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Infinitely many solutions for quasilinear Schrödinger equation with general superlinear nonlinearity

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Abstract

In this article, we study the quasilinear Schrödinger equation

$$-\Delta(u) + V(x)u - \Delta(u^2)u = g(x,u), \quad x \in \mathbb{R}^N,$$

where the potential $V(x)$ and the primitive of $g(x,u)$ are allowed to be sign-changing. Under more general superlinear conditions on g, we obtain the existence of infinitely many nontrivial solutions by using the mountain pass theorem. Recent results in the literature are significantly improved.

MSC: 35J20; 35J62; 35Q55

Keywords: Quasilinear Schrödinger equation; Sign-changing potential; Mountain pass theorem

1 Introduction

In this paper, we study the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - \triangle\left(u^2\right)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $V \in C(\mathbb{R}^N, \mathbb{R}), g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Its solutions are related to the existence of standing wave solutions of the following quasilinear Schrödinger equation:

$$i\frac{\partial \Gamma}{\partial t} = -\Delta\Gamma + W(x)\Gamma - k\triangle\left(\theta\left(|\Gamma|^2\right)\right)\theta'\left(|\Gamma|^2\right)\Gamma - g(x,\Gamma), \quad \forall x \in \mathbb{R}^N. \tag{1.2}$$

In recent years, many scholars studied the standing wave solutions of quasilinear Schrödinger equation via variational methods, such as [1–6]. At that time, the classical semilinear elliptic equation was widely studied under certain conditions of V and g, see [7–9]. In many works, problem (1.1) cannot be solved directly by the variational method, but a change of variables can solve this problem. The main difficulty in solving problem (1.2) is that there is no suitable space to define the energy functional corresponding to equation (1.1), see for example [10]. Liu, Wang, and Wang [11], He and Qian [12] studied the existence of solutions for quasilinear Schrödinger equation. They transformed the quasilinear equation into the semilinear equation in a common Sobolev space framework by using a change of variables. Colin, Jeanjean [13], and Willem [14] also used the same method in Orlicz space framework. From [15], we know that a change of variables has some shortcomings. The existence and multiplicity of nontrivial solutions are proved by the minimax method, the Nehari method, a change of variables, and the perturbation method in [16]. By using the perturbation method, Wu and Wu [17] obtained the existence of positive solutions, negative solutions, and a sequence of high energy solutions; Liu, Liu, and Wang [15] obtained the existence of ground state positive solution for a quasilinear elliptic equation. But the perturbation method is not as simple as a change of variables. It is more suitable to solve the problem of the existence of a single solution, but has some limitations in dealing with the problem of multiple solutions. A change of variables is simple and effective in solving problems, but it depends on the specific expression of an equation to a great extent and cannot transform a more general quasilinear equation into a semilinear equation. Liu, Liu, and Wang [18], Liu and Chen [19], Wang and Chen [20] considered the quasilinear Schrödinger equation with critical growth. Wang [21] used the perturbation method to consider the quasilinear elliptic equations with critical growth. Liu, Liu, and Wang [15] used the perturbation method to consider the more general quasilinear critical problem. The more general quasilinear critical problem was also considered by Dong Fang and Szulkin [22], Chen, Tang, and Cheng [23], Xue and Tang [24].

Many authors always assumed that the potential V is positive. If the potential is sign-changing, then the existence of the negative part of the potential function increases the difficulty of proving the boundedness of (PS) sequence and improves the energy level of the corresponding functional. $\Phi(u)$ will donate an energy functional of solution u.

More precisely, Zhang, Tang, and Zhang [25] studied problem (1.1) with sign-changing potential and obtained the existence of infinitely many solutions under superlinear assumptions. They obtained the following theorem.

Theorem 1.1 *Assume that V and g satisfy the following conditions*:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ *and* $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$;

(V_2) *There exists a constant* $r > 0$ *such that*

$$\lim_{|y|\to+\infty} \operatorname{meas}\bigl(\{x \in \mathbb{R}^N : |x-y| \le r, V(x) \le M\}\bigr) = 0, \quad \forall M > 0;$$

(G_0) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, *and there exist constants* $c_1, c_2 > 0$ *and* $4 < p < 22^*$ *such that*

$$\bigl|g(x,u)\bigr| \le c_1|u| + c_2|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(G_1) $\lim_{|u|\to\infty} \frac{G(x,u)}{u^4} = \infty$ *uniformly in* x, *and there exists* $r_0 \ge 0$ *such that* $G(x,u) \ge 0$ *for any* $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ *and* $|u| \ge r_0$, *where* $G(x,u) = \int_0^u g(x,s)\,ds$;

(G_2) $\widetilde{G}(x,u) := \frac{1}{4}g(x,u)u - G(x,u) \ge 0$, *and there exist* $c_0 > 0$ *and* $\sigma > \max\{1, \frac{2N}{N+2}\}$ *such that*

$$\bigl|G(x,u)\bigr|^\sigma \le c_0|u|^{2\sigma}\widetilde{G}(x,u)$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ *with* u *large enough*;

(G_3) *$g(x,-u) = -g(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.*

Then problem (1.1) has infinitely many nontrivial solutions $\{u_n\}$ such that $\|u_n\| \to \infty$ and $\Phi(u_n) \to \infty$.

