\mathcal{Y}(x, 1, s, t) ds \\ 0 \\ 0 \\ \frac{1}{s}\mathcal{Y}_J \end{pmatrix}, \\ \mathcal{B}Z &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k_2\vartheta \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Utilizing the Fourier transform to (2.4), we get

$$\begin{cases} \widehat{Z}_t + i\mathcal{A}\widehat{Z} + \mathcal{L}\widehat{Z} = -i^2\mathcal{B}\widehat{Z}, \\ \widehat{Z}(t, 0) = \widehat{Z}_0(t), \end{cases} \tag{2.6}$$

where $\widehat{Z}(t, t) = (\widehat{r}, \widehat{g}, \widehat{v}, \widehat{w}, \widehat{\phi}, \widehat{\varpi}, \widehat{\vartheta}, \widehat{\sigma}, \widehat{\mathcal{Y}})^T(t, t)$. The equation (2.6)₁ can be stated as

$$\begin{cases} \widehat{r}_t - i\widehat{g} + \widehat{w} + l\widehat{\varpi} = 0, \\ \widehat{g}_t - ir\widehat{r} - k_0l\widehat{\phi} = 0, \\ \widehat{v}_t - ai\widehat{v} + \widehat{w} = 0, \\ \widehat{w}_t - aii\widehat{v} - \widehat{r} + mi\widehat{\vartheta} = 0, \\ \widehat{\phi}_t - k_0i\widehat{r} + k_0l\widehat{g} = 0, \\ \widehat{\varpi}_t - k_0ii\widehat{\phi} - l\widehat{r} + \aleph_1\widehat{\varpi} + \int_{\varphi_1}^{\varphi_2} \aleph_2(s)\widehat{\mathcal{Y}}(t, 1, s, t) ds = 0, \\ \widehat{\vartheta}_t - k_1ii\widehat{\sigma} + \beta\widehat{w} + i^2k_2\widehat{\vartheta} = 0, \\ \widehat{\sigma}_t - ii\widehat{\vartheta} = 0, \\ s\widehat{\mathcal{Y}}_t + \widehat{\mathcal{Y}}_J = 0. \end{cases} \tag{2.7}$$

Lemma 2.1 Suppose that (1.8) holds. Assume that $\widehat{Z}(t, t)$ is the solution of (2.6), then the energy functional $\widehat{V}(t, t)$ is stated as

$$\widehat{V}(t, t) = \frac{\beta}{2} \left\{ |\widehat{r}|^2 + |\widehat{g}|^2 + |\widehat{v}|^2 + |\widehat{w}|^2 + |\widehat{\phi}|^2 + |\widehat{\varpi}|^2 + \frac{m}{\beta}|\widehat{\vartheta}|^2 + \frac{mk_1}{\beta}|\widehat{\sigma}|^2 \right\}$$

$$+ \frac{\beta}{2} \int_0^1 \int_{\wp_1}^{\wp_2} s |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ, \quad (2.8)$$

satisfies

$$\frac{d\widehat{V}(\iota, t)}{dt} \leq -C_1 |\widehat{\varpi}|^2 - k_2 m \iota^2 |\widehat{\vartheta}|^2 \leq 0, \quad (2.9)$$

where $C_1 = \beta (\mathfrak{N}_1 - \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds) > 0$.

Proof First of all, multiplying (2.7)_{1,2,3,4,5,6} by $\beta\bar{r}$, $\beta\bar{g}$, $\beta\bar{v}$, $\beta\bar{w}$, $\beta\bar{\phi}$, and $\beta\bar{\sigma}$, respectively. Further, multiplying (2.7)_{7,8} by $m\bar{\vartheta}$ and $k_1 m\bar{\sigma}$. Then by adding these equalities and taking the real part, we obtain

$$\begin{aligned} & \frac{\beta}{2} \frac{d}{dt} \left[|\widehat{r}|^2 + |\widehat{g}|^2 + |\widehat{v}|^2 + |\widehat{w}|^2 + |\widehat{\phi}|^2 + |\widehat{\vartheta}|^2 + \frac{m}{\beta} |\widehat{\vartheta}|^2 + \frac{mk_1}{\beta} |\widehat{\sigma}|^2 \right] dx \\ & + k_2 m \iota^2 |\widehat{\vartheta}|^2 + \beta \mathfrak{N}_1 |\widehat{\vartheta}|^2 + \Re e \left\{ \beta \int_{\wp_1}^{\wp_2} \mathfrak{N}_2(s) \overline{\widehat{\vartheta}} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds \right\} = 0. \end{aligned} \quad (2.10)$$

In second step, by multiplying (2.7)₉ by $\overline{\widehat{\mathcal{Y}}}|\mathfrak{N}_2(s)|$ and integrating the result over $(0, 1) \times (\wp_1, \wp_2)$

$$\begin{aligned} & \frac{d}{dt} \frac{\beta}{2} \int_0^1 \int_{\wp_1}^{\wp_2} s |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ & = -\frac{\beta}{2} \int_0^1 \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| \frac{d}{dJ} |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ & = \frac{\beta}{2} \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| (|\widehat{\mathcal{Y}}(\iota, 0, s, t)|^2 - |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2) ds \\ & = \frac{\beta}{2} \left(\int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds \right) |\widehat{\vartheta}|^2 - \frac{\beta}{2} \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds, \end{aligned} \quad (2.11)$$

utilizing Young's inequality, we get

$$\begin{aligned} & \Re e \left\{ \beta \int_{\wp_1}^{\wp_2} \mathfrak{N}_2(s) \overline{\widehat{\vartheta}} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds \right\} \\ & \leq \frac{\beta}{2} \left(\int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds \right) |\widehat{\vartheta}|^2 + \frac{\beta}{2} \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds, \end{aligned} \quad (2.12)$$

by substituting (2.11) and (2.12) into (2.10), we find

$$\frac{d\widehat{V}(\iota, t)}{dt} \leq -\beta \left(\mathfrak{N}_1 - \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds \right) |\widehat{\vartheta}|^2 - k_2 m \iota^2 |\widehat{\vartheta}|^2,$$

then, by (1.8), $\exists C_1 = \beta (\mathfrak{N}_1 - \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds) > 0$ such that

$$\frac{d\widehat{V}(\iota, t)}{dt} \leq -C_1 |\widehat{\vartheta}|^2 - k_2 m \iota^2 |\widehat{\vartheta}|^2 \leq 0. \quad (2.13)$$

Hence, we get the required result. \square

The following Lemma is required in order to get the main result.

Lemma 2.2 *The functional*

$$\mathcal{D}_1(\iota, t) := \Re e \{ i\iota (\widehat{\varpi} \bar{\phi} + l \widehat{\phi} \bar{w}) \}, \quad (2.14)$$

satisfies the following for any $\varepsilon_1 > 0$

$$\begin{aligned} \frac{d\mathcal{D}_1(\iota, t)}{dt} &\leq -\frac{k_0}{2} \iota^2 |\widehat{\phi}|^2 + 2\varepsilon_1 \frac{\iota^2}{1+\iota^2} |\widehat{g}|^2 + c(\varepsilon_1)(1+\iota^2) |\widehat{\varpi}|^2 \\ &\quad + c(\varepsilon_1)(1+\iota^2) |\widehat{w}|^2 + c|\widehat{\vartheta}|^2 \\ &\quad + c \int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \end{aligned} \quad (2.15)$$