In this paper, inspired by [11, 25–27], we study the sign-changing potential case for problem (1.1) by the mountain pass theorem and establish the existence of infinitely many solutions under more general superlinear assumptions.

Now, we are ready to state the main results of this paper.

Theorem 1.2 *Assume that* (V_1)–(V_2), (G_0)–(G_1), *and* (G_3) *are satisfied. Furthermore, assume that V and g satisfy the following conditions*:

(G_4) *There exist $\mu > 4$, $r_1 > 0$, and $\varsigma > 0$ such that*

$$\mu G(x,u) \leq ug(x,u) + \varsigma u^2, \quad \forall (x,u) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}, |u| \geq r_1;$$

(G_5) *There exists $r_2 > r_0$ such that*

$$g(x,u)u \geq 0, \quad \forall (x,u) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}, |u| \geq r_2.$$

Then problem (1.1) has infinitely many nontrivial solutions $\{u_n\}$ such that $\|u_n\| \to \infty$ and $\Phi(u_n) \to \infty$.

Example 1.1 Let $g(x,u) = a(x)[\frac{1}{5}u^5 - \frac{1}{2}u^2 \sin u + u \cos u]$, where $a \in C(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ and $0 < \inf_{\mathbb{R}^{\mathbb{N}}} a \leq \sup_{\mathbb{R}^{\mathbb{N}}} a < \infty$. It is easy to check that the superlinear function g does not satisfy Theorem 1.1, but it satisfies Theorem 1.2.

2 Variational setting and preliminaries

From (V_1), we can see that there exists a constant $V_0 > 0$ such that $\widetilde{V}(x) := V(x) + V_0 > 0$ for any $x \in \mathbb{R}^N$. Let $\widetilde{g}(x,u) := g(x,u) + V_0 u$ and study the following new equation:

$$-\triangle u + \widetilde{V}(x)u - \triangle(u^2)u = \widetilde{g}(x,u), \quad x \in \mathbb{R}^N. \tag{2.1}$$

So we can study the equivalent problem (2.1) of problem (1.1). Assume that V and G satisfy conditions (V_1)–(V_2), (G_0)–(G_1), and (G_4); it is easy to get that $\widetilde{V}$ and $\widetilde{g}$ still satisfy conditions (V_1)–(V_2), (G_0)–(G_1), and (G_4). Hence, we make the following assumption:

($\widetilde{V}_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) > 0$.

As usual, for $1 \leq s < +\infty$, we let

$$\|u\|_s = \left(\int_{\mathbb{R}^N} |u(x)|^s \, dx \right)^{1/s}, \quad u \in L^s(\mathbb{R}^N),$$
$$H^1(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N) \}$$

and the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right)^{1/2}.$$

Under assumption $(\widetilde{\mathrm{V}}_1)$, we consider the following working space:

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty \right\}$$

with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} \big(\nabla u \cdot \nabla v + V(x)uv\big)\,dx$$

and the norm

$$\|u\|_E = (u,u)_E^{\frac{1}{2}}.$$

As we all know, under assumption $(\widetilde{\mathrm{V}}_1)$, the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2,2^*]$, and the embedding $E \hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for $s \in [2,2^*)$, i.e., there exist constants $a_s > 0$ such that

$$\|u\|_s \le a_s \|u\|_E, \quad \forall u \in E, s \in [2,2^*].$$

Lemma 2.1 ([17]) *Under assumptions* $(\widetilde{\mathrm{V}}_1)$ *and* (V_2), *the embedding* $E \hookrightarrow L^s(\mathbb{R}^N)$ *is compact for* $s \in [2,2^*)$.

To solve problem (1.1), define the natural energy functional $\Phi : E \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)\,dx + \frac{1}{4}\int_{\mathbb{R}^N} \big(\big|\nabla\big(u^2\big)\big|^2\big)\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

Clearly,

$$\frac{1}{4}\int_{\mathbb{R}^N} \big(\big|\nabla\big(u^2\big)\big|^2\big)\,dx = \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,dx.$$

Therefore

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(\big(1 + 2|u|^2\big)|\nabla u|^2\big)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

As we all know, Φ cannot be well defined in E generally. To overcome this difficulty, we make the change of variables by Liu et al. [11] and Colin, Jean [13] as

$$v = f^{-1}(u),$$

where f is defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2|f(t)|^2}} \quad \text{on } [0,+\infty)$$

and

$$f(-t) = -f(t) \quad \text{on } (-\infty,0].$$

Let us recall some properties of variables $f:\mathbb{R}\to\mathbb{R}$, the proof of which can be found in [11, 13, 28].