Proof By differentiating \mathcal{D}_1 and using (2.7), we get

$$\begin{aligned} \frac{d\mathcal{D}_1(\iota, t)}{dt} &= \Re e \{ i\iota \widehat{\varpi}_t \bar{\phi} - i\iota \widehat{\phi}_t \bar{\varpi} + i\iota l \widehat{\phi}_t \bar{w} - i\iota l \widehat{w}_t \bar{\phi} \} \\ &= -k_0 \iota^2 |\widehat{\phi}|^2 + k_0 \iota^2 |\widehat{\varpi}|^2 - \Re e \{ i\aleph_1 \iota \widehat{\varpi} \bar{\phi} \} + \Re e \{ a l \iota^2 \widehat{\nu} \bar{\phi} \} \\ &\quad + \Re e \{ i k_0 l \iota \widehat{g} \bar{\varpi} \} - \Re e \{ k_0 l \iota^2 \widehat{\varpi} \bar{w} \} - \Re e \{ i k_0 l^2 \iota \widehat{g} \bar{w} \} \\ &\quad - \Re e \{ m l \iota^2 \widehat{\vartheta} \bar{\phi} \} - \Re e \left\{ i\iota \int_{\wp_1}^{\wp_2} \aleph_2(s) \bar{\phi} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds \right\}. \end{aligned} \quad (2.16)$$

The terms in the RHS of (2.16) are obtained by utilizing the Young's inequality. For any $\varepsilon_1, \delta_1, \delta_2 > 0$, we have

$$\begin{aligned} -\Re e \{ i\aleph_1 \iota \widehat{\varpi} \bar{\phi} \} &\leq \delta_1 \iota^2 |\widehat{\phi}|^2 + c(\delta_1) |\widehat{\varpi}|^2, \\ \Re e \{ i k_0 l \iota \widehat{g} \bar{\varpi} \} &\leq \varepsilon_1 \frac{\iota^2}{1+\iota^2} |\widehat{g}|^2 + c(\varepsilon_1)(1+\iota^2) |\widehat{\varpi}|^2, \\ -\Re e \{ l k_0 \iota \widehat{\varpi} \bar{w} \} &\leq c \iota^2 |\widehat{w}|^2 + c |\widehat{\varpi}|^2, \\ -\Re e \{ i k_0 l^2 \iota \widehat{g} \bar{w} \} &\leq \varepsilon_1 \frac{\iota^2}{1+\iota^2} |\widehat{g}|^2 + c(\varepsilon_1)(1+\iota^2) |\widehat{w}|^2, \\ \Re e \{ a l \iota^2 \widehat{\nu} \bar{\phi} \} &\leq \delta_1 \iota^2 |\widehat{\phi}|^2 + c(\delta_1) \iota^2 |\widehat{\nu}|^2, \\ -\Re e \{ m l \iota^2 \widehat{\vartheta} \bar{\phi} \} &\leq \delta_1 \iota^2 |\widehat{\phi}|^2 + c(\delta_1) \iota^2 |\widehat{\vartheta}|^2, \\ -\Re e \left\{ i\iota \int_{\wp_1}^{\wp_2} \aleph_2(s) \bar{\phi} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds \right\} &\leq \delta_2 \aleph_1 \iota^2 |\widehat{\phi}|^2 + c(\delta_2) \int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \end{aligned} \quad (2.17)$$

Inserting the above estimates (2.17) into (2.16) and by letting $\delta_1 = \frac{k_0}{12}$, $\delta_2 = \frac{k_0}{4\aleph_1}$, we get the required (2.15). \square

Lemma 2.3 *The functional*

$$\mathcal{D}_2(\iota, t) := \Re e \{ i\iota (a k_1 \widehat{\vartheta} \bar{\sigma} + a \beta \widehat{\vartheta} \bar{w} + 2 k_1 \widehat{v} \bar{\sigma}) \}, \quad (2.18)$$

satisfies the following for any $\varepsilon_2, \varepsilon_3 > 0$

$$\begin{aligned} \frac{d\mathcal{D}_2(\iota, t)}{dt} \leq & -\frac{ak_1^2}{2}\iota^2|\widehat{\sigma}|^2 - \frac{a\beta^2}{2}\iota^2|\widehat{w}|^2 + \varepsilon_2\iota^2|\widehat{r}|^2 + \varepsilon_3\iota^2|\widehat{v}|^2 \\ & + c(\varepsilon_2, \varepsilon_3)(1 + \iota^2 + \iota^4)|\widehat{\vartheta}|^2. \end{aligned} \quad (2.19)$$

Proof By differentiating \mathcal{D}_2 and using (2.7), we get

$$\begin{aligned} \frac{\mathcal{D}_2(\iota, t)}{dt} = & \Re e\{iuak_1\widehat{\vartheta}_t\overline{\widehat{\sigma}} - iuak_1\widehat{\sigma}_t\overline{\widehat{\vartheta}} - iu\beta a\widehat{\vartheta}_t\overline{\widehat{w}} + iu\beta a\widehat{w}_t\overline{\widehat{\vartheta}}\} \\ & + \Re e\{2iu\beta k_1\widehat{v}_t\overline{\widehat{\sigma}} - 2iu\beta k_1\widehat{\sigma}_t\overline{\widehat{v}}\} \\ = & -ak_1\iota^2|\widehat{\sigma}|^2 - a\beta^2\iota^2|\widehat{w}|^2 + a(k_1 + m\beta)\iota^2|\widehat{\vartheta}|^2 \\ & + \Re e\{\beta\iota^2(2k_1 - a^2)\widehat{v}\overline{\widehat{\vartheta}}\} + \Re e\{ia\beta\iota\widehat{r}\overline{\widehat{\vartheta}}\} \\ & - \Re e\{iaak_1k_2\iota^3\widehat{\vartheta}\overline{\widehat{\sigma}}\} + \Re e\{ia\beta k_2\iota^3\widehat{\vartheta}\overline{\widehat{w}}\}. \end{aligned} \quad (2.20)$$

The terms in the RHS of (2.20) are obtained by utilizing Young's inequality. Next, for any $\varepsilon_2, \varepsilon_3, \delta_3, \delta_4 > 0$, we can find

$$\begin{aligned} \Re e\{ia\beta\iota\widehat{r}\overline{\widehat{\vartheta}}\} & \leq \varepsilon_2\iota^2|\widehat{r}|^2 + c(\varepsilon_2)|\widehat{\vartheta}|^2, \\ \Re e\{\beta\iota^2(2k_1 - a^2)\widehat{v}\overline{\widehat{\vartheta}}\} & \leq \varepsilon_3\iota^2|\widehat{v}|^2 + c(\varepsilon_3)|\widehat{\vartheta}|^2, \\ -\Re e\{iaak_1k_2\iota^3\widehat{\vartheta}\overline{\widehat{\sigma}}\} & \leq \delta_3\iota^2|\widehat{\sigma}|^2 + c(\delta_3)\iota^4|\widehat{\vartheta}|^2, \\ \Re e\{ia\beta k_2\iota^3\widehat{\vartheta}\overline{\widehat{w}}\} & \leq \delta_4\iota^2|\widehat{w}|^2 + c(\delta_4)\iota^4|\widehat{\vartheta}|^2. \end{aligned} \quad (2.21)$$

By substituting (2.21) into (2.20) and letting $\delta_3 = \frac{ak_1^2}{2}, \delta_4 = \frac{a\beta^2}{2}$, we get (2.19). \square

Lemma 2.4 *The functional*

$$\mathcal{D}_3(\iota, t) := \Re e\{\widehat{\phi}\overline{\widehat{g}}\}, \quad (2.22)$$

satisfies the below for any $\varepsilon_4 > 0$

$$\frac{d\mathcal{D}_3(\iota, t)}{dt} \leq -\frac{k_0l^2}{2}|\widehat{g}|^2 + \varepsilon_4|\widehat{r}|^2 + c\iota^2|\widehat{\vartheta}|^2 + c(\varepsilon_4)(1 + \iota^2)|\widehat{\phi}|^2. \quad (2.23)$$

Proof By differentiating \mathcal{D}_3 and using (2.7), we have

$$\begin{aligned} \frac{\mathcal{D}_3(\iota, t)}{dt} = & \Re e\{\widehat{\phi}_t\overline{\widehat{g}} + \widehat{g}_t\overline{\widehat{\phi}}\} \\ = & -k_0l|\widehat{g}|^2 + k_0l|\widehat{\phi}|^2 \\ & + \Re e\{ik_0\iota\widehat{\vartheta}\overline{\widehat{g}}\} + \Re e\{iu\widehat{r}\overline{\widehat{\phi}}\}. \end{aligned} \quad (2.24)$$

The last two terms in the RHS of (2.24) are obtained by Young's inequality, which we solve for any $\varepsilon_4, \delta_5 > 0$

$$\begin{aligned}\Re e\{ik_0\iota\widehat{\varpi}\bar{g}\} &\leq \delta_5|\bar{g}|^2 + c(\delta_5)\iota^2|\widehat{\varpi}|^2, \\ \Re e\{i\iota\widehat{r}\bar{\phi}\} &\leq \varepsilon_4|\widehat{r}|^2 + c(\varepsilon_4)\iota^2|\widehat{\phi}|^2.\end{aligned}\tag{2.25}$$

By substituting (2.25) into (2.24) and letting $\delta_5 = \frac{k_0\iota^2}{2}$, we obtained (2.23). \square

Next, we have the following lemma.

Lemma 2.5 *The functional*

$$\mathcal{D}_4(\iota, t) := a\iota\mathcal{F}_1(\iota, t) - \iota^2\mathcal{F}_2(\iota, t),\tag{2.26}$$

where

$$\mathcal{F}_1(\iota, t) := \Re e\{i\iota(l\widehat{w}\bar{\nu} + \widehat{v}\bar{\varpi})\} \quad \text{and} \quad \mathcal{F}_2(\iota, t) := \Re e\{(\widehat{w}\bar{\nu} + a\widehat{g}\bar{\nu})\},\tag{2.27}$$

satisfies

(1) For $a = 1$. Then,

$$\begin{aligned}\frac{d\mathcal{D}_4(\iota, t)}{dt} &\leq -\frac{a^2\iota^2}{2}\iota^2|\widehat{\nu}|^2 - \frac{1}{2}\iota^2|\widehat{r}|^2 + c|\widehat{\varpi}|^2 + (1+a^2\iota^2)\iota^2|\widehat{w}|^2 \\ &\quad + c(\iota^2 + \iota^4)|\widehat{\vartheta}|^2 + c\int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds.\end{aligned}\tag{2.28}$$

(2) For $a \neq 1$. Then, for any $\varepsilon_5 > 0$

$$\begin{aligned}\frac{d\mathcal{D}_4(\iota, t)}{dt} &\leq -\frac{a^2\iota^2}{2}\iota^2|\widehat{\nu}|^2 - \frac{1}{2}\iota^2|\widehat{r}|^2 + \varepsilon_5\frac{\iota^4}{(1+\iota^2)^2}|\widehat{g}|^2 + c(\varepsilon_5)\iota^2(1+\iota^2)^2|\widehat{w}|^2 \\ &\quad + c(1+\iota^2)|\widehat{\varpi}|^2 + c(\iota^2 + \iota^4)|\widehat{\vartheta}|^2 \\ &\quad + c\int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds.\end{aligned}\tag{2.29}$$

Proof Firstly, by differentiating $\mathcal{F}_1, \mathcal{F}_2$ and using (2.7), we get

$$\begin{aligned}\frac{d\mathcal{F}_1(\iota, t)}{dt} &= \Re e\{i\iota l\widehat{w}_t\bar{\nu} - i\iota\widehat{v}_t\bar{\varpi} + i\iota\widehat{v}_t\bar{\varpi} - i\iota\widehat{\varpi}_t\bar{\nu}\} \\ &= -a\iota l^2|\widehat{\nu}|^2 + a\iota l^2|\widehat{w}|^2 + \Re e\{i\aleph_1\iota\widehat{\varpi}\bar{\nu}\} - \Re e\{a\iota^2\widehat{w}\bar{\varpi}\} \\ &\quad + \Re e\{k_0\iota^2\widehat{\phi}\bar{\nu}\} + \Re e\{ml\iota^2\widehat{\vartheta}\bar{\nu}\} \\ &\quad + \Re e\left\{i\iota\int_{\wp_1}^{\wp_2} \aleph_2(s)\bar{\nu}\widehat{\mathcal{Y}}(\iota, 1, s, t) ds\right\},\end{aligned}\tag{2.30}$$

and

$$\frac{d\mathcal{F}_2(\iota, t)}{dt} = \Re e\{\widehat{w}_t\bar{\nu} + \widehat{r}_t\bar{\varpi} + a\widehat{v}_t\bar{g} + a\widehat{g}_t\bar{\nu}\}$$

$$\begin{aligned}
&= -|\widehat{w}|^2 + |\widehat{r}|^2 + \Re\{i(a^2 - 1)\iota \widehat{w} \overline{\widehat{g}}\} - \Re\{im\iota \widehat{v} \overline{\widehat{r}}\} \\
&\quad - \Re\{l \widehat{w} \overline{\widehat{w}}\} + \Re\{al k_0 \phi \overline{\widehat{v}}\}.
\end{aligned} \tag{2.31}$$

Now, differentiating \mathcal{D}_4 and by (2.30) and (2.31), we have

$$\begin{aligned}
\frac{d\mathcal{D}_4(\iota, t)}{dt} &= -a^2 l^2 \iota^2 |\widehat{v}|^2 - \iota^2 |\widehat{r}|^2 + (1 + a^2 l^2) \iota^2 |\widehat{w}|^2 + \Re\{ial \aleph_1 \iota \widehat{w} \overline{\widehat{v}}\} \\
&\quad + \Re\{i(1 - a^2) \iota^3 \widehat{w} \overline{\widehat{g}}\} + \Re\{im \iota^3 \widehat{v} \overline{\widehat{r}}\} + \Re\{l(1 - a^2) \iota^2 \widehat{w} \overline{\widehat{w}}\} \\
&\quad + \Re\{am l^2 \iota^2 \widehat{v} \overline{\widehat{v}}\} + \Re\left\{ial \iota \int_{\wp_1}^{\wp_2} \aleph_2(s) \overline{\widehat{v}} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds\right\}.
\end{aligned} \tag{2.32}$$

At this point, we discuss two cases:

Case 1. ($a = 1$).

In this case, by applying the Young's inequality to the terms on the RHS of (2.32). Then, for any $\delta_6, \delta_7, \delta_8 > 0$, we get

$$\begin{aligned}
\Re\{ial \aleph_1 \iota \widehat{w} \overline{\widehat{v}}\} &\leq \delta_6 \iota^2 |\widehat{v}|^2 + c(\delta_6) |\widehat{w}|^2, \\
\Re\{im \iota^3 \widehat{v} \overline{\widehat{r}}\} &\leq \delta_7 \iota^2 |\widehat{r}|^2 + c(\delta_7) \iota^4 |\widehat{v}|^2, \\
\Re\{am l^2 \iota^2 \widehat{v} \overline{\widehat{v}}\} &\leq \delta_6 \iota^2 |\widehat{v}|^2 + c(\delta_6) \iota^2 |\widehat{v}|^2, \\
\Re\left\{ial \iota \int_{\wp_1}^{\wp_2} \aleph_2(s) \overline{\widehat{v}} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds\right\} \\
&\leq \delta_8 \aleph_1 \iota^2 |\widehat{v}|^2 + c(\delta_8) \int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds.
\end{aligned} \tag{2.33}$$

Inserting the above estimates of (2.33) into (2.32).