Lemma 2.2 *The function $f(t)$ and its derivative enjoy the following properties*:

(f_1) *f is uniquely defined, C^∞, and invertible*;

(f_2) $|f'(t)|\le 1$ *for all* $t\in\mathbb{R}$;

(f_3) $|f(t)|\le |t|$ *for all* $t\in\mathbb{R}$;

(f_4) $f(t)/t\to 1$ *as* $t\to 0$;

(f_5) $f(t)/\sqrt{t}\to 2^{1/4}$ *as* $t\to+\infty$;

(f_6) $f(t)/2\le tf'(t)\le f(t)$ *for all* $t>0$;

(f_7) $f^2(t)/2\le tf(t)f'(t)\le f^2(t)$ *for all* $t\in\mathbb{R}$;

(f_8) $|f(t)|\le 2^{\frac14}|t|^{\frac12}$ *for all* $t\in\mathbb{R}$;

(f_9) *There exists a positive constant C such that*

$$\bigl|f(t)\bigr|\ge\begin{cases}C|t|, & |t|\le 1,\\ C|t|^{\frac12}, & |t|\ge 1;\end{cases}$$

(f_{10}) *For any $\alpha>0$, there exists a positive constant $C(\alpha)$ such that*

$$\bigl|f(\alpha t)\bigr|^2\le C(\alpha)\bigl|f(t)\bigr|^2;$$

(f_{11})

$$\bigl|f(t)f'(t)\bigr|\le 1/\sqrt{2}.$$

Therefore, after the change of variables, we get the following functional:

$$\Psi(v)=\frac12\int_{\mathbb{R}^N}|\nabla v|^2\,dx+\frac12\int_{\mathbb{R}^N}V(x)f^2(v)\,dx-\int_{\mathbb{R}^N}G\bigl(x,f(v)\bigr)\,dx. \tag{2.2}$$

It is easy to check that the functional Ψ is well defined in E. Our hypotheses mean that $\Psi\in C^1(E,\mathbb{R})$, we have

$$\bigl\langle\Psi'(v),\omega\bigr\rangle=\int_{\mathbb{R}^N}\nabla v\nabla\omega\,dx+\int_{\mathbb{R}^N}V(x)f(v)f'(v)\omega\,dx-\int_{\mathbb{R}^N}g\bigl(x,f(v)\bigr)f'(v)\omega\,dx \tag{2.3}$$

for any $\omega\in E$. It is clear that the critical points of Ψ are the weak solutions of the following equation:

$$-\triangle v=\frac{1}{\sqrt{1+2|f(v)|^2}}\bigl(g\bigl(x,f(v)\bigr)-V(x)f(v)\bigr)\quad\text{in }\mathbb{R}^N.$$

We also observe that if v is a critical point of Ψ, then $u=f(v)$ is a critical point of Φ, i.e., $u=f(v)$ is a solution of problem (2.1). Recall that a sequence $\{v_n\}\subset E$ is called a $(C)_c$-sequence if $\Psi(v_n)\to c$ and $(1+\|v_n\|_E)\Psi'(v_n)\to 0$, Ψ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

Proposition 2.1 ([29]) *Let X be an infinite dimensional Banach space, $X=Y\oplus Z$, where Y is finite dimensional. If $\varphi\in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition for all $c>0$ and*

(I_1) $\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ *for all* $u \in X$;
(I_2) *There exist positive constants* θ *and* α *such that* $\varphi|_{\partial B_\theta \cap Z} \geq \alpha$;
(I_3) *For any finite dimensional subspace* $\widetilde{X} \subset X$, *there is* $R = R(\widetilde{X}) > 0$ *such that* $\varphi(u) \leq 0$ *on* $\widetilde{X} \backslash B_R$.

Then φ *possesses an unbounded sequence of critical values.*

Lemma 2.3 *Suppose that* ($\widetilde{V}_1$), (V_2), (G_0)–(G_1), *and* (G_4) *are satisfied. Then any* $(C)_c$-*sequence of* Ψ *is bounded in* E.

Proof Let $\{v_n\} \subset E$ be such that

$$\Psi(v_n) \to c, \qquad \bigl(1 + \|v_n\|_E\bigr)\Psi'(v_n) \to 0. \tag{2.4}$$

Then there is a constant $C_1 > 0$ such that

$$\Psi(v_n) - \frac{2}{\mu}\Psi'(v_n)v_n \leq C_1. \tag{2.5}$$

First, we prove that there exists $C_2 > 0$ such that

$$\int_{\mathbb{R}^N} \bigl(|\nabla v_n|^2 + V(x)f^2(v_n)\bigr)\,dx \leq C_2.$$

Suppose to the contrary that

$$\|v_n\|_0^2 := \int_{\mathbb{R}^N} \bigl(|\nabla v_n|^2 + V(x)f^2(v_n)\bigr)\,dx \to \infty.$$

Let $\tilde{f}(v_n) := f(v_n)/\|v_n\|_0$, then $\|\tilde{f}(v_n)\|_E \leq 1$. Passing to a subsequence, we may assume that $\tilde{f}(v_n) \rightharpoonup \omega$ in E, $\tilde{f}(v_n) \to \omega$ in $L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*)$, and $\tilde{f}(v_n) \to \omega$ a.e. on $\mathbb{R}^N$.