Finally, by letting $\delta_6 = \frac{a^2 l^2}{8}$, $\delta_7 = \frac{1}{2}$, $\delta_8 = \frac{a^2 l^2}{4 \aleph_1}$, we obtained (2.28).

Case 2. ($a \neq 1$).

In this case, using the Young's inequality to the terms on the RHS of (2.32) for any $\varepsilon_5, \delta_9, \delta_{10}, \delta_{11} > 0$ gives

$$\begin{aligned}
\Re\{ial \aleph_1 \iota \widehat{w} \overline{\widehat{v}}\} &\leq \delta_9 \iota^2 |\widehat{v}|^2 + c(\delta_9) |\widehat{w}|^2, \\
\Re\{i(1 - a^2) \iota^3 \widehat{w} \overline{\widehat{g}}\} &\leq \varepsilon_5 \frac{\iota^4}{(1 + \iota^2)^2} |\widehat{g}|^2 + c(\varepsilon_5) \iota^2 (1 + \iota^2)^2 |\widehat{w}|^2, \\
\Re\{im \iota^3 \widehat{v} \overline{\widehat{r}}\} &\leq \delta_{10} \iota^2 |\widehat{r}|^2 + c(\delta_{10}) \iota^4 |\widehat{v}|^2, \\
\Re\{l(1 - a^2) \iota^2 \widehat{w} \overline{\widehat{w}}\} &\leq c \iota^2 |\widehat{w}|^2 + c \iota^2 |\widehat{w}|^2, \\
\Re\{am l^2 \iota^2 \widehat{v} \overline{\widehat{v}}\} &\leq \delta_9 \iota^2 |\widehat{v}|^2 + c(\delta_9) \iota^2 |\widehat{v}|^2, \\
\Re\left\{ial \iota \int_{\wp_1}^{\wp_2} \aleph_2(s) \overline{\widehat{v}} \widehat{\mathcal{Y}}(\iota, 1, s, t) ds\right\} \\
&\leq \delta_{11} \aleph_1 \iota^2 |\widehat{v}|^2 + c(\delta_{11}) \int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds.
\end{aligned} \tag{2.34}$$

Inserting (2.34) into (2.32), and letting $\delta_9 = \frac{a^2 l^2}{8}$, $\delta_{10} = \frac{1}{2}$, $\delta_{11} = \frac{a^2 l^2}{4 \aleph_1}$, we get (2.29). The proof of Lemma 2.5 is completed. \square

Now, introducing the following functional.

Lemma 2.6 *The functional*

$$\mathcal{D}_5(\iota, t) := \int_0^1 \int_{\wp_1}^{\wp_2} s e^{-sJ} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ,$$

satisfies

$$\begin{aligned} \frac{d\mathcal{D}_5(\iota, t)}{dt} &\leq -\zeta_1 \int_0^1 \int_{\wp_1}^{\wp_2} s |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ + \mathfrak{N}_1 |\widehat{\varpi}|^2 \\ &\quad - \zeta_1 \int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds, \end{aligned} \quad (2.35)$$

where $\zeta_1 > 0$.

Proof By differentiating \mathcal{D}_5 with respect to t and utilizing (2.7)₉, we have

$$\begin{aligned} \frac{d\mathcal{D}_5(\iota, t)}{dt} &= - \int_0^1 \int_{\wp_1}^{\wp_2} s e^{-sJ} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ &\quad - \int_{\wp_1}^{\wp_2} e^{-s} |\mathfrak{N}_2(s)| [e^{-s} |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 - |\widehat{\mathcal{Y}}(\iota, 0, s, t)|^2] ds. \end{aligned}$$

Using $\mathcal{Y}(\iota, 0, s, t) = \mathfrak{Y}_t(\iota, t) = \varpi$, & $e^{-s} \leq e^{-sJ} \leq 1$, $\forall 0 < J < 1$, we have

$$\begin{aligned} \frac{d\mathcal{D}_5(\iota, t)}{dt} &\leq - \int_0^1 \int_{\wp_1}^{\wp_2} s e^{-s} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ &\quad - \int_{\wp_1}^{\wp_2} e^{-s} |\mathfrak{N}_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds + \left(\int_{\wp_1}^{\wp_2} |\mathfrak{N}_2(s)| ds \right) |\widehat{\varpi}|^2. \end{aligned}$$

Next, we have $-e^{-s} \leq -e^{-\wp_2}$, for all $s \in [\wp_1, \wp_2]$, since $-e^{-s}$ is an increasing function. Assuming that $\zeta_1 = e^{-\wp_2}$ and remembering (1.8), we obtain (2.35). \square

We define the Lyapunov functionals at this point

- For $\alpha = 1$:

$$\begin{aligned} \mathcal{K}_1(\iota, t) &:= N \widehat{V}(\iota, t) + N_1 \frac{\iota^4}{(1 + \iota^2)^3} \mathcal{D}_1(\iota, t) + N_2 \frac{\iota^2}{(1 + \iota^2)^2} \mathcal{D}_2(\iota, t) \\ &\quad + N_3 \frac{\iota^6}{(1 + \iota^2)^4} \mathcal{D}_3(\iota, t) + N_4 \frac{\iota^2}{(1 + \iota^2)^2} \mathcal{D}_4(\iota, t) + N_5 \mathcal{D}_5(\iota, t). \end{aligned} \quad (2.36)$$

- For $\alpha \neq 1$:

$$\begin{aligned} \mathcal{K}_2(\iota, t) &:= M \widehat{V}(\iota, t) + M_1 \frac{\iota^4}{(1 + \iota^2)^6} \mathcal{D}_1(\iota, t) + M_2 \frac{\iota^2}{(1 + \iota^2)^3} \mathcal{D}_2(\iota, t) \\ &\quad + M_3 \frac{\iota^6}{(1 + \iota^2)^7} \mathcal{D}_3(\iota, t) + M_4 \frac{\iota^2}{(1 + \iota^2)^5} \mathcal{D}_4(\iota, t) + M_5 \mathcal{D}_5(\iota, t), \end{aligned} \quad (2.37)$$

where $N, M, N_i, M_i, i = 1, \dots, 5$ are positive constants and will be selected later.

Lemma 2.7 There exist $\mu_i > 0, i = 1, \dots, 6$ such that the functionals $\mathcal{K}_1(\iota, t)$ and $\mathcal{K}_2(\iota, t)$ given by (2.36) and (2.37) satisfies

- For $\alpha = 1$:

$$\begin{cases} \mu_1 \widehat{V}(\iota, t) \leq \mathcal{K}_1(\iota, t) \leq \mu_2 \widehat{V}(\iota, t), \\ \mathcal{K}'_1(\iota, t) \leq -\mu_3 J_1(\iota) \mathcal{K}_1(\iota, t), \quad \forall t > 0. \end{cases} \quad (2.38)$$

- For $\alpha \neq 1$:

$$\begin{cases} \mu_4 \widehat{V}(\iota, t) \leq \mathcal{K}_2(\iota, t) \leq \mu_5 \widehat{V}(\iota, t), \\ \mathcal{K}'_2(\iota, t) \leq -\mu_6 J_2(\iota) \mathcal{K}_2(\iota, t), \quad \forall t > 0, \end{cases} \quad (2.39)$$

where

$$J_1(\iota) = \frac{\iota^6}{(1 + \iota^2)^4} \quad \text{and} \quad J_2(\iota) = \frac{\iota^6}{(1 + \iota^2)^7}. \quad (2.40)$$

Proof First, by differentiating (2.36) and using (2.9), (2.15), (2.19), (2.23), (2.28) and (2.35) with the fact that $\frac{\iota^2}{1+\iota^2} \leq \min\{1, \iota^2\}$ and $\frac{1}{1+\iota^2} \leq 1$, we have

$$\begin{aligned} \mathcal{K}'_1(\iota, t) &\leq -\frac{\iota^6}{(1 + \iota^2)^4} \left[\frac{k_0 l^2}{2} N_3 - 2\varepsilon_1 N_1 \right] |\widehat{g}|^2 \\ &\quad - \frac{\iota^4}{(1 + \iota^2)^2} \left[\frac{1}{2} N_4 - \varepsilon_2 N_2 - \varepsilon_4 N_3 \right] |\widehat{r}|^2 \\ &\quad - \frac{\iota^4}{(1 + \iota^2)^2} \left[\frac{a\beta^2}{2} N_2 - c(\varepsilon_1) N_1 - c N_4 \right] |\widehat{w}|^2 \\ &\quad - \frac{\iota^4}{(1 + \iota^2)^2} \left[\frac{a^2 l^2}{2} N_4 - \varepsilon_3 N_2 \right] |\widehat{v}|^2 \\ &\quad - \frac{\iota^6}{(1 + \iota^2)^3} \left[\frac{k_0}{2} N_1 - c(\varepsilon_4) N_3 \right] |\widehat{\phi}|^2 - \frac{\iota^4}{(1 + \iota^2)^2} \left[\frac{a k_1^2}{2} N_2 \right] |\widehat{\sigma}|^2 \\ &\quad - \iota^2 [m k_2 N - c N_1 - c(\varepsilon_2, \varepsilon_3) N_2 - c N_4] |\widehat{\vartheta}|^2 \\ &\quad - [C_1 N - c(\varepsilon_1) N_1 - c N_3 - c N_4 - \aleph_1 N_5] |\widehat{\omega}|^2 \\ &\quad - \zeta_1 N_5 \int_0^1 \int_{\mathcal{S}^1}^{\mathcal{S}^2} s |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ &\quad - [\zeta_1 N_5 - c N_1 - c N_4] \int_{\mathcal{S}^1}^{\mathcal{S}^2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \end{aligned} \quad (2.41)$$

By setting

$$\varepsilon_1 = \frac{k_0 l^2 N_3}{8 N_1}, \quad \varepsilon_2 = \frac{N_4}{8 N_2}, \quad \varepsilon_3 = \frac{a^2 l^2 N_4}{4 N_2}, \quad \varepsilon_4 = \frac{N_4}{8 N_3}.$$

We obtain the following

$$\mathcal{K}'_1(\iota, t) \leq -\frac{\iota^6}{(1 + \iota^2)^4} \left[\frac{k_0 l^2}{4} N_3 \right] |\widehat{g}|^2 - \frac{\iota^4}{(1 + \iota^2)^2} \left[\frac{1}{4} N_4 \right] |\widehat{r}|^2$$

$$\begin{aligned}
& - \frac{\iota^4}{(1+\iota^2)^2} \left[\frac{a\beta^2}{2} N_2 - c(N_1, N_3)N_1 - cN_4 \right] |\widehat{w}|^2 \\
& - \frac{\iota^4}{(1+\iota^2)^2} \left[\frac{a^2 l^2}{4} N_4 \right] |\widehat{v}|^2 - \frac{\iota^4}{(1+\iota^2)^2} \left[\frac{a k_1^2}{2} N_2 \right] |\widehat{\sigma}|^2 \\
& - \frac{\iota^6}{(1+\iota^2)^3} \left[\frac{k_0}{2} N_1 - c(N_3, N_4)N_3 \right] |\widehat{\phi}|^2 \\
& - \iota^2 [mk_2 N - cN_1 - c(N_2, N_4)N_2 - cN_4] |\widehat{\vartheta}|^2 \\
& - [C_1 N - c(N_1, N_3)N_1 - cN_3 - cN_4 - \zeta_1 N_5] |\widehat{\omega}|^2 \\
& - \zeta_1 N_5 \int_0^1 \int_{\wp_1}^{\wp_2} s |\wp_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\
& - [\zeta_1 N_5 - cN_1 - cN_4] \int_{\wp_1}^{\wp_2} |\wp_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \tag{2.42}
\end{aligned}$$

Next, we fix N_3, N_4 and choose N_1 large enough such that

$$\frac{k_0}{2} N_1 - c(N_3, N_4)N_3 > 0,$$

then, we pick N_2 and N_5 large enough in such a way that

$$\begin{aligned}
& \frac{a\beta^2}{2} N_2 - c(N_1, N_3)N_1 - cN_4 > 0, \\
& \zeta_1 N_5 - cN_1 - cN_4 > 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{K}'_1(\iota, t) \leq & -\alpha_0 \frac{\iota^6}{(1+\iota^2)^4} |\widehat{g}|^2 - \alpha_5 \frac{\iota^6}{(1+\iota^2)^3} |\widehat{\phi}|^2 - \iota^2 [mk_2 N - c] |\widehat{\vartheta}|^2 \\
& - \frac{\iota^4}{(1+\iota^2)^2} (\alpha_1 |\widehat{r}|^2 + \alpha_2 |\widehat{w}|^2 + \alpha_3 |\widehat{v}|^2 + \alpha_4 |\widehat{\sigma}|^2) - [C_1 N - c] |\widehat{\omega}|^2 \\
& - \alpha_6 \int_0^1 \int_{\wp_1}^{\wp_2} s |\wp_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ. \tag{2.43}
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
|\mathcal{K}_1(\iota, t) - N \widehat{\mathcal{V}}(\iota, t)| = & N_1 \frac{\iota^4}{(1+\iota^2)^3} |\mathcal{D}_1(\iota, t)| + N_2 \frac{\iota^2}{(1+\iota^2)^2} |\mathcal{D}_2(\iota, t)| \\
& + N_3 \frac{\iota^6}{(1+\iota^2)^4} |\mathcal{D}_3(\iota, t)| + N_4 \frac{\iota^2}{(1+\iota^2)^2} |\mathcal{D}_4(\iota, t)| + N_5 |\mathcal{D}_5(\iota, t)| \\
\leq & \alpha N_1 \frac{\iota^4}{(1+\iota^2)^3} |\Re e\{iu(\widehat{\omega}\bar{\phi} + l\bar{\phi}\widehat{w})\}| \\
& + N_2 \frac{\iota^2}{(1+\iota^2)^2} |\Re e\{iu(ak_1\widehat{\vartheta}\bar{\sigma} + a\beta\widehat{\vartheta}\bar{w} + 2k_1\widehat{v}\bar{\sigma})\}| \\
& + N_3 \frac{\iota^6}{(1+\iota^2)^4} |\Re e\{\widehat{\phi}\bar{g}\}|
\end{aligned}$$

$$\begin{aligned}
& + N_4 \frac{\iota^2}{(1+\iota^2)^2} |\Re e\{i\iota(l\widehat{w}\bar{\nu} + \widehat{v}\bar{\omega})\}| \\
& + N_4 \frac{\iota^2}{(1+\iota^2)^2} |\Re e\{(\widehat{w}\bar{r} + a\widehat{g}\bar{\nu})\}| \\
& + N_5 \int_0^1 \int_{\wp_1}^{\wp_2} s e^{-sj} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, j, s, t)|^2 ds dj.
\end{aligned}$$

By utilizing Young's inequality, the fact that $\frac{\iota^2}{1+\iota^2} \leq \min\{1, \iota^2\}$ and $\frac{1}{1+\iota^2} \leq 1$, we find

$$|\mathcal{K}_1(\iota, t) - N\widehat{V}(\iota, t)| \leq c\widehat{V}(\iota, t).$$

Hence, we get

$$(N - c)\widehat{V}(\iota, t) \leq \mathcal{K}_1(\iota, t) \leq (N + c)\widehat{V}(\iota, t). \quad (2.44)$$

Now, we choose N large enough in such a way that

$$N - c > 0, \quad C_1 N - c > 0, \quad m k_2 N - c > 0,$$

and utilizing (2.8), estimates (2.43) and (2.44), respectively.