Case one $\omega = 0$, according to the definition of f and (f_1) (see Lemma 2.2), we have

$$f(-t) = -f(t), \qquad f'(-t) = f'(t), \quad \forall t \in \mathbb{R}. \tag{2.6}$$

If $v_n \geq 0$ and $|f(v_n)| \geq r_2$, according to (G_5) and the definition of f, we have

$$g\bigl(x, f(v_n)\bigr) \geq 0. \tag{2.7}$$

Since (f_6) and (2.7), one sees that

$$\int_{\mathbb{R}^N} g\bigl(x, f(v_n)\bigr)f'(v_n)v_n\,dx \geq \frac{1}{2}\int_{\mathbb{R}^N} g\bigl(x, f(v_n)\bigr)f(v_n)\,dx. \tag{2.8}$$

If $v_n < 0$ and $|f(v_n)| \geq r_2$, according to (G_3), (G_5), (f_6), (2.6), (2.8), and the definition of f, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} g\bigl(x, f(v_n)\bigr)f'(v_n)v_n\,dx &= \int_{\mathbb{R}^N} g\bigl(x, f(-v_n)\bigr)f'(-v_n)(-v_n)\,dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} g\bigl(x, f(-v_n)\bigr)f(-v_n)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} g\bigl(x, f(v_n)\bigr)f(v_n)\,dx.
\end{aligned} \tag{2.9}$$

Let $r = \max\{r_0, r_1, r_2\}$. Because v_n is a Cerami sequence of Ψ, from (G_0), (G_4), (f_7), (2.5), (2.8), and (2.9), we obtain

$$\begin{aligned}
C_3 &\geq \Psi(v_n) - \frac{2}{\mu}\langle \Psi'(v_n), v_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx - \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\,dx \\
&\quad - \frac{2}{\mu}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \\
&\quad - \frac{2}{\mu}\int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)v_n\,dx + \frac{2}{\mu}\int_{\mathbb{R}^N} g\big(x, f(v_n)\big)f'(v_n)v_n\,dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx - \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\,dx \\
&\quad - \frac{2}{\mu}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \\
&\quad - \frac{2}{\mu}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx + \frac{2}{\mu}\int_{\mathbb{R}^N} g\big(x, f(v_n)\big)f'(v_n)v_n\,dx \\
&= \frac{\mu-4}{2\mu}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{\mu-4}{2\mu}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx - \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\,dx \\
&\quad + \frac{2}{\mu}\int_{\mathbb{R}^N} g\big(x, f(v_n)\big)f'(v_n)v_n\,dx \\
&= \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\,dx + \frac{2}{\mu}\int_{\mathbb{R}^N} g\big(x, f(v_n)\big)f'(v_n)v_n\,dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\,dx + \frac{1}{\mu}\int_{\mathbb{R}^N} g\big(x, f(v_n)\big)f(v_n)\,dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 + \int_{\{x||f(v_n)|\geq r, x\in\mathbb{R}^N\}} \left(\frac{1}{\mu}g\big(x, f(v_n)\big)f(v_n) - G\big(x, f(v_n)\big)\right)dx \\
&\quad - \int_{\{x||f(v_n)|< r, x\in\mathbb{R}^N\}} \left(\frac{1}{\mu}g\big(x, f(v_n)\big)f(v_n) - G\big(x, f(v_n)\big)\right)dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 + \int_{\{x||f(v_n)|\geq r, x\in\mathbb{R}^N\}} \left(\frac{1}{\mu}g\big(x, f(v_n)\big)f(v_n) - G\big(x, f(v_n)\big)\right)dx \\
&\quad - \int_{\{x||f(v_n)|< r, x\in\mathbb{R}^N\}} \left(\left|\frac{1}{\mu}g\big(x, f(v_n)\big)f(v_n)\right| + \left|G\big(x, f(v_n)\big)\right|\right)dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \frac{\varsigma}{\mu}\int_{\{x||f(v_n)|\geq r, x\in\mathbb{R}^N\}} f^2(v_n)\,dx \\
&\quad - \int_{\{x||f(v_n)|< r, x\in\mathbb{R}^N\}} \left[\frac{1}{\mu}\big(c_1|f(v_n)|^2 + c_2|f(v_n)|^p\big) + \frac{c_1}{2}|f(v_n)|^2 + \frac{c_2}{p}|f(v_n)|^p\right]dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \frac{\varsigma}{\mu}\|f(v_n)\|_2^2 \\
&\quad - \int_{\{x||f(v_n)|< r, x\in\mathbb{R}^N\}} \left[\frac{c_1(2+\mu)}{2\mu}|f(v_n)|^2 + \frac{c_2(p+\mu)}{p\mu}|f(v_n)|^p\right]dx \\
&= \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \frac{\varsigma}{\mu}\|f(v_n)\|_2^2 \\
&\quad - \int_{\{x||f(v_n)|< r, x\in\mathbb{R}^N\}} \left[\frac{c_1(2+\mu)}{2\mu} + \frac{c_2(p+\mu)}{p\mu}|f(v_n)|^{p-2}\right]|f(v_n)|^2\,dx
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \frac{\varsigma}{\mu}\|f(v_n)\|_2^2 \\
&\quad - \left[\frac{c_1(2+\mu)}{2\mu} + \frac{c_2(p+\mu)}{p\mu} r^{p-2}\right] \int_{\{x||f(v_n)|<r, x\in\mathbb{R}^N\}} |f(v_n)|^2\,dx \\
&\geq \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \frac{\varsigma}{\mu}\|f(v_n)\|_2^2 - \left[\frac{c_1(2+\mu)}{2\mu} + \frac{c_2(p+\mu)}{p\mu} r^{p-2}\right]\|f(v_n)\|_2^2 \\
&= \frac{\mu-4}{2\mu}\|v_n\|_0^2 - \left[\frac{\varsigma}{\mu} + \frac{c_1(2+\mu)}{2\mu} + \frac{c_2(p+\mu)}{p\mu} r^{p-2}\right]\|f(v_n)\|_2^2,
\end{aligned}$$

where $C_3 > 0$. Thus,

$$1 \leq \frac{2(\varsigma + \frac{c_1(2+\mu)}{2} + \frac{c_2(p+\mu)}{p} r^{p-2})}{\mu-4} \limsup_{n\to\infty} \|\tilde{f}(v_n)\|_2^2 = 0, \tag{2.10}$$

which is a contradiction.

Set

$$\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \leq |f(v_n(x))| < b\}, \quad 0 \leq a < b.$$

The second case $\omega \neq 0$, then $\mathrm{meas}(\Omega) > 0$, where $\Omega := \{x \in \mathbb{R}^N : \omega \neq 0\}$. For any $x \in \Omega$, we have $|f(v_n)| \to \infty$ as $n \to \infty$. Therefore, we have $\Omega \subset \Omega_n(r_0, \infty)$ for large $n \in \mathbb{N}$, where r_0 is defined in (G_1). By (G_1), we know that

$$\frac{G(x,f(v_n))}{|f(v_n)|^4} \to +\infty \quad \text{as } n \to \infty.$$

Using Fatou's lemma, we have

$$\int_\Omega \frac{G(x,f(v_n))}{|f(v_n)|^4}\,dx \to +\infty \quad \text{as } n \to \infty. \tag{2.11}$$

Since (2.4) and (2.11), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|v_n\|_0^2} \\
&= \lim_{n\to\infty} \frac{\Psi(v_n)}{\|v_n\|_0^2} \\
&= \lim_{n\to\infty} \frac{1}{\|v_n\|_0^2}\left(\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x) f^2(v_n)\right)dx - \int_{\mathbb{R}^N} G(x,f(v_n))\,dx\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{G(x,f(v_n))}{|f(v_n)|^2}|\tilde{f}(v_n)|^2\,dx - \int_{\Omega_n(r_0,\infty)} \frac{G(x,f(v_n))}{|f(v_n)|^2}|\tilde{f}(v_n)|^2\,dx\right) \\
&\leq \frac{1}{2} + \limsup_{n\to\infty}\left(\left(c_1 + c_2 r_0^{p-2}\right)\int_{\mathbb{R}^N} |\tilde{f}(v_n)|^2\,dx - \int_{\Omega_n(r_0,\infty)} \frac{G(x,f(v_n))}{|f(v_n)|^2}|\tilde{f}(v_n)|^2\,dx\right) \\
&\leq C_4 - \liminf_{n\to\infty} \int_\Omega \frac{G(x,f(v_n))}{|f(v_n)|^4}|f(v_n)\tilde{f}(v_n)|^2\,dx \\
&= -\infty,
\end{aligned}$$

where $C_4 > 0$, which is a contradiction. Therefore, there exists $C_2 > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)f^2(v_n)\right) dx \le C_2.$$

Next, to prove $\{v_n\}$ is bounded in E, we just need to show that there exists $C_5 > 0$ such that

$$\|v_n\|_0^2 := \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)f^2(v_n)\right) dx \ge C_5 \|v_n\|_E^2. \tag{2.12}$$