One can find a positive constant $\alpha > 0$, then $\forall t > 0$ & $\forall \iota \in \mathbb{R}$, we obtain

$$\mu_1 \widehat{V}(\iota, t) \leq \mathcal{K}_1(\iota, t) \leq \mu_2 \widehat{V}(\iota, t). \quad (2.45)$$

and

$$\begin{aligned}
\mathcal{K}'_1(\iota, t) & \leq -\alpha \frac{\iota^6}{(1+\iota^2)^4} \left(|\widehat{g}|^2 + |\widehat{\phi}|^2 + |\widehat{\vartheta}|^2 + |\widehat{r}|^2 + |\widehat{w}|^2 + |\widehat{\nu}|^2 + |\widehat{\sigma}|^2 + |\widehat{\omega}|^2 \right. \\
& \quad \left. + \int_0^1 \int_{\wp_1}^{\wp_2} s |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, j, s, t)|^2 ds dj \right), \quad (2.46)
\end{aligned}$$

then

$$\mathcal{K}'_1(\iota, t) \leq -\lambda_1 J_1(\iota) \widehat{V}(\iota, t), \quad \forall t \geq 0. \quad (2.47)$$

Therefore, for some positive constant $\mu_3 = \frac{\lambda_1}{\mu_2} > 0$, we get

$$\mathcal{K}'_1(\iota, t) \leq -\mu_3 J_1(\iota) \mathcal{K}_1(\iota, t), \quad \forall t \geq 0, \quad (2.48)$$

where $J_1(\iota) = \frac{\iota^6}{(1+\iota^2)^4}$, for some $\lambda_1, \mu_i > 0, i = 1, 2, 3$. The proof of the first result (2.38) is finished.

Before the proof of the second result (2.39). In the estimates (2.21), we used the inequalities

$$\begin{aligned}
\Re e\{ia\beta\iota\widehat{r}\bar{\vartheta}\} & \leq \varepsilon_2 \frac{\iota^2}{(1+\iota^2)^2} |\widehat{r}|^2 + c(\varepsilon_2)(1+\iota^2)^2 |\widehat{\vartheta}|^2, \\
\Re e\{\beta\iota^2(2k_1 - a^2)\bar{v}\bar{\vartheta}\} & \leq \varepsilon_3 \frac{\iota^2}{(1+\iota^2)^2} |\widehat{v}|^2 + c(\varepsilon_3)(1+\iota^2)^2 |\widehat{\vartheta}|^2.
\end{aligned} \quad (2.49)$$

Hence, the estimate (2.19) can also be written as

$$\begin{aligned} \frac{d\mathcal{D}_2(\iota, t)}{dt} &\leq -\frac{ak_1^2}{2}\iota^2|\widehat{\sigma}|^2 - \frac{a\beta^2}{2}\iota^2|\widehat{w}|^2 + \varepsilon_2\frac{\iota^2}{(1+\iota^2)^2}|\widehat{r}|^2 \\ &\quad + \varepsilon_3\frac{\iota^2}{(1+\iota^2)^2}|\widehat{v}|^2 + c(\varepsilon_2, \varepsilon_3)(1+\iota^2)^2|\widehat{\vartheta}|^2. \end{aligned} \quad (2.50)$$

Similarly, we can prove the second result.

So, we derive (2.37) and by using (2.9), (2.15), (2.50), (2.23), (2.29) and (2.35) with the fact that $\frac{\iota^2}{1+\iota^2} \leq \min\{1, \iota^2\}$ and $\frac{1}{1+\iota^2} \leq 1$, we get

$$\begin{aligned} \mathcal{K}'_2(\iota, t) &\leq -\frac{\iota^6}{(1+\iota^2)^7}\left[\frac{k_0l^2}{2}M_3 - 2\varepsilon_1M_1 - \varepsilon_5M_4\right]|\widehat{g}|^2 \\ &\quad - \frac{\iota^4}{(1+\iota^2)^5}\left[\frac{1}{2}M_4 - \varepsilon_2M_2 - \varepsilon_4M_3\right]|\widehat{r}|^2 \\ &\quad - \frac{\iota^4}{(1+\iota^2)^3}\left[\frac{a\beta^2}{2}M_2 - c(\varepsilon_1)M_1 - cM_4\right]|\widehat{w}|^2 \\ &\quad - \frac{\iota^4}{(1+\iota^2)^5}\left[\frac{a^2l^2}{2}M_4 - \varepsilon_3M_2\right]|\widehat{v}|^2 \\ &\quad - \frac{\iota^6}{(1+\iota^2)^6}\left[\frac{k_0}{2}M_1 - c(\varepsilon_4)M_3\right]|\widehat{\phi}|^2 - \frac{\iota^4}{(1+\iota^2)^3}\left[\frac{ak_1^2}{2}M_2\right]|\widehat{\sigma}|^2 \\ &\quad - \iota^2[mk_2M - cM_1 - c(\varepsilon_2, \varepsilon_3)M_2 - cM_4]|\widehat{\vartheta}|^2 \\ &\quad - [C_1M - c(\varepsilon_1)M_1 - cM_3 - cM_4 - \aleph_1M_5]|\widehat{w}|^2 \\ &\quad - \xi_1M_5 \int_0^1 \int_{\wp_1}^{\wp_2} s |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\ &\quad - [\xi_1M_5 - cM_1 - cM_4] \int_{\wp_1}^{\wp_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \end{aligned} \quad (2.51)$$

By setting

$$\varepsilon_1 = \frac{k_0l^2M_3}{16M_1}, \quad \varepsilon_2 = \frac{M_4}{8M_2}, \quad \varepsilon_3 = \frac{a^2l^2M_4}{4M_2}, \quad \varepsilon_4 = \frac{M_4}{8M_3}, \quad \varepsilon_5 = \frac{k_0l^2M_3}{8M_4},$$

we obtain the following

$$\begin{aligned} \mathcal{K}'_2(\iota, t) &\leq -\frac{\iota^6}{(1+\iota^2)^7}\left[\frac{k_0l^2}{4}M_3\right]|\widehat{g}|^2 - \frac{\iota^4}{(1+\iota^2)^5}\left[\frac{1}{4}M_4\right]|\widehat{r}|^2 \\ &\quad - \frac{\iota^4}{(1+\iota^2)^3}\left[\frac{a\beta^2}{2}M_2 - c(M_1, M_3)M_1 - cM_4\right]|\widehat{w}|^2 \\ &\quad - \frac{\iota^4}{(1+\iota^2)^5}\left[\frac{a^2l^2}{4}M_4\right]|\widehat{v}|^2 - \frac{\iota^4}{(1+\iota^2)^3}\left[\frac{ak_1^2}{2}M_2\right]|\widehat{\sigma}|^2 \\ &\quad - \frac{\iota^6}{(1+\iota^2)^6}\left[\frac{k_0}{2}M_1 - c(M_3, M_4)M_3\right]|\widehat{\phi}|^2 \\ &\quad - \iota^2[mk_2M - cM_1 - c(M_2, M_4)M_2 - cM_4]|\widehat{\vartheta}|^2 \end{aligned}$$

$$\begin{aligned}
& - [C_1 M - c(M_1, M_3)M_1 - cM_3 - cM_4 - \zeta_1 M_5] |\widehat{\omega}|^2 \\
& - \zeta_1 M_5 \int_0^1 \int_{\varphi_1}^{\varphi_2} s |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \\
& - [\zeta_1 M_5 - cM_1 - cM_4] \int_{\varphi_1}^{\varphi_2} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, 1, s, t)|^2 ds. \tag{2.52}
\end{aligned}$$