We can assume that $v_n \not\equiv 0$ (if not, the result is obvious). If this conclusion is not true, for a subsequence, we have $\frac{\|v_n\|_0^2}{\|v_n\|_E^2} \to 0$. Let $\omega_n = \frac{v_n}{\|v_n\|_E}$ and $j_n = \frac{f^2(v_n)}{\|v_n\|_E^2}$, then

$$\int_{\mathbb{R}^N} \left(|\nabla \omega_n|^2 + V(x) j_n(x)\right) dx \to 0.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla \omega_n|^2 \, dx \to 0,$$
$$\int_{\mathbb{R}^N} V(x) j_n(x) \, dx \to 0,$$
$$\int_{\mathbb{R}^N} V(x) \omega_n^2 \, dx \to 1.$$

Similar to the idea of [27], we support that for any $\varepsilon > 0$, $\operatorname{meas}(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |v_n(x)| \ge C_6\}$, $C_6 > 0$ is independent of n. If not, there exist $\varepsilon_0 > 0$ and $\{v_{n_k}\} \subset \{v_n\}$ such that

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^N : \left|v_{n_k}(x)\right| \ge k\right\}\right) \ge \varepsilon_0 > 0,$$

where $k > 0$ is an integer. Set $\Omega_{n_k} := \{x \in \mathbb{R}^N : |v_{n_k}(x)| \ge k\}$. From (f_9) and $(\widetilde{V}_1)$, there exists $M' > 0$ such that

$$\|v_{n_k}\|_0^2 \ge \int_{\mathbb{R}^N} V(x) f^2(v_{n_k}) \, dx \ge \int_{\Omega_{n_k}} V(x) f^2(v_{n_k}) \, dx \ge M' k \varepsilon_0 \to +\infty \quad \text{as } k \to \infty,$$

which is a contradiction. Hence our conclusion is true. Notice that as $|v_n(x)| \le C_6$, from (f_9) and (f_{10}), we have

$$\frac{C}{C_6^2} v_n^2 \le f^2\left(\frac{1}{C_6} v_n\right) \le C_7 f^2(v_n),$$

where $C_7 > 0$ is a constant. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \Omega_n} V(x) \omega_n^2 \, dx &\le C_8 \int_{\mathbb{R}^N \setminus \Omega_n} V(x) \frac{f^2(v_n)}{\|v_n\|_E^2} \, dx \\ &\le C_8 \int_{\mathbb{R}^N} V(x) j_n(x) \, dx \to 0,\end{aligned} \tag{2.13}$$

where $C_8 > 0$ is a constant. On the other hand, by absolute continuity of integral, there exists $\varepsilon > 0$ such that

$$\int_{\Omega'} V(x)\omega_n^2\,dx \le \frac{1}{2}, \tag{2.14}$$

where $\Omega' \subset \mathbb{R}^N$ and $\operatorname{meas}(\Omega') < \varepsilon$. Combining (2.13) and (2.14), we obtain

$$\int_{\mathbb{R}^N} V(x)\omega_n^2\,dx = \int_{\mathbb{R}^N\setminus\Omega_n} V(x)\omega_n^2\,dx + \int_{\Omega_n} V(x)\omega_n^2\,dx \le \frac{1}{2} + o(1),$$

which means that $1 \le \frac{1}{2}$, a contradiction. Then (2.12) holds. This completes the proof. □

Lemma 2.4 *Assume that* $(\widetilde{\mathrm{V}}_1)$, (V_2), (G_0)–(G_1), *and* (G_4) *hold, then* Ψ *satisfies* $(C)_c$-*condition.*

Proof According to Lemma 2.3, we know that $\{v_n\}$ is bounded in E. For a subsequence, we may assume that $v_n \rightharpoonup v$ in E. From Lemma 2.1, we have $v_n \to v$ in $L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*)$, and $v_n \to v$ a.e. on $\mathbb{R}^N$. We claim that there exists $C_9 > 0$ such that

$$\int_{\mathbb{R}^N} \bigl(\bigl|\nabla(v_n - v)\bigr|^2 + V(x)\bigl(f(v_n)f'(v_n) - f(v)f'(v)\bigr)(v_n - v)\bigr)\,dx \ge C_9\|v_n - v\|_E^2. \tag{2.15}$$

We may assume that $v_n \ne v$ (otherwise the conclusion is trivial). Set

$$\widetilde{\omega}_n = \frac{v_n - v}{\|v_n - v\|_E}, \qquad \widetilde{j}_n = \frac{f(v_n)f'(v_n) - f(v)f'(v)}{v_n - v}.$$

Argue by contradiction and assume that

$$\int_{\mathbb{R}^N} \bigl(|\nabla\widetilde{\omega}_n|^2 + V(x)\widetilde{j}_n(x)\widetilde{\omega}_n^2\bigr)\,dx \to 0.$$

Since

$$\frac{d}{dt}\bigl(f(t)f'(t)\bigr) = f(t)f''(t) + \bigl(f'(t)\bigr)^2 = \frac{1}{(1 + 2f^2(t))^2} > 0$$

$f(t)f'(t)$ is strictly increasing, for any $C_{10} > 0$, there exists $\delta_1 > 0$ such that

$$\frac{d}{dt}\bigl(f(t)f'(t)\bigr) \ge \delta_1$$

as $|t| \le C_{10}$. Hence, we know that $\widetilde{j}_n(x) > 0$. Therefore

$$\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}_n|^2\,dx \to 0, \qquad \int_{\mathbb{R}^N} V(x)\widetilde{j}_n(x)\widetilde{\omega}_n^2\,dx \to 0, \qquad \int_{\mathbb{R}^N} V(x)\widetilde{\omega}_n^2\,dx \to 1.$$

By a similar argument as (2.13) and (2.14), we can conclude a contradiction.

On the other hand, it follows from (f_2), (f_3), (f_8), (f_{11}), and (G_0) that there is $C_{11} > 0$ such that

$$\left|\int_{\mathbb{R}^N} \bigl(g\bigl(x, f(v_n)\bigr)f'(v_n) - g\bigl(x, f(v)\bigr)f'(v)\bigr)(v_n - v)\,dx\right|$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^N} C_{11}\big(|v_n| + |v_n|^{\frac{p}{2}-1} + |v| + |v|^{\frac{p}{2}-1}\big)|v_n - v|\,dx \\
&\leq C_{11}\Big(\big(\|v_n\|_2 + \|v\|_2\big)\|v_n - v\|_2 + \big(\|v_n\|_{\frac{p}{2}}^{\frac{p-2}{2}} + \|v\|_{\frac{p}{2}}^{\frac{p-2}{2}}\big)\|v_n - v\|_{\frac{p}{2}}\Big) \\
&= o(1). \qquad (2.16)
\end{aligned}
$$

Then, by (2.15) and (2.16), we get

$$
\begin{aligned}
o(1) &= \big\langle \Psi'(v_n) - \Psi'(v), v_n - v\big\rangle \\
&= \int_{\mathbb{R}^N} \Big(\big|\nabla(v_n - v)\big|^2 + V(x)\big(f(v_n)f'(v_n) - f(v)f'(v)\big)(v_n - v)\Big)\,dx \\
&\quad - \int_{\mathbb{R}^N} \big(g\big(x,f(v_n)\big)f'(v_n) - g\big(x,f(v)\big)f'(v)\big)(v_n - v)\,dx \\
&\geq C_9\|v_n - v\|_E^2 + o(1).
\end{aligned}
$$

Therefore, we obtain $\|v_n - v\|_E \to 0$ as $n \to \infty$. This completes the proof. □

3 Proof of the main results

Let $\{e_j\}$ be a total orthonormal basis of E, define

$$
X_j = \mathbb{R}e_j, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k+1}^{\infty} X_j},
$$

where $k \in \mathbb{Z}$ and Y_k is finite dimensional.

Lemma 3.1 ([25]) *Under assumptions* $(\widetilde{\mathrm{V}}_1)$, (V_2), *for* $s \in [2, 2^*)$,

$$
\beta_k(s) := \sup_{v \in Z_k, \|v\|=1} \|v\|_s \to 0, \quad k \to \infty.
$$

We need to prove that there exists $C_{12} > 0$ such that

$$
\int_{\mathbb{R}^N} \big(|\nabla v|^2 + V(x)f^2(v)\big)\,dx \geq C_{12}\|v\|_E^2, \quad \forall v \in S_\theta, \qquad (3.1)
$$

where $S_\theta := \{v \in E : \|v\| = \theta\}$. Similar to the proof of (2.12), we can get that (3.1) is true. And according to Lemma 3.1, we can choose an integer $m \geq 1$ such that

$$
\|v\|_2^2 \leq \frac{C_{12}}{4c_1}\|v\|_E^2, \qquad \|v\|_{\frac{p}{2}}^{\frac{p}{2}} \leq \frac{C_{12}}{4c_2}\|v\|_E^{\frac{p}{2}}, \quad \forall v \in Z_m. \qquad (3.2)
$$

Lemma 3.2 *Assume that* $(\widetilde{\mathrm{V}}_1)$, (V_2), *and* (G_0) *hold, then there exist positive constants* θ *and* α *such that*

$$
\Psi|_{S_\theta \cap Z_m} \geq \alpha.
$$

Proof For every $v \in Z_m$ and $\|v\|_E = \theta < 1$, from (f_3), (f_8), (3.1), and (3.2), we obtain