Next, we fix M_3, M_4 and choose M_1 large enough such that

$$\frac{k_0}{2} M_1 - c(M_3, M_4) M_3 > 0,$$

then, we select M_2, M_5 large enough such that

$$\frac{a\beta^2}{2} M_2 - c(M_1, M_3) M_1 - cM_4 > 0,$$

$$\zeta_1 M_5 - cM_1 - cM_4 > 0.$$

Hence, we arrive at

$$\begin{aligned}
\mathcal{K}'_2(\iota, t) & \leq -\kappa_0 \frac{\iota^6}{(1+\iota^2)^7} |\widehat{g}|^2 - \kappa_5 \frac{\iota^6}{(1+\iota^2)^6} |\widehat{\phi}|^2 - \iota^2 [mk_2 M - c] |\widehat{\vartheta}|^2 \\
& - \frac{\iota^4}{(1+\iota^2)^5} (\kappa_1 |\widehat{r}|^2 + \kappa_3 |\widehat{v}|^2) - \frac{\iota^4}{(1+\iota^2)^3} (\kappa_2 |\widehat{w}|^2 + \kappa_4 |\widehat{\sigma}|^2) \\
& - [C_1 M - c] |\widehat{\omega}|^2 - \kappa_6 \int_0^1 \int_{\varphi_1}^{\varphi_2} s |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ. \tag{2.53}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\mathcal{K}_2(\iota, t) - M \widehat{V}(\iota, t)| & = M_1 \frac{\iota^4}{(1+\iota^2)^6} |\mathcal{D}_1(\iota, t)| + M_2 \frac{\iota^2}{(1+\iota^2)^3} |\mathcal{D}_2(\iota, t)| \\
& + M_3 \frac{\iota^6}{(1+\iota^2)^7} |\mathcal{D}_3(\iota, t)| + M_4 \frac{\iota^2}{(1+\iota^2)^5} |\mathcal{D}_4(\iota, t)| + M_5 |\mathcal{D}_5(\iota, t)| \\
& \leq a M_1 \frac{\iota^4}{(1+\iota^2)^6} |\Re e\{iu(\widehat{\omega}\bar{\phi} + l\phi\bar{w})\}| \\
& + M_2 \frac{\iota^2}{(1+\iota^2)^3} |\Re e\{iu(ak_1\widehat{\vartheta}\bar{\sigma} + a\beta\widehat{\vartheta}\bar{w} + 2k_1\widehat{v}\bar{\sigma})\}| \\
& + M_3 \frac{\iota^6}{(1+\iota^2)^7} |\Re e\{\widehat{\phi}\bar{g}\}| \\
& + M_4 \frac{\iota^2}{(1+\iota^2)^5} |\Re e\{iu(l\widehat{w}\bar{v} + \widehat{v}\bar{\sigma})\}| \\
& + M_4 \frac{\iota^2}{(1+\iota^2)^5} |\Re e\{(\widehat{w}\bar{r} + a\widehat{g}\bar{v})\}| \\
& + M_5 \int_0^1 \int_{\varphi_1}^{\varphi_2} s e^{-sJ} |\aleph_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ.
\end{aligned}$$

Utilizing Young's inequality, and the fact that $\frac{\iota^2}{1+\iota^2} \leq \min\{1, \iota^2\}$ and $\frac{1}{1+\iota^2} \leq 1$, we find

$$|\mathcal{K}_2(\iota, t) - M\widehat{V}(\iota, t)| \leq c\widehat{V}(\iota, t).$$

Hence, we get

$$(M - c)\widehat{V}(\iota, t) \leq \mathcal{K}_2(\iota, t) \leq (M + c)\widehat{V}(\iota, t). \quad (2.54)$$

Now, we choose M large enough in such a way that

$$M - c > 0, \quad C_1M - c > 0, \quad mk_2M - c > 0,$$

using (2.8), we get (2.53) and (2.54), respectively. One can find a positive constant $\kappa > 0$, then $\forall t > 0$ & $\forall \iota \in \mathbb{R}$, we get

$$\mu_4\widehat{V}(\iota, t) \leq \mathcal{K}_2(\iota, t) \leq \mu_5\widehat{V}(\iota, t). \quad (2.55)$$

and

$$\begin{aligned} \mathcal{K}'_2(\iota, t) &\leq -\kappa \frac{\iota^6}{(1+\iota^2)^7} \left(|\widehat{g}|^2 + |\widehat{\phi}|^2 + |\widehat{\vartheta}|^2 + |\widehat{r}|^2 + |\widehat{w}|^2 + |\widehat{\nu}|^2 + |\widehat{\sigma}|^2 + |\widehat{\varpi}|^2 \right. \\ &\quad \left. + \int_0^1 \int_{\wp_1}^{\wp_2} s |\mathfrak{X}_2(s)| |\widehat{\mathcal{Y}}(\iota, J, s, t)|^2 ds dJ \right). \end{aligned} \quad (2.56)$$

then

$$\mathcal{K}'_2(\iota, t) \leq -\lambda_2 J_2(\iota) \widehat{V}(\iota, t), \quad \forall t \geq 0. \quad (2.57)$$

Therefore, for some positive constant $\mu_6 = \frac{\lambda_2}{\mu_5} > 0$, we get

$$\mathcal{K}'_2(\iota, t) \leq -\mu_6 J_2(\iota) \mathcal{K}_2(\iota, t), \quad \forall t \geq 0, \quad (2.58)$$

where $J_2(\iota) = \frac{\iota^6}{(1+\iota^2)^7}$, for some $\lambda_2, \mu_i > 0, i = 4, 5, 6$. The proof of the second result (2.39) is finished. \square

The pointwise estimates of the functional $\widehat{V}(\iota, t)$ are given in the following result.

Proposition 2.8 Suppose (1.8) holds. Then, for any $t \geq 0$ and $\iota \in \mathbb{R}$, there exist a positive constants $d_1, d_2 > 0$ such that the energy functional stated by (2.8) holds

$$\begin{cases} \widehat{V}(\iota, t) \leq d_1 \widehat{V}(\iota, 0) e^{-\mu_3 J_1(\iota)t} & \text{if } a = 1, \\ \widehat{V}(\iota, t) \leq d_2 \widehat{V}(\iota, 0) e^{-\mu_6 J_2(\iota)t} & \text{if } a \neq 1, \end{cases} \quad (2.59)$$

where $J_1(\iota) = \frac{\iota^6}{(1+\iota^2)^4}$, $J_2(\iota) = \frac{\iota^6}{(1+\iota^2)^7}$.

Proof From (2.38)₂ and (2.39)₂, we have

$$\mathcal{K}_1(\iota, t) \leq \mathcal{K}_1(\iota, 0)e^{-\mu_3 J_1(\iota)t}, \quad \forall t \geq 0, \text{ if } \alpha = 1 \quad (2.60)$$

$$\mathcal{K}_2(\iota, t) \leq \mathcal{K}_2(\iota, 0)e^{-\mu_6 J_2(\iota)t}, \quad \forall t \geq 0, \text{ if } \alpha \neq 1. \quad (2.61)$$

Hence, according of (2.38)₁, (2.39)₁ and (2.60), (2.61), we established (2.59). \square

2.2 Decay estimates

Now, we will show the following important result.