$$\begin{aligned}
\Psi(v) &= \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla v|^2 + V(x)f^2(v)\big)\,dx - \int_{\mathbb{R}^N} G\big(x,f(v)\big)\,dx \\
&\geq \frac{C_{12}}{2}\|v\|_E^2 - \int_{\mathbb{R}^N}\big(c_1\big|f(v)\big|^2 + c_2\big|f(v)\big|^p\big)\,dx \\
&\geq \frac{C_{12}}{2}\|v\|_E^2 - \int_{\mathbb{R}^N}\big(c_1|v|^2 + c_2|v|^{\frac{p}{2}}\big)\,dx \\
&\geq \frac{C_{12}}{2}\|v\|_E^2 - \frac{C_{12}}{4}\|v\|_E^2 - \frac{C_{12}}{4}\|v\|_E^{\frac{p}{2}} \\
&= \frac{C_{12}}{4}\|v\|_E^2\big(1 - \|v\|_E^{\frac{p-4}{2}}\big) > 0,
\end{aligned}$$

where $p \in (4, 22^*)$. This completes the proof.

Lemma 3.3 *Assume that* ($\widetilde{V}_1$), (V_2), (G_0), *and* (G_1) *hold, then for any finite dimensional subspace* $\tilde{E} \subset E$, *there exists* $R = R(\widetilde{E}) > 0$ *such that*

$$\Psi(v) \leq 0, \quad \forall v \in \widetilde{E} \setminus B_R.$$

Proof For any finite dimensional subspace $\widetilde{E} \subset E$, there exists an integer $m > 0$ such that $\widetilde{E} \subset E_m$. To the contrary, there is a sequence $\{v_n\} \subset \widetilde{E}$ such that $\|v_n\|_E \to \infty$ and $\Psi(v_n) > 0$. Hence

$$\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla v_n|^2 + V(x)f^2(v_n)\big)\,dx > \int_{\mathbb{R}^N} G\big(x,f(v_n)\big)\,dx. \tag{3.3}$$

Let $\overline{\omega}_n = \frac{v_n}{\|v_n\|_E}$, for a subsequence, we can assume that $\overline{\omega}_n \rightharpoonup \overline{\omega}$ in E, $\overline{\omega}_n \to \overline{\omega}$ in $L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*)$, and $\overline{\omega}_n \to \overline{\omega}$ a.e. on $\mathbb{R}^N$. Let $\Omega_1 := \{x \in \mathbb{R}^N : \overline{\omega}(x) \neq 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^N : \overline{\omega}(x) = 0\}$. If $\operatorname{meas}(\Omega_1) > 0$, according to ($G_1$), ($f_5$), and Fatou's lemma, we obtain

$$\int_{\Omega_1}\frac{G(x,f(v_n))}{\|v_n\|_E^2}\,dx = \int_{\Omega_1}\frac{G(x,f(v_n))}{(f(v_n))^4}\frac{(f(v_n))^4}{v_n^2}\overline{\omega}_n^2\,dx \to +\infty.$$

By (G_0) and (G_1), there exists $C_{13} > 0$ such that

$$G(x,t) \geq -C_{13}t^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{\Omega_2}\frac{G(x,f(v_n))}{\|v_n\|_E^2}\,dx \geq -C_{13}\int_{\Omega_2}\frac{f^2(v_n)}{\|v_n\|_E^2}\,dx \geq -C_{13}\int_{\Omega_2}\overline{\omega}_n^2\,dx.$$

Because $\overline{\omega}_n \to \overline{\omega}$ in $L^2(\mathbb{R}^N)$, then

$$\liminf_{n\to\infty}\int_{\Omega_2}\frac{G(x,f(v_n))}{\|v_n\|_E^2}\,dx \geq 0.$$

Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{G(x,f(v_n))}{\|v_n\|_E^2}\,dx = +\infty.$$

By (3.3), we get $\frac{1}{2} > +\infty$, which is a contradiction. So $\mathrm{meas}(\Omega_1) = 0$, i.e., $\overline{\omega}(x) = 0$ a.e. on $\mathbb{R}^N$. According to the equivalency of all norms in $\tilde{E}$, there exists $\iota > 0$ such that

$$\|v\|_2^2 \geq \iota \|v\|_E^2, \quad \forall v \in \tilde{E}.$$

Hence

$$0 = \lim_{n\to\infty} \|\overline{\omega}_n\|_2^2 \geq \lim_{n\to\infty} \iota \|\overline{\omega}_n\|_E^2 = \iota,$$

a contradiction. This completes the proof. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Set $X = E$, $Y = Y_m$ and $Z = Z_m$. Clearly, by $\Psi(0) = 0$ and (G_3), we get Ψ is even. According to Lemma 2.4, Lemma 3.2, and Lemma 3.3, we know that all the conditions of Proposition 2.1 are satisfied. Therefore, problem (2.1) possesses infinitely many nontrivial solutions sequence $\{v_n\}$ such that $\Psi(v_n) \to \infty$ as $n \to \infty$, then problem (1.1) also possesses infinitely many nontrivial solutions sequence $\{u_n\}$ such that $\Phi(u_n) \to \infty$ as $n \to \infty$. □

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The authors declare no competing interests.

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