Theorem 2.9 Let s be a nonnegative integer, and $Z_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution Z of problem (2.2)–(2.3) holds, $\forall t \geq 0$ the following decay estimates

- For $\alpha = 1$

$$\|\partial_x^k Z(t)\|_2 \leq C \|Z_0\|_1 (1+t)^{-\frac{1}{12} - \frac{k}{6}} + C(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} Z_0\|_2 \quad (2.62)$$

- For $\alpha \neq 1$

$$\|\partial_x^k Z(t)\|_2 \leq C \|Z_0\|_1 (1+t)^{-\frac{1}{12} - \frac{k}{6}} + C(1+t)^{-\frac{\ell}{8}} \|\partial_x^{k+\ell} Z_0\|_2, \quad (2.63)$$

where ℓ and k are nonnegative integers $k + \ell \leq s$ and $C > 0$ is a positive constant.

Proof From (2.8), we get $|\widehat{Z}(\iota, t)|^2 \sim \widehat{V}(\iota, t)$.

- If $\alpha = 1$, then by using the Plancherel theorem and (2.59)₁, we have

$$\begin{aligned} \|\partial_x^k Z(t)\|_2^2 &= \int_{\mathbb{R}} |\iota|^{2k} |\widehat{Z}(\iota, t)|^2 d\iota \\ &\leq c \int_{\mathbb{R}} |\iota|^{2k} e^{-\mu_3 J_1(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c \underbrace{\int_{|\iota| \leq 1} |\iota|^{2k} e^{-\mu_3 J_1(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota}_{R_1} \\ &\quad + c \underbrace{\int_{|\iota| \geq 1} |\iota|^{2k} e^{-\mu_3 J_1(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota}_{R_2}. \end{aligned} \quad (2.64)$$

Now, we estimate R_1, R_2 , the low-frequency part $|\iota| \leq 1$ and the high-frequency part $|\iota| \geq 1$, respectively. First, we have $J_1(\iota) \geq \frac{1}{16}\iota^6$, for $|\iota| \leq 1$. Then

$$\begin{aligned} R_1 &\leq c \int_{|\iota| \leq 1} |\iota|^{2k} e^{-\frac{\mu_3}{16}|\iota|^6 t} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c \sup_{|\iota| \leq 1} \{|\widehat{Z}(\iota, 0)|^2\} \int_{|\iota| \leq 1} |\iota|^{2k} e^{-\frac{\mu_3}{16}|\iota|^6 t} d\iota, \end{aligned} \quad (2.65)$$

by utilizing Lemma 1.1, we get

$$R_1 \leq c \sup_{|\iota| \leq 1} \{|\widehat{Z}(\iota, 0)|^2\} (1+t)^{-\frac{k}{3} - \frac{1}{6}}$$

$$\leq c\|Z_0\|_1^2(1+t)^{-\frac{k}{3}-\frac{1}{6}}. \quad (2.66)$$

Secondly, we have $J_1(\iota) \geq \frac{1}{16}\iota^{-2}$, for $|\iota| \geq 1$. Then

$$R_2 \leq c \int_{|\iota| \geq 1} |\iota|^{2k} e^{-\frac{\mu_3}{16}|\iota|^{-2}t} |\widehat{Z}(\iota, 0)|^2 d\iota, \quad \forall t \geq 0. \quad (2.67)$$

Then, through the inequality

$$\sup_{|\iota| \geq 1} \{|\iota|^{-2\ell} e^{-c\frac{1}{16}|\iota|^{-2}t}\} \leq C(1+t)^{-\ell}, \quad (2.68)$$

we get that

$$\begin{aligned} R_2 &\leq c \sup_{|\iota| \geq 1} \{|\iota|^{-2\ell} e^{-\frac{\mu_3}{16}|\iota|^{-2}t}\} \int_{|\iota| \geq 1} |\iota|^{2(k+\ell)} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c(1+t)^{-\ell} \|\partial_x^{k+\ell} Z(x, 0)\|_2^2, \quad \forall t \geq 0. \end{aligned} \quad (2.69)$$

Substituting (2.66) and (2.69) into (2.64), we find (2.62).

- If $\alpha \neq 1$, similar to the first estimate, we apply the Plancherel theorem and using (2.59)₂, we get

$$\begin{aligned} \|\partial_x^k Z(t)\|_2^2 &= \int_{\mathbb{R}} |\iota|^{2k} |\widehat{Z}(\iota, t)|^2 d\iota \\ &\leq c \int_{\mathbb{R}} |\iota|^{2k} e^{-\mu_6 J_2(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c \underbrace{\int_{|\iota| \leq 1} |\iota|^{2k} e^{-\mu_6 J_2(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota}_{R_3} \\ &\quad + c \underbrace{\int_{|\iota| \geq 1} |\iota|^{2k} e^{-\mu_6 J_2(\iota)t} |\widehat{Z}(\iota, 0)|^2 d\iota}_{R_4}. \end{aligned} \quad (2.70)$$

Now, we estimate R_3, R_4 , the low-frequency part $|\iota| \leq 1$ and the high-frequency part $|\iota| \geq 1$, respectively. First, we have $J_2(\iota) \geq \frac{1}{64}\iota^6$, for $|\iota| \leq 1$. Then

$$\begin{aligned} R_3 &\leq c \int_{|\iota| \leq 1} |\iota|^{2k} e^{-\frac{\mu_6}{64}|\iota|^6 t} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c \sup_{|\iota| \leq 1} \{|\widehat{Z}(\iota, 0)|^2\} \int_{|\iota| \leq 1} |\iota|^{2k} e^{-\frac{\mu_6}{64}|\iota|^6 t} d\iota, \end{aligned} \quad (2.71)$$

by utilizing Lemma 1.1, we get

$$\begin{aligned} R_3 &\leq c \sup_{|\iota| \leq 1} \{|\widehat{Z}(\iota, 0)|^2\} (1+t)^{-\frac{k}{3}-\frac{1}{6}} \\ &\leq c\|Z_0\|_1^2(1+t)^{-\frac{k}{3}-\frac{1}{6}}. \end{aligned} \quad (2.72)$$

Secondly, we have $J_2(\iota) \geq \frac{1}{64}\iota^{-8}$, for $|\iota| \geq 1$. Then

$$R_4 \leq c \int_{|\iota| \geq 1} |\iota|^{2k} e^{-\frac{\mu_6}{64}|\iota|^{-8}t} |\widehat{Z}(\iota, 0)|^2 d\iota, \quad \forall t \geq 0. \quad (2.73)$$

By (2.68), we find

$$\begin{aligned} R_4 &\leq c \sup_{|\iota| \geq 1} \left\{ |\iota|^{-2\ell} e^{-\frac{\mu_6}{64}|\iota|^{-8}t} \right\} \int_{|\iota| \geq 1} |\iota|^{2(k+\ell)} |\widehat{Z}(\iota, 0)|^2 d\iota \\ &\leq c(1+t)^{-\frac{\ell}{4}} \left\| \partial_x^{k+\ell} Z(x, 0) \right\|_2^2, \quad \forall t \geq 0. \end{aligned} \quad (2.74)$$

□

Substituting (2.72) and (2.74) into (2.70), we obtain (2.63).

3 Conclusion

The investigation of the general decay estimate of Bresse–Fourier system solutions with respect to the distributed delay term is the goal of this work, which employs the energy technique in Fourier space.

The different process that results from the distributed delay, which determines the formation of this term in the system in Fourier space, is what concerns us in the current work.

In the upcoming works, we will try the same approach in the same systems, but with various memory types; we anticipate getting results that are comparable.

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Competing interests

The authors declare no competing interests.

Author contributions

A. Choucha conceptualized, investigated, analyzed and validated the research while Salah Boulaaras, Rashid Jan and Rafik Guefaifia formulated, investigated, numerically examined, reviewed and supervised this research work.

